Crossing Symmetry, Current Algebra, and the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin Relation

A. N. KAMAL*
Rutherford High Energy Laboratory, Chilton, Didcot, Berkshire, England
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The implications of crossing symmetry and current algebra on the KSRF relation have been investigated.

USING current algebra (CA) and an extra assumption regarding the equal-time commutator $[\partial_\mu A_\mu(x), A_0(y)]$, Weinberg[1] derived the π-π scattering amplitude[2]

$$T=F^{-2}[\delta_{\alpha\beta}\delta_{\gamma\delta}(s-1)+\delta_{\alpha\gamma}\delta_{\beta\delta}(t-1)+\delta_{\alpha\delta}\delta_{\beta\gamma}(u-1)], \quad (1)$$

where $\alpha\beta$ and $\gamma\delta$ are the initial and final isospin indices; s, t, and u are the conventional Mandelstam variables, and F is the pion-decay constant with an experimental value of 94 MeV. Projection of T in the three isospin states and the lowest angular momenta in these states gives

$$T_0^{(0)}(\nu)=7/F^2+O(\nu), \quad (2)$$

$$T_1^{(1)}(\nu)=(4/3F^2)\nu+O(\nu^2), \quad (3)$$

$$T_0^{(2)}(\nu)=-2/F^2+O(\nu). \quad (4)$$

On the other hand, one has the KSRF[3] relation

$$f_\rho^2=m_\rho^2/2F^2. \quad (5)$$

The ρ width Γ_ρ is related to f_ρ via

$$\Gamma_\rho=\tfrac{2}{3}(f_\rho^2/4\pi)k_\rho^3/m_\rho^2, \quad (6)$$

where

$$4k_\rho^2=m_\rho^2-4.$$

If one uses the experimental values[4] for m_ρ and F, one gets[5]

$$(f_\rho^2/4\pi)\simeq 2.66 \quad (7)$$

and hence

$$\Gamma_\rho\simeq 130\text{ MeV}, \quad (8)$$

leading to the belief that the KSRF relation is numerically correct. Recently Gounaris and Sakurai[6] evaluated a finite-width correction to Γ_ρ. They parametrize the p-wave phase shift in an effective-range fashion à la Chew and Mandelstam[7]

$$(k/\sqrt{s})\cot\delta_1=k^2h(s)+a+bk^2, \quad (9)$$

with

$$h(s)=\frac{2}{\pi}\frac{k}{\sqrt{s}}\ln\left(\frac{\sqrt{s}+2k}{2}\right).$$

The resonance condition and the width are defined through

$$\cot\delta_1|_{s=m_\rho^2}=0, \quad (10)$$

$$\left.\frac{d\delta_1}{ds}\right|_{s=m_\rho^2}=\frac{1}{\Gamma_\rho m_\rho}. \quad (11)$$

If m_ρ and Γ_ρ were known, these two conditions would serve to determine a and b in Eq. (9). Their purpose is, however, to evaluate Γ_ρ, with m_ρ given. Thus one extra condition has to be invoked. This they do, following Brown and Goble,[8] by matching $\cot\delta_1$ to the CA value given by Eq. (3) at threshold. One then gets[6]

$$\Gamma_\rho=\frac{k_\rho^5}{3\pi F^2m_\rho^2}\left(1-\frac{k_\rho^4h'(m_\rho^2)}{3\pi F^2}\right)^{-1}, \quad (12)$$

where $h'(s)\equiv dh/ds$. If one ignores the variation of $h(s)$ in the neighborhood $s=m_\rho^2$, one gets the Brown-Goble[8] result

$$\Gamma_\rho=k_\rho^5/3\pi F^2m_\rho^2. \quad (13)$$

Comparing Eq. (13) with the formula for ρ decay,

$$\Gamma_\rho=\tfrac{2}{3}(f\rho^2/4\pi)k_\rho^2/m_\rho^2, \quad (14)$$

one finds

$$f_\rho^2=2k_\rho^2/F^2=(m_\rho^2/2F^2)(1-4/m_\rho^2). \quad (15)$$

Ignoring the square of the pion mass compared to the square of the ρ mass, one gets the KSRF relation[3]

$$f_\rho^2=m_\rho^2/2F^2. \quad (16)$$

The purpose of the present work is to investigate the above results and see what is involved.

