fortunately, either of these possibilities would greatly reduce the effectiveness of the hard-pion scheme as a powerful calculation method for treating low-energy reactions.

Note added in proof. We have been informed by Dr. A. Brody of a numerical error in the SLAC analysis.[4] The corrected experimental value is $(g_T/g_L)^2=0.64\pm0.25$. In terms of the linear hard-pion model of SW,[1] this implies $-2.32<\delta<-1.14$ and $\Gamma_A\leqslant 41$ MeV. In the nonlinear model that we consider this value of $(g_T/g_L)^2$ corresponds to $\Gamma_A=90_{-14}^{+10}$ MeV, to be compared with $\Gamma_A=140\pm30$ MeV of Ref. 4. It is amusing that the nonlinear hard-pion model described here is in better agreement with present experiment than either Schnitzer-Weinberg[1] or Gilman-Harari.[3] Brown and West inform us that the data on radiative pion decay are also in better agreement with the nonlinear hard-pion model than with SW.[1]

We wish to acknowledge valuable discussions with Professor K. Wilson. After the completion of this work we have come to know of similar results obtained by S. Brown and G. West[11] within their pole-dominance framework. We thank them for a conversation.

[11] S. G. Brown and G. B. West, this issue, Phys. Rev. **180**, 1613 (1969).

PHYSICAL REVIEW VOLUME 180, NUMBER 5 25 APRIL 1969

Resonant Solutions of the Low Equation from Fixed-Point Theorems*

H. McDaniel†
Illinois Institute of Technology, Chicago, Illinois 60616

AND

R. L. Warnock‡
CERN, Geneva, Switzerland

(Received 3 July 1968)

Fixed-point theorems are used to prove the existence of a class of solutions to the one-meson Low equation of the static-baryon model. The main result is that there exist solutions involving an arbitrary choice of narrow resonances. This is true for any crossing matrix with a finite number of channels, and for any cutoff function of a large class. For sufficiently small coupling constants, the solutions can be constructed by a convergent iteration procedure. The stable particles and the arbitrarily chosen resonances are associated with Castillejo-Dalitz-Dyson poles of an appropriate denominator function. The methods used do not suffice to show that solutions of the bootstrap type exist. Our earlier work is improved in that resonances are allowed and a bigger range of coupling constants and a weaker cutoff are permitted. The analysis is based on a crossing-symmetric N/D formulation of the Low equation.

1. INTRODUCTION

In a recent publication,[1] one of us showed that the one-meson Low equation of the static-baryon model has a solution for arbitrary crossing matrix and cutoff, provided that the coupling constant is sufficiently small. The solution corresponds to an "elementary" baryon, in the sense that the baryon does not participate in a bootstrap. Equivalently, one may say that the baryon is associated with a Castillejo-Dalitz-Dyson (CDD) pole. Another property of the solution discussed in Ref. 1 is that it does not have resonances, and therefore does not relate to observed meson-baryon scattering.

In the present paper, our method is improved so that we are able to prove existence of solutions with narrow resonances, and to weaken the requirements on coupling constants and cutoffs. The resonant solutions proved to exist are still not of the bootstrap type. They are directly connected with CDD poles. Non-CDD resonances, part of a bootstrap or not, are beyond our reach. The CDD poles can be prescribed at will, provided their residues are sufficiently small. One can have, therefore, as many narrow resonances as he likes, at arbitrarily chosen positions. This persistence of the CDD ambiguity in models with arbitrary crossing matrix is not especially surprising, in view of experience with soluble models[2] and the work of Lovelace[3] and Atkinson[4] on more general models. The present paper seems to contain the first complete proof for general crossing matrix, however, since Lovelace does not touch the existence question, and Atkinson leaves aside some technical difficulties concerning ghost poles.

It would seem to be of importance to the bootstrap program to decide whether the Low equation has (in

* Work partially supported by the National Science Foundation, and performed in part under the auspices of the U. S. Atomic Energy Commission.

† National Aeronautical and Space Administration Trainee.
‡ Permanent address: Illinois Institute of Technology, Chicago, Ill., and Argonne National Laboratory, Argonne, Ill.

[1] R. L. Warnock, Phys. Rev. **170**, 1323 (1968); **174**, 2169(E) (1968).

[2] A. W. Martin and W. D. McGlinn, Phys. Rev. **136**, B1515 (1964); A. A. Cunningham, J. Math. Phys. **8**, 716 (1967); J. T. Cushing (to be published).

[3] C. Lovelace, Commun. Math. Phys. **4**, 261 (1967).

[4] D. Atkinson, J. Math. Phys. **8**, 2281 (1967).

some appropriate precise sense) a bootstrap solution. As we remarked in Ref. 1, the proposals for approximate bootstrap solutions found in the literature do not convince one that there is an exact bootstrap solution. Furthermore, Huang and Mueller[5] proved that with particular crossing matrices and a certain class of cutoffs, there is, in a reasonable technical sense, no bootstrap solution.

It seems likely that levitation by the bootstrap procedure will be as hard for theoretical physicists as it is for ordinary people. At present it is not clear whether any of the known methods of nonlinear analysis will be powerful enough to take us into the region of large coupling constants where the bootstrap solutions are supposed to lie. The contraction mapping principle and Schauder's theorem, used in the present paper, seem to be inadequate. Possibly the Newton-Kantorovich method will be more suitable, as Amatuni[6] has suggested.

In Sec. 2, we begin by substituting an equivalent N/D equation for the Low equation. This is just the usual equation for the N function, but with our full account of crossing symmetry it is a nonlinear functional equation. It may be regarded as having the form $N=AN$, where A is a nonlinear operator on an appropriate function space. The question of the existence of solutions of the Low equation is then a matter of the existence of fixed points of A (i.e., members of the function space left invariant by A) together with the matter of ruling out ghost zeros of the D function. In Ref. 1, the operator A was taken to be the integral operator which occurs in the Low equation itself. By identifying A as the N/D operator we gain two advantages. First, the CDD poles are incorporated naturally, and we can ask about the existence of solutions with prescribed CDD poles. Second, we can do the proofs for larger coupling constants than in the direct attack on the Low equation.

Section 3 contains a statement of two standard fixed-point theorems, together with a description of what one must do to apply the theorems in our example.

Section 4 gives the main analysis of the operator A in the case without CDD poles. It is verified that the iterative fixed-point theorem (contraction mapping principle) applies when the coupling constant and cutoff are suitably restricted. Ghost zeros of the D function are ruled out.

Section 5 repeats the work of Sec. 4, but with account of CDD terms. It is found that the arguments are practically unperturbed if the CDD residues are sufficiently small. The required smallness of the residues implies a corresponding narrowness of our resonances. The solutions which are proved to exist have certain known qualitative properties. For instance, the N function differs from the Born term by less than a prescribed amount, and has the same sign as the Born term. The sign of the CDD pole residues is also the same as that of the Born term. It is proved that with the opposite sign of residues ghost poles are inevitable. The solutions satisfy the Levinson relation in the general form including a CDD term.

In Sec. 6 we find that under conditions weaker than those of Sec. 5 we can still prove existence of a solution, irrespective of uniqueness or a means of computation. Here Schauder's fixed-point theorem is used, instead of the iterative method.

Section 7 is devoted to conclusions and the outlook for further applications of nonlinear analysis in S-matrix theory.

2. N/D EQUATION SUBSTITUTED FOR THE LOW EQUATION

The static-baryon Low equation in the one-meson approximation is written as follows[7]:

$$f_\alpha(z)=\frac{\lambda_\alpha}{z}+\frac{1}{\pi}\int_1^\infty\frac{d\omega\,\rho(\omega)\,|f_\alpha(\omega+i0)|^2}{\omega-z}+\sum_{\beta=1}^{n}c_{\alpha\beta}\frac{1}{\pi}\int_1^\infty\frac{d\omega\,\rho(\omega)\,|f_\beta(\omega+i0)|^2}{\omega+z}\,. \quad (2.1)$$

The meson energy is ω in a system of units where the meson mass is 1. The amplitude f_α for scattering of pseudoscalar mesons in p waves is related to the phase shift at physical ω by the formula

$$f_\alpha(\omega+i0)=\sin\delta_\alpha(\omega)e^{i\delta_\alpha(\omega)}/\rho(\omega)\,,\quad \omega\geqslant 1$$
$$\rho(\omega)=k^3v^2(k)/12\pi\,,\qquad k=(\omega^2-1)^{1/2}\,. \quad (2.2)$$

The cutoff function $v(k)$ is the Fourier transform of the baryon source density.[7] The crossing matrix $c=[c_{\alpha\beta}]=c^*$ obeys $c^2=1$, but otherwise it is arbitrary. The number of channels labeled by the index α is arbitrary but finite. The real parameters λ_α are sums of direct and crossed-channel baryon pole residues

$$\lambda_\alpha=-g_\alpha{}^2+\sum_\beta c_{\alpha\beta}g_\beta{}^2\,. \quad (2.3)$$

We write the N/D system as an integral equation for the N function,[8,9] where because of crossing the kernel is a functional of N itself. We take a D function which obeys an unsubtracted dispersion relation, which is normalized to 1 at infinity, and which has in general a finite number of first-order CDD poles at finite,

[5] K. Huang and A. H. Mueller, Phys. Rev. Letters **14**, 396 (1965); Phys. Rev. **140**, B365 (1965).
[6] A.Ts. Amatuni, Nuovo Cimento **58A**, 321 (1968).
[7] E. M. Henley and W. Thirring, *Elementary Quantum Field Theory* (McGraw-Hill Book Co., Inc., New York, 1962).
[8] J. L. Uretsky, Phys. Rev. **123**, 1459 (1961).
[9] G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963).

physical energies $\omega_{\alpha i}>1$. Thus,

$$f_\alpha(z)=\frac{N_\alpha(z)}{D_\alpha(z)}=N_\alpha(z)\Big/\left(1-\sum_{i=1}^{m(\alpha)}\frac{c_{\alpha i}}{\omega_{\alpha i}-z}-\frac{1}{\pi}\int_1^\infty\frac{\rho(\omega)N_\alpha(\omega)d\omega}{\omega-z}\right),\quad(2.4)$$

where the residues $c_{\alpha i}$ are real. Equation (2.4) is not the most general N/D representation, and hence it does not lead to the most general integral equations. For example, we have not allowed for the possibility of a CDD pole at infinity, or for an infinite number of CDD poles. Our object here is not to discuss all solutions of the Low equation, but rather to show that the CDD ambiguity persists when crossing symmetry is imposed, and to illustrate techniques which are likely to be useful in more interesting physical problems. After the change of variable

$$t=1/\omega,$$

the integral equation is as follows:

$$N(t)=B(t)-\sum_{i=1}^{m}c_it_it\frac{B(t)-B(t_i)}{t-t_i}-\frac{t}{\pi}\int_0^1\frac{B(t)-B(\tau)}{t-\tau}\frac{\rho(\tau)}{\tau}N(\tau)d\tau,\quad(2.5)$$

$$B_\alpha(t)=\lambda_\alpha t+\frac{t}{\pi}\sum_\beta c_{\alpha\beta}\int_0^1\frac{\rho(\tau)|f_\beta(\tau)|^2d\tau}{\tau(\tau+t)},\quad(2.6)$$

$$|f(t)|^2=N^2(t)\Big/\left\{\left[1+\sum\frac{c_it_it}{t_i-t}+\frac{t}{\pi}P\int_0^1\frac{\rho(\tau)N(\tau)d\tau}{\tau(\tau-t)}\right]^2+\rho^2(t)N^2(t)\right\}.\quad(2.7)$$

The index α is suppressed where unnecessary, and we write $N(t)$ for $N(\omega(t))$, etc. Notice that the direct-channel baryon pole is regarded as a "left-hand singularity," i.e., it appears in the N function. This means that when the D function has no zero on the physical sheet (as will be the case in our work of Secs. 4 and 5), then D has the representation[9,10]

$$D(z)=R(z)\mathfrak{D}(z)=\prod_{i=1}^{m}\frac{1}{z-\omega_i}\exp\left[\frac{-z}{\pi}\int_1^\infty\frac{\delta(\omega)d\omega}{\omega(\omega-z)}\right].\quad(2.8)$$

We have not yet learned how to approach existence questions for stable baryon poles generated dynamically as zeros of D.