If one assumes that the p-wave amplitude is *dominated near threshold* by the ρ-meson pole in the direct channel then one gets, near threshold,

$$\frac{1}{\nu}T_1^{(1)}(\nu)\simeq\frac{8}{3}\frac{f_\rho^2}{m_\rho^2-4}+O(\nu), \quad (17)$$

* On leave from University of Alberta, Edmonton, Canada.
[1] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).
[2] We are using $S=1+iT(2\pi)^4\delta(\sum_i P_i)$ and

$$T^{(I)}=\sum_{l=0}^{\infty}(2l+1)T^{(I)}(\nu)P_l(\cos\theta),$$

where $4\nu=s-4$, $m_\pi=1$.
[3] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. **147**, 1071 (1966). (Hereafter referred to as KSRF.)
[4] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **40**, 77 (1968).
[5] J. J. Sakurai, Phys. Rev. Letters **17**, 1021 (1966).
[6] G. J. Gounaris and J. J. Sakurai, Phys. Rev. Letters **21**, 244 (1968).
[7] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[8] L. S. Brown and R. L. Goble, Phys. Rev. Letters **20**, 346 (1968).

where f_ρ is introduced via

$$\mathcal{L}_{\rho\pi\pi}=\tfrac{1}{2}f_\rho\epsilon^{\alpha\beta\gamma}\rho_\gamma{}^\mu\phi^\alpha\overleftrightarrow{\partial}_\mu\phi^\beta. \tag{18}$$

Comparing Eq. (17) with Eq. (3), it is immediately clear that

$$f_\rho{}^2=2k_\rho{}^2/F^2. \tag{19}$$

This in essence is the Brown-Goble[8] derivation of the KSRF relation.

In chiral dynamics one can understand the ρ dominance of the p-wave amplitude at threshold as follows. Weinberg[9] has derived an effective Lagrangian

$$\mathcal{L}=(1/8F^2)(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2+(1/4F^2)\boldsymbol{\phi}^2(\partial_\mu\boldsymbol{\phi})^2+f_\rho\boldsymbol{\rho}^\mu\cdot(\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi})-(1/4F^2)(\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi})^2. \tag{20}$$

The first two terms reproduce CA result, and it can be shown quite easily that the π-π amplitude resulting from the ρ pole in all channels is exactly canceled by the last term. One can, however, emphasize the role of the ρ meson by using the vector identity

$$(\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi})^2+(\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi})^2=\boldsymbol{\phi}^2(\partial_\mu\boldsymbol{\phi})^2. \tag{21}$$

Then Eq. (20) takes the form

$$\mathcal{L}=(1/8F^2)(\boldsymbol{\phi}\cdot\boldsymbol{\phi})^2+(1/4F^2)(\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi})^2+f_\rho\boldsymbol{\rho}^\mu\cdot(\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi}). \tag{22}$$

Now the ρ meson appears as the dominant contributor to the π-π scattering. The p-wave contribution from the second term exactly cancels the p-wave contribution from ρ *exchanges*, leaving the ρ meson in the direct channel as the sole contributor to the p waves at and near threshold. As for the S waves, the first two terms largely cancel in the $I=0$ channel, giving a combined contribution of $-1/F^2$ at threshold. The ρ exchange provides $8/F^2$, giving a total in $I=0$ at threshold of $7/F^2$. In the $I=2$ state the second term in Eq. (22) does not contribute at threshold. The first term yields $2/F^2$ while ρ exchange gives $-4/F^2$, giving a total of $-2/F^2$. It must be emphasized that a model which has only $\rho\pi\pi$ interaction and a symmetric ϕ^4 interaction will not be consistent with the KSRF relation and the CA result. An additional derivative coupling is essential to do the trick. In the present paper we are searching for a dynamical explanation (dynamical in the sense of analyticity) for the validity of the KSRF relation consistent with CA and crossing symmetry, that is, the mechanism that provides the magic cancellations. These cancellations in Weinberg's model are provided by $(\boldsymbol{\phi}\cdot\partial_\mu\boldsymbol{\phi})^2$ terms which at low energies simulate the combined effect of complicated mechanisms including what goes on at higher energies.