If N is a solution of (2.5)–(2.7), then by the usual argument of the N/D method, Eq. (2.4) furnishes a solution of the Low equation provided that D has no zero on the physical sheet. We shall find that zeros of D can be forbidden by limiting the values of coupling constants and CDD residues.

The right-hand side of (2.5) is to be regarded as the result of a nonlinear operator A acting on N. Thus, (2.5) reads

$$N=AN,\quad(2.9)$$

and our object is to investigate fixed points of A in an appropriate function space.

Equations (2.1) and (2.5) may be generalized to allow several kinds of baryons and mesons with unequal masses. Then unitarity would take a matrix form, and we would use the matrix N/D method. The general outlines of the following proofs would be retained, however, and one can expect that similar existence theorems would hold. In particular, there is no reason to expect that the CDD ambiguity would be eliminated or reduced by inclusion of additional channels.

3. FIXED-POINT THEOREMS AND CHOICE OF FUNCTION SPACE

We make use of two of the most common tools of nonlinear analysis,[11–14] namely, the contraction mapping principle and Schauder's fixed-point principle.

The contraction mapping principle is concerned with metric spaces. A space K is called a *metric space* if to every pair φ,ψ of points there is assigned a real number $\delta(\varphi,\psi)\geqslant 0$ called the *distance* between these points such that

(a) $\delta(\varphi,\psi)=0$, if and only if $\varphi=\psi$

(b) $\delta(\varphi,\psi)=\delta(\psi,\varphi)$,

(c) $\delta(\varphi,\eta)+\delta(\eta,\psi)\geqslant\delta(\varphi,\psi)$,

for all φ, ψ, and η in K. Limits in K will be defined with respect to the distance; i.e., a sequence $\{\varphi_n\}$ tends to a limit φ if $\delta(\varphi_n,\varphi)$ tends to zero. We write $\varphi_n\to\varphi$. A convergent sequence satisfies the Cauchy condition, as is easily seen:

$$\delta(\varphi_n,\varphi_m)\leq\delta(\varphi_n,\varphi)+\delta(\varphi_m,\varphi)<\epsilon,\quad n,m>N.\quad(3.1)$$

However, a sequence meeting the Cauchy condition (a "Cauchy sequence") is not necessarily convergent. A metric space K is called *complete* if every Cauchy sequence has a limit point in K. The *contraction mapping principle* (or Banach-Cacciopoli fixed-point theorem) is as follows: In a complete metric space K let A be an

[10] M. Sugawara and A. Kanazawa, Phys. Rev. **126**, 2251 (1962).

[11] M. A. Krasnosel'skii, *Topological Methods in the Theory of Non-linear Integral Equations* (Pergamon Press, Inc., Oxford, England, 1964).

[12] J. Cronin, *Fixed Points and Topological Degree in Non-linear Analysis* (American Mathematical Society, Providence, R. I., 1964).

[13] W. Pogorzelski, *Integral Equations and Their Applications* (Pergamon Press, Inc., Oxford, England, 1966), Vol. I.

[14] T. L. Saaty and J. Bram, *Non-linear Mathematics* (McGraw-Hill Book Co. Inc., New York, 1964).

operator such that $A(K)\subset K$, and such that

$$\delta(A\varphi,A\psi)\leq\beta\delta(\varphi,\psi)\,,\quad 0\leq\beta<1 \tag{3.2}$$

for all φ, $\psi\in K$ and β independent of φ, ψ. Then the operator A has a unique fixed point φ in K: $\varphi=A\varphi$. The sequence $\varphi_n=A\varphi_{n-1}$ converges to φ for any initial point $\varphi_0\in K$. The error at the nth iteration is bounded in terms of that at the first iteration:

$$\delta(\varphi,\varphi_n)\leq[\beta^n/(1-\beta)]\delta(\varphi_1,\varphi_0)\,. \tag{3.3}$$

An operator satisfying (3.2) is often called a *contraction mapping*. The proof of the preceding theorem is quite easy.[11–14]

If we are to apply the contraction-mapping idea to the N/D equation (2.9), how are we to choose the metric space K? It should be chosen so as to include physically interesting solutions, be a subset of the domain of A, and lend itself to practical proofs of $A(K)\subset K$ and the contraction property (3.2). One can think of more than one possibility, but it seems satisfactory to choose all real n-component functions $\varphi(t)=[\varphi_1(t),\cdots,\varphi_n(t)]$ which meet the following conditions:

$$|\varphi_\alpha(t)|\leq at\,,\qquad 0\leq t\leq 1 \tag{3.4}$$

$$|\varphi_\alpha(t)-\varphi_\alpha(t')|\leq b|t-t'|^\mu\,,\quad 0\leq t,t'\leq 1\,,\quad 0<\mu<1. \tag{3.5}$$

Here a and b are constants independent of φ_α, which must eventually be determined so that the hypotheses of the contraction-mapping theorem are verified. The distance is chosen to be

$$\delta(\varphi,\psi)=\sup_{\alpha,t}|\varphi_\alpha(t)-\psi_\alpha(t)|+H[\varphi-\psi]\,, \tag{3.6}$$

$$H[\chi]=\sup_{\substack{\alpha,t,t'\\ t\neq t'}}\left|\frac{\chi_\alpha(t)-\chi_\alpha(t')}{|t-t'|^\mu}\right|\,. \tag{3.7}$$

The inclusion of the minimum Hölder coefficient H in (3.6) has to do with the necessity of estimating the principal-value integral in Eq. (2.7). It may immediately be checked that (3.6) has the properties of a distance. That K is complete is proved in Appendix A. We learned to use Eq. (3.6) from Ref. 13, p. 582.

To introduce Schauder's principle we mention a classical theorem which has a nice intuitive appeal. It is Brouwer's fixed-point theorem,[14] which states that a continuous mapping of a closed sphere in R_n into itself has at least one fixed point. *Schauder's theorem* is an infinite-dimensional generalization: Let K be a convex closed subset of a linear normed space. Suppose that A is a continuous mapping of K into itself such that $A(K)$ is compact. Then A has at least one fixed point in K.

The definitions involved in the theorem are summarized in Ref. 1, or in Ref. 12, Chap. 3. Proofs of the Brouwer and Schauder theorems, not very difficult, may be found in Ref. 14. For the set K in our application of Schauder's theorem we once again take the set of functions defined in (3.4) and (3.5). For the norm we could take $\|\varphi\|=\delta(\varphi,0)$ from (3.6), but for our purposes it will be more economical to use the following norm:

$$\|\varphi\|=\sup_{\alpha,t}|\varphi_\alpha(t)|\,. \tag{3.8}$$

To apply Schauder's theorem we do not need such detailed estimates of principal-value integrals, so the H term in (3.6) can be dropped. According to Ref. 1, the conditions for Schauder's theorem are met in our case if

(a) $A(K)\subset K$,

(b) A is continuous in its action on K,

i.e., $\|A\varphi-A\psi\|<\epsilon$ if $\|\varphi-\psi\|<\delta(\epsilon)$. (3.9)

4. ANALYSIS OF THE NON-LINEAR N/D OPERATOR

In this section we apply the contraction-mapping theorem to prove existence (by construction) of a unique solution to the N/D equation in the set K [Eqs. (3.4) and (3.5)]. We also show that the corresponding D function is free of ghost zeros, hence that we have a solution of the Low equation. To avoid obscuring the argument, we first omit CDD poles; they are reinstated in Sec. 5.

Let φ be any member of the set K defined in (3.4) and (3.5). We study $\psi=A\varphi$, where the operator A defined in (2.5)–(2.7) can be written out in the following way:

$$\psi(t)=A\varphi(t)=B(t)-\frac{t}{\pi}\int_0^1 G(t,\tau)\frac{\rho(\tau)}{\tau}\varphi(\tau)d\tau\,, \tag{4.1}$$

$$B_\alpha(t)=\lambda_\alpha t+\sum c_{\alpha\beta}\frac{t}{\pi}\int_0^1\frac{\rho(\tau)|f_\beta(\tau)|^2d\tau}{\tau(\tau+t)}\,, \tag{4.2}$$

$$G_\alpha(t,\tau)=\frac{B_\alpha(t)-B_\alpha(\tau)}{t-\tau}=\lambda_\alpha+\sum_\beta c_{\alpha\beta}\frac{1}{\pi}\int_0^1\frac{\rho(\sigma)|f_\beta(\sigma)|^2d\sigma}{(\sigma+t)(\sigma+\tau)}\,, \tag{4.3}$$

$$|f(t)|^2=\varphi^2(t)\Bigg/\left\{\left[1+\frac{t}{\pi}P\int_0^1\frac{\rho(\tau)\varphi(\tau)d\tau}{\tau(\tau-t)}\right]^2+\rho^2(t)\varphi^2(t)\right\}\,. \tag{4.4}$$

Our first task is to show that $\psi\in K$ whenever $\varphi\in K$; i.e., $A(K)\subset K$. As a preliminary to that, we get a bound on $|f|^2$. From (3.4),

$$|f(t)|^2\leq\left\{at\Bigg/\left[1-\sup\left|\frac{t}{\pi}P\int_0^1\frac{\rho(\tau)\varphi(\tau)d\tau}{\tau(\tau-t)}\right|\right]\right\}^2\,. \tag{4.5}$$

Here and in the following "sup" without subscripts means supremum with respect to whatever variables are free, either $\alpha=1,2,\cdots,n$ or t, $0\leq t\leq 1$, or both. The principal-value integral in (4.5) is bounded by splitting it into two parts and applying (3.4) and (3.5):

$$\left|\frac{t}{\pi}P\int_0^1\frac{\rho(\tau)\varphi(\tau)d\tau}{\tau(\tau-t)}\right|$$
$$=\left|\frac{t}{\pi}\int_0^1\rho(\tau)\left[\frac{\varphi(\tau)-\varphi(t)}{\tau-t}\right]\frac{d\tau}{\tau}+\frac{t\varphi(t)}{\pi}P\int_0^1\frac{\rho(\tau)d\tau}{\tau(\tau-t)}\right|$$
$$\leq b\sup\left|\frac{t}{\pi}\int_0^1\frac{\rho(\tau)d\tau}{\tau|\tau-t|^{1-\mu}}\right|+a\sup\left|\frac{t^2}{\pi}P\int_0^1\frac{\rho(\tau)d\tau}{\tau(\tau-t)}\right|$$
$$=bA_1+aA_2. \tag{4.6}$$

The constants A_1, A_2 exist if we require some reasonable behavior of the cutoff function, namely,

$$|\rho(t)-\rho(\tau)|\leq k|t-\tau|^{\nu},\quad k,\nu>0 \tag{4.7a}$$

$$\rho(1)=\rho(0)=0. \tag{4.7b}$$

Taking (4.7a) and (4.7b) together, we have

$$|\rho(t)|\leq kt^{\nu}=k\omega^{-\nu},\quad \nu>0 \tag{4.8}$$

which allows a weaker cutoff than was required in Ref. 1. We choose a and b to be so small that

$$bA_1+aA_2<1. \tag{4.9}$$

In that case

$$|f(t)|^2\leq\left(\frac{at}{1-bA_1-aA_2}\right)^2\equiv a^2\xi(a,b)t^2. \tag{4.10}$$