Looking at the problem purely from a dynamical viewpoint, there are difficulties in understanding Eq. (19) obtained in the manner explained. Since the comparison with the CA result is being made at threshold, there is no *a priori* reason to believe that the direct ρ pole would dominate the p-wave amplitude. More important, for the sake of crossing symmetry one would be forced to consider the ρ-exchange graphs also. If one does so, then the total amplitude is

$$T_1{}^{(1)}(\nu)\simeq\frac{8}{3}\frac{f_\rho{}^2\nu}{m_\rho{}^2-4}+f_\rho{}^2\frac{m_\rho{}^2+s+4\nu}{2\nu}\times\left[-2+\frac{m_\rho{}^2+2\nu}{2\nu}\ln\left(1+\frac{4\nu}{m_\rho{}^2}\right)\right]. \tag{23}$$

Near threshold

$$\frac{1}{\nu}T_1{}^{(1)}(\nu)\simeq4\frac{f_\rho{}^2}{m_\rho{}^2}\left(1+\frac{4}{m_\rho{}^2}\right). \tag{24}$$

If Eq. (24) is now matched with the CA result of Eq. (3), one gets

$$f_\rho{}^2\simeq(m_\rho{}^2/3F^2)(1-4/m_\rho{}^2); \tag{25}$$

or, to 15% accuracy,

$$f_\rho{}^2\simeq m_\rho{}^2/3F^2. \tag{26}$$

This value is $\frac{2}{3}$ the KSRF value and the width of the ρ meson, on the basis of the argument *presented so far*, would also be $\frac{2}{3}$ the width produced by the KSRF relation.

The parametrization for δ_1 used by Gounaris and Sakurai[6] and by Brown and Goble,[8] Eq. (9), corresponds to an amplitude generated by a direct chain summation with an interaction[10]:

$$\mathcal{L}_{\rho\pi\pi}=\lambda_1(\boldsymbol{\phi}\times\partial_\mu\boldsymbol{\phi})^2. \tag{27}$$

In the resonance approximation it simply corresponds to the ρ-meson pole graph in the direct channel. One expects, therefore, that the effective-range parametrization is valid near $s=m_\rho{}^2$. Again from the dynamical viewpoint the crossed channels are expected to contribute near threshold and there is no *a priori* reason why this parametrization ought to be good near threshold.

One can, in principle, maintain crossing symmetry and yet have the ρ meson dominate the p-wave amplitude at low energies. For this to happen, the p-wave projections of $I=0$ and 2 exchanges will have to cancel the p-wave projections of ρ exchange. This is feasible since the crossing matrix

$$\beta=\begin{pmatrix}\frac{1}{3} & 1 & 5/3\\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6}\\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{pmatrix} \tag{28}$$

has a structure such that the p-wave projection of $I=2$ exchange could, in principle, cancel the p-wave projection of $I=0$ and 1 exchanges (i.e., $\frac{1}{3}+\frac{1}{2}-\frac{5}{6}=0$).

[9] S. Weinberg, Phys. Rev. **166**, 1568 (1968).

[10] A. N. Kamal, Proc. Phys. Soc. (London) **77**, 917 (1961); Y. Miyamoto, Progr. Theoret. Phys. (Kyoto) **24**, 840 (1960).

Such a cancellation must be taking place since the KSRF relation appears to be numerically correct.