The dependence of ξ on a and b will be essentially irrelevant for our work, provided (4.9) holds. Bounds on B and G follow directly from (4.2), (4.3), and (4.10):

$$|B(t)|\leq t[|\lambda|+\sum|c|a^2\xi A_3],$$
$$|G(t,\tau)|\leq|\lambda|+\sum|c|a^2\xi A_3,$$
$$A_3=\frac{1}{\pi}\int_0^1\rho(t)dt. \tag{4.11}$$

We have introduced the notations $|\lambda|=\sup_\alpha|\lambda_\alpha|$, $\sum|c|=\sup_\alpha\sum_\beta|c_{\alpha\beta}|$. From (4.1) and (4.11) we have the inequality

$$|\psi(t)|\leq t[|\lambda|+\sum|c|a^2\xi A_3][1+aA_3]. \tag{4.12}$$

Thus, to meet the first condition (3.4) for $A(K)\subset K$ we can require

$$[|\lambda|+\sum|c|a^2\xi A_3][1+aA_3]\leq a. \tag{4.13}$$

To deal with the second condition (3.5) we evaluate the difference quotient

$$\frac{\psi(t)-\psi(t')}{t-t'}=G(t,t')$$
$$-\frac{1}{\pi}\int_0^1\left[\frac{tG(t,\tau)-t'G(t',\tau)}{t-t'}\right]\frac{\rho(\tau)}{\tau}\varphi(\tau)d\tau. \tag{4.14}$$

From (4.3) the expression in brackets is seen to have the form

$$\frac{tG_\alpha(t,\tau)-t'G_\alpha(t',\tau)}{t-t'}=\lambda_\alpha$$
$$+\sum_\beta c_{\alpha\beta}\frac{1}{\pi}\int_0^1\frac{\rho(\sigma)\sigma|f_\beta(\sigma)|^2d\sigma}{(\sigma+t)(\sigma+t')(\sigma+\tau)}. \tag{4.15}$$

An upper bound of the integral in (4.15) is obtained by discarding t, t', and τ in the denominators, and we find from (4.11), (4.14), and (4.15) that the increment of ψ obeys the bound

$$|\psi(t)-\psi(t')|$$
$$\leq|t-t'|[|\lambda|+\sum|c|a^2\xi A_3][1+aA_3]. \tag{4.16}$$

Since $|t-t'|\leq|t-t'|^{\mu}$, we guarantee the second condition (3.5), provided

$$[|\lambda|+\sum|c|a^2\xi A_3][1+aA_3]\leq b. \tag{4.17}$$

The last step of the proof is to show that A is a contraction operator, i.e., to demonstrate (3.2) when the distance is given by (3.6). We have to demonstrate that there is a fixed $\beta<1$ such that

$$\sup|\psi-\bar{\psi}|+H[\psi-\bar{\psi}]$$
$$\leq\beta(\sup|\varphi-\bar{\varphi}|+H[\varphi-\bar{\varphi}]), \tag{4.18}$$

for all φ, $\bar{\varphi}\in K$, where $\psi=A\varphi$, $\bar{\psi}=A\bar{\varphi}$. We estimate the two terms on the left side of (4.18) separately. Each estimate is in the form of a linear combination of the two terms that occur on the right side. This is the reason that the H term is included in the distance. Thus, our upper bound for the left side will take the form $a_1\sup|\varphi-\bar{\varphi}|+a_2H[\varphi-\bar{\varphi}]$, and $\max(a_1,a_2)$ will be a satisfactory β provided it can be made less than 1.

From (4.1) we have

$$\sup|\psi-\bar{\psi}|\leq\sup\Bigg\{|B-\bar{B}|$$
$$+\left|\frac{t}{\pi}\int_0^1[G(t,\tau)\varphi(\tau)-\bar{G}(t,\tau)\bar{\varphi}(\tau)]\frac{\rho(\tau)}{\tau}d\tau\right|\Bigg\}. \tag{4.19}$$

By the definition (4.2) of B, the first term on the right has the bound

$$\sup|B-\bar{B}|$$
$$\leq\sum|c|\frac{1}{\pi}\int_0^1\frac{\rho(\tau)d\tau}{\tau(\tau+1)}\sup_\beta||f|^2-|\bar{f}|^2|. \tag{4.20}$$

Here we were able to replace $t/(\tau+t)$, an increasing function of t, by its maximum value $1/(\tau+1)$. From (4.4) one sees that $|f|^2-|\bar{f}|^2$ has the form

$$\frac{x}{y}-\frac{\bar{x}}{\bar{y}}=\frac{1}{y}(x-\bar{x})+\frac{\bar{x}}{y\bar{y}}(\bar{y}-y), \tag{4.21}$$

where

$$x=\varphi^2,$$

$$y=\left[1+\frac{\tau}{\pi}P\int_0^1\frac{\rho(t)\varphi(t)dt}{t(t-\tau)}\right]^2+\rho^2(\tau)\varphi^2(\tau),$$

$$x-\bar{x}=(\varphi+\bar{\varphi})\Delta,$$

$$y-\bar{y}=\tau\frac{P}{\pi}\int_0^1\frac{\rho(t)\Delta(t)dt}{t(t-\tau)}\times\left[2+\tau\frac{P}{\pi}\int_0^1\frac{\rho(t)[\varphi(t)+\bar{\varphi}(t)]dt}{t(t-\tau)}\right]+\rho^2(\tau)[\varphi(\tau)+\bar{\varphi}(\tau)]\Delta(\tau). \tag{4.22}$$

In (4.22) we use the notation $\Delta=\varphi-\bar{\varphi}$. When the principal-value integrals in (4.22) are decomposed in the manner of (4.6), we get the following majorizations:

$$\left|\frac{\tau^2}{\pi}P\int_0^1\frac{\rho(t)\Delta(t)dt}{t(t-\tau)}\right|\leq H[\Delta]A_4+\sup|\Delta|A_2, \tag{4.23}$$

$$\left|\frac{\tau}{\pi}P\int_0^1\frac{\rho(t)[\varphi(t)+\bar{\varphi}(t)]dt}{t(t-\tau)}\right|\leq 2bA_1+2aA_2, \tag{4.24}$$

$$A_4=\sup\left|\frac{t^2}{\pi}\int_0^1\frac{\rho(\tau)d\tau}{\tau|\tau-t|^{1-\mu}}\right|. \tag{4.25}$$

By (4.22)–(4.25) and (4.10) we find

$$||f|^2-|\bar{f}|^2|\leq 2a\xi\tau[a\xi(1+bA_1+aA_2)(A_4H[\Delta]+A_2\sup|\Delta|)+(1+a^2\xi A_5)\sup|\Delta|], \tag{4.26}$$

where

$$A_5=\sup|t\rho(t)|^2. \tag{4.27}$$

That is,

$$||f|^2-|\bar{f}|^2|\leq\tau(R_1\sup|\Delta|+R_2H[\Delta]), \tag{4.28}$$

where R_1 and R_2 are certain constants. Introducing this result in (4.20) we get our estimate of the first term in (4.19):

$$\sup|B-\bar{B}|\leq\sum|c|(R_1\sup|\Delta|+R_2H[\Delta])A_6,$$

$$A_6=\frac{1}{\pi}\int_0^1\frac{\rho(t)dt}{t+1}. \tag{4.29}$$

To handle the second term in (4.19), we write the expression in the square bracket as

$$G\varphi-\bar{G}\bar{\varphi}=G\Delta+\bar{\varphi}(G-\bar{G}). \tag{4.30}$$

The first term on the right of (4.30) is majorized by means of (4.11). By (4.3) and (4.28) the second term obeys the inequality

$$\left|\bar{\varphi}(\tau)\frac{1}{\pi}\sum_\beta c_{\alpha\beta}\int_0^1 d\sigma\,\rho(\sigma)\frac{|f_\beta|^2-|\bar{f}_\beta|^2}{(\sigma+\tau)(\sigma+t)}\right|\leq a\sum|c|A_6(R_1\sup|\Delta|+R_2H[\Delta]). \tag{4.31}$$

By combining (4.29)–(4.31) we can now limit both of the terms on the right side of (4.19):

$$\sup|\psi-\bar{\psi}|\leq\sum|c|(R_1\sup|\Delta|+R_2H[\Delta])A_6(1+aA_7)+A_7\sup|\Delta|(|\lambda|+\sum|c|a^2\xi A_3),$$

$$A_7=\frac{1}{\pi}\int_0^1\frac{\rho(t)dt}{t}. \tag{4.32}$$

The right side of this inequality is a linear combination of the two terms that appear in $\delta(\varphi,\bar{\varphi})$:

$$\sup|\psi-\bar{\psi}|\leq M_1\sup|\Delta|+M_2H[\Delta]. \tag{4.33}$$

Next, we must derive an inequality of the same type for $H[\psi-\bar{\psi}]$. Equation (4.14) gives a start:

$$H[\psi-\bar{\psi}]=\sup_{\alpha tt'}\left|\frac{\psi(t)-\psi(t')}{t-t'}-\frac{\bar{\psi}(t)-\bar{\psi}(t')}{t-t'}\right|\leq\sup_{\alpha tt'}|G(t,t')-\bar{G}(t,t')|+\sup_{\alpha tt'}\left|\frac{1}{\pi}\int_0^1\frac{d\tau}{\tau}\rho(\tau)\left[\varphi(\tau)\frac{tG(t,\tau)-t'G(t',\tau)}{t-t'}-\bar{\varphi}(\tau)\frac{t\bar{G}(t,\tau)-t'\bar{G}(t',\tau)}{t-t'}\right]\right|. \tag{4.34}$$

The first term on the right is estimated by (4.31), modulo a factor $\bar{\varphi}$. By means of (4.15) the expression within the brackets in the second term, call it X, is handled as follows:

$$X=\left|\lambda_\alpha\Delta_\alpha(\tau)+\sum_\beta c_{\alpha\beta}\frac{1}{\pi}\int_0^1\frac{d\sigma\,\rho(\sigma)\sigma}{(\sigma+t)(\sigma+t')(\sigma+\tau)}\times[\varphi_\alpha(\tau)|f_\beta(\sigma)|^2-\bar{\varphi}_\alpha(\tau)|\bar{f}_\beta(\sigma)|^2]\right|$$
$$\leq|\lambda|\sup|\Delta|+\sum|c|[a^2\xi A_3\sup|\Delta|+aA_6(R_1\sup|\Delta|+R_2H[\Delta])]. \tag{4.35}$$

The integral in (4.35) was treated by writing

$$\varphi|f|^2-\bar{\varphi}|\bar{f}|^2=\varphi(|f|^2-|\bar{f}|^2)+|\bar{f}|^2(\varphi-\bar{\varphi}),$$

with a subsequent application of (4.10) and (4.28). Now from (4.34), (4.31), and (4.35) we can assemble the result

$$H[\psi-\bar{\psi}]\leq\sum |c|A_7(R_1\sup|\Delta|+R_2H[\Delta]) +A_7\{|\lambda|\sup|\Delta|+\sum |c|a[a\xi A_3\sup|\Delta| +A_6(R_1\sup|\Delta|+R_2H[\Delta])]\}. \quad (4.36)$$