Maintaining crossing symmetry, the s-wave projections of the ρ exchanges are

$$T_s^{(0)}(\nu)=2f_\rho^2\left[-2+\frac{s+m_\rho^2+4\nu}{2\nu}\times\ln\left(1+\frac{4\nu}{m_\rho^2}\right)\right]\underset{\nu\to 0}{\simeq}16\frac{f_\rho^2}{m_\rho^2}, \quad (29)$$

$$T_s^{(2)}(\nu)=-\tfrac{1}{2}T_s^{(0)}(\nu)\underset{\nu\to 0}{\simeq}\frac{-8f_\rho^2}{m_\rho^2}. \quad (30)$$

Comparing Eq. (29) with Eq. (2), one gets

$$f_\rho^2=\tfrac{7}{16}m_\rho^2/F^2, \quad (31)$$

a value remarkably close to the KSRF value. The reason for this is that ρ exchange provides almost all of the CA amplitude at threshold—it provides a contribution of $+0.18$ to the scattering length with f_ρ^2 given by Eq. (7). This would suggest that all other contributions to the $I=0$, s-wave amplitude at threshold should either largely cancel each other or be individually close to zero. *If one assumes that there is a resonance in the $I=0$ channel* then the threshold contribution to the $I=0$ s-wave amplitude from this resonance in both direct and crossed channels would be reasonably large and greater than zero, since $\beta_{0,0}$ element of the crossing matrix is $+\frac{1}{3}$. In order to cancel this positive contribution the $I=2$ exchange would have to provide a repulsion in the direct $I=0$, s-wave channel. Since $\beta_{2,0}$ is $+5/3$, this implies that the space part of the s-wave projection of $I=2$ exchanges would have to be negative. If $I=2$ exchange were thought of as a single-particle exchange, this would imply a negative metric. Recall, however, that $\beta_{2,2}$ is $-\frac{5}{6}$ and that for the cancellations in the p wave (discussed earlier) the space part of the p-wave projection of the $I=2$ exchanges would have to be positive. The only way to reconcile cancellations in s and p waves with an $I=0$ resonance would be to assume that the s-channel discontinuity due to $I=2$ exchanges for physical energies but unphysical angles[11] must oscillate—positive near the origin so that the tail would be suppressed in a p-wave projection, $Q_1(z)$ dropping faster than $Q_0(z)$ for large z.[12] If such a mechanism were operational, then one could allow fair-sized s-wave contributions with somewhat precarious cancellations among them. The other alternative is that the s-wave contributions arising from mechanisms other than the ρ exchange are individually vanishingly small at threshold. If this is true then it is hard to understand the KSRF relation within the context of crossing symmetry.

If one equates Eq. (30) with Eq. (4), one gets

$$f_\rho^2=\tfrac{1}{4}m_\rho^2/F^2, \quad (32)$$

which is exactly one-half the KSRF value. The reason is not far to seek. The ρ exchange provides a scattering length $a_2\simeq-0.09$ with f_ρ^2 given by Eq. (7), while CA suggests $a_2\simeq-0.06$. Though the discrepancy is as much as 50%, one is dealing with small numbers and it would be perfectly possible for the $I=0$ and $I=2$ exchanges to provide a small positive contribution (~ 0.03) to the scattering length since both $\beta_{0,2}$ and $\beta_{2,2}$ have the right signature. The cancellation mechanism leaving a small positive residue is also feasible if the crossed-channel discontinuity oscillates. Thus the KSRF relation can only be consistent with crossing symmetry if there are rather fine cancellations in the s and the p waves. We are proposing that *if there is a low energy $I=0$, s-wave resonance* such a cancellation can be brought about by an oscillating discontinuity in the $I=2$ crossed channel. If these cancellations do not occur and the s waves arising from mechanisms other than ρ exchange are vanishingly small, it is hard to reconcile the KSRF relation with crossing symmetry.

[11] See Ref. 7, Eq. (IV.10).

[12] A. Froissart-Gribov representation {V. N. Gribov, Zh. Eksperim. i Teor Fiz. 41, 1962 (1961) [English transl.: Sov. Phys.—JETP **14**, 1395 (1962)]; R. Omnès and M. Froissart, *Mandelstam Theory and Regge Poles* (W. A. Benjamin, Inc., New York, 1963), Chap. 7} has been used for all partial waves.