In other words, there are constants M_3 and M_4 such that

$$H[\psi-\bar{\psi}]\leq M_3\sup|\Delta|+M_4H[\Delta]. \quad (4.37)$$

Finally, from (4.33) and (4.37) we obtain the inequality which is relevant for the contraction property:

$$\delta(\psi,\bar{\psi})\leq\beta\delta(\varphi,\bar{\varphi}), \quad \beta=\max(M_1+M_3, M_2+M_4). \quad (4.38)$$

We can now summarize a set of sufficient conditions for the N/D equation to have a unique solution in the set K. The conditions are

Eqs. (4.7a), (4.7b), (4.9), (4.13), (4.17), $\beta<1$ in (4.38). (4.39)

In order to show that these requirements can be met we define a parameter x,

$$a=\alpha_1 x, \quad b=\alpha_2 x, \quad |\lambda|=\alpha_3 x. \quad (4.40)$$

Then the inequalities (4.9), (4.13), and (4.17) which imply $A(K)\subset K$ may be expressed as

$$O(x)<1, \quad \alpha_3+O(x)\leq\alpha_1, \quad \alpha_3+O(x)\leq\alpha_2, \quad (4.41)$$

where $O(x)$ stands for a function bounded by a constant times x as x tends to zero with the α's fixed. By (4.32) and (4.36) we also have $\beta=O(x)$, so it is clear that all of the conditions (4.39) are met when x is small enough, provided we choose $\alpha_3<\alpha_1$, $\alpha_3<\alpha_2$ and take a cutoff satisfying (4.7).

We now know that with appropriate restrictions on the coupling constant, the cutoff, and the parameters a and b which define K, there is a unique $\varphi=N$ in K which satisfies the N/D equation (2.5) without CDD poles. This N function is obtained by iteration, $\varphi_{n+1}=A\varphi_n$, beginning with any member φ_0 of K. An error bound for the nth iteration is provided by (3.3). To show that the scattering amplitude constructed from this N function by (2.4) without pole terms is a solution of the Low equation, we must be sure that D has no zeros on the physical sheet. Zeros on the real axis will be ruled out by a bound on $D(\omega+i0)-1$. The same bound will hold at complex z, because of the Phragmén-Lindelöf theorem[15]: Let $f(z)$ be analytic in the half-plane $y>0$, continuous in the closed half-plane $y\geqslant 0$, and bounded on the real axis $[|f(x)|\leqslant M]$. Suppose also that $f(z)=O(e^{r\beta})$, $\beta<1$, uniformly in θ for a sequence $r=r_n\to\infty$. Then $|f(z)|\leq M$ for $y\geqslant 0$.

To apply the Phragmén-Lindelöf theorem we return to the original energy plane. General complex values of the energy are denoted by z, and real values by ω. Suppose that $D(z)$ is constructed from our iterative solution $N(\omega)$,

$$D(z)=1-\frac{1}{\pi}\int_1^\infty\frac{\rho(\omega)N(\omega)d\omega}{\omega-z}. \quad (4.42)$$

According to (3.4) and (4.8)

$$\rho(\omega)N(\omega)=O(\omega^{-1-\nu}), \quad \nu>0, \quad \omega\to\infty \quad (4.43)$$

and by (3.5) and (4.7), we know that ρN is Hölder-continuous on any finite interval. It follows that $D(z)-1$ is analytic and bounded in the open cut plane, and continuous in the closed cut plane. The continuity follows from the Plemelj-Privalov theorem on boundary values of Cauchy integrals.[16] We can apply the Phragmén-Lindelöf theorem to $f(z)=D(z)-1$ in the upper half-plane. By (4.6) we know that

$$|D(\omega+i0)-1| \leq[(bA_1+aA_2)^2+A_5a^2]^{1/2}, \quad \omega\geqslant 1. \quad (4.44)$$

For $\omega<1$ we have a smaller bound:

$$|D(\omega+i0)-1|\leq a\left|\frac{1}{\pi}\int_1^\infty\frac{\rho(\omega')d\omega'}{(\omega'-1)+(1-\omega)}\right| \leq a\left|\frac{1}{\pi}\int_1^\infty\frac{\rho(\omega')d\omega'}{\omega'-1}\right| =a\left|\frac{1}{\pi}\int_0^1\frac{\rho(t)dt}{t(t-1)}\right|\leq aA_2. \quad (4.45)$$

Thus, $|D(z)-1|$ is bounded as in (4.44) in the entire cut plane. We choose a and b so that the right side of (4.44) is less than 1. This implies an inequality like the first one of (4.41), which can always be satisfied. Then $D(z)$ has no zero in the closed half-plane. Also there is no zero in the other half-plane because of $D(z^*)=D(z)^*$.

5. ANALYSIS OF N/D OPERATOR INCLUDING CDD POLES

For notational simplicity we write the equations including just one CDD pole. All of the following arguments go through in the same way, however, if there is any finite number of poles at distinct energies. If the CDD term in $D(\omega)$ is $-\hat{c}/(\hat{\omega}-\omega)$, then we must investigate $\psi=A\varphi$, where

$$\psi(t)=B(t)-t\hat{t}\hat{c}G(t,\hat{t})-\frac{t}{\pi}\int_0^1 G(t,\tau)\frac{\rho(\tau)}{\tau}\varphi(\tau)d\tau. \quad (5.1)$$

[15] R. P. Boas, Jr., *Entire Functions* (Academic Press Inc., New York, 1954), p. 3.

[16] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff, Ltd. Groningen, The Netherlands, 1953).

B and G have the same expressions as before in terms of $|f|^2$ [cf. Eqs. (4.2) and (4.3)], while $|f|^2$ is changed to the following:

$$|f(t)|^2=\varphi^2(t)\Bigg/\left\{\left[1+\frac{t\hat{t}\hat{c}}{\hat{t}-t}+\frac{t}{\pi}P\int_0^1\frac{\rho(\tau)\varphi(\tau)d\tau}{\tau(\tau-t)}\right]^2+\rho^2(t)\varphi^2(t)\right\}. \quad (5.2)$$

We assume $\hat{t}<1$. If $\hat{c}$ is small, the term in square brackets in the denominator of (5.2) may vanish at a point $t=t_*$ very close to $t=\hat{t}$. If φ is a solution of the N/D equation, t_*^{-1} is the energy of a resonance: $\text{Re}D(t_*)=0$. At $t=t_*$ we do not have our usual small bound on $|f|^2$ as in (4.10). Instead of a bound proportional to a^2 we have merely the unitarity limit $|f|^2\leq 1/\rho^2$. The violation of the a^2 bound is only a local matter, however, and the smaller the residue $\hat{c}$ the smaller the region in which it occurs. The idea of the following discussion is to show that by choosing $\hat{c}$ sufficiently small, the effects of the pole are so localized as not to change essentially our previous arguments.

We denote by Ω a small interval around $t=\hat{t}$ such that $\Omega\subset(0,1)$. $\bar{\Omega}$ denotes its complement.

$$\Omega=\left[\frac{1}{\hat{\omega}+\epsilon},\frac{1}{\hat{\omega}-\epsilon}\right],\quad \hat{\omega}-\epsilon>1,\quad \epsilon>0$$

$$\bar{\Omega}=[0,1]-\Omega. \quad (5.3)$$

Now we have the bounds

$$\begin{aligned}|f(t)|^2&\leq\frac{a^2t^2}{[1-|\hat{c}/\epsilon|-aA_2-bA_1]^2}\equiv\hat{\xi}a^2t^2,\quad t\in\bar{\Omega}\\ &\leq\rho^{-2}(t),\quad t\in\Omega.\end{aligned} \quad (5.4)$$

Here we have applied (4.6), and have assumed the following inequality in place of (4.9):

$$|\hat{c}/\epsilon|+bA_1+aA_2<1. \quad (5.5)$$

We have also assumed that ρ has no zero near $t=\hat{t}$. We can majorize B and G by treating separately the integrals over Ω and $\bar{\Omega}$. For the integral occurring in B we have

$$\begin{aligned}\frac{1}{\pi}\int_0^1\frac{\rho(\tau)|f(\tau)|^2d\tau}{\tau(\tau+t)}&\leq\frac{a^2\hat{\xi}}{\pi}\int_{\bar{\Omega}}\rho(\tau)d\tau+\frac{1}{\pi}\int_\Omega\frac{d\tau}{\tau^2\rho(\tau)}\\ &\leq a^2\hat{\xi}\hat{A}_3+2\epsilon/\pi\hat{\rho},\end{aligned} \quad (5.6)$$

where

$$\hat{A}_3=\frac{1}{\pi}\int_{\bar{\Omega}}\rho(\tau)d\tau,\quad \frac{1}{\hat{\rho}}=\sup_{t\in\Omega}\frac{1}{\rho(t)}. \quad (5.7)$$

Here and in the following $\hat{A}_i$ denotes the quantity A_i of Sec. 4, but with the region of integration $\bar{\Omega}$ replacing the unit interval. Referring to (4.2) and (4.3), we get

$$|B(t)|\leq t[|\lambda|+\sum|c|(a^2\hat{\xi}\hat{A}_3+2\epsilon/\pi\hat{\rho})], \quad (5.8)$$

$$|G(t,\tau)|\leq[|\lambda|+\sum|c|(a^2\hat{\xi}\hat{A}_3+2\epsilon/\pi\hat{\rho})]. \quad (5.9)$$

By introducing these results in (5.1), one finds the inequalities that replace (4.12) and (4.13):

$$|\psi(t)|\leq t[|\lambda|+\sum|c|(a^2\hat{\xi}\hat{A}_3+2\epsilon/\pi\hat{\rho})]\times(1+aA_3+|\hat{c}|\hat{t}), \quad (5.10)$$

$$[|\lambda|+\sum|c|(a^2\hat{\xi}\hat{A}_3+2\epsilon/\pi\hat{\rho})]\times(1+aA_3+|\hat{c}|\hat{t})\leq a. \quad (5.11)$$

By means of (4.14), (4.15), and (5.4) we get a similar result for the increment of ψ:

$$\begin{aligned}&|\psi(t)-\psi(t')|\\ &\leq|t-t'|[|\lambda|+\sum|c|(a^2\hat{\xi}\hat{A}_3+2\epsilon/\pi\hat{\rho})]\times(1+aA_3+|\hat{c}|\hat{t}),\end{aligned} \quad (5.12)$$

$$[|\lambda|+\sum|c|(a^2\hat{\xi}\hat{A}_3+2\epsilon/\pi\hat{\rho})]\times(1+aA_3+|\hat{c}|\hat{t})\leq b. \quad (5.13)$$

$A(K)\subset K$ is guaranteed by (5.5), (5.11), and (5.13).

When we go on to investigate the contraction property of A there is a bit of trouble, because $1/y$ in (4.21) is not necessarily bounded in Ω. If we should have $\varphi(t_*)=0$ simultaneously with $\text{Re}D(t_*)=0$, then $1/y(t_*)$ would be infinite. A way out of the difficulty is to add another condition to our definition of the set K in such a way that zeros of $\varphi(t)$ are forbidden entirely, except at $t=0$. Namely, to (3.4) and (3.5) we add the requirement

$$|\varphi_\alpha(t)-\lambda_\alpha t|\leq\gamma t,\quad \gamma<\inf_\alpha|\lambda_\alpha|. \quad (5.14)$$

Thus, φ_α is required to be so close to the Born term $\lambda_\alpha t$ that it cannot have a zero except at infinite energy: $|\varphi_\alpha|\geqslant(|\lambda_\alpha|-\gamma)t$. This gives us a new condition to meet to ensure $A(K)\subset K$. From (5.1), (5.8), and (5.9) we have

$$\begin{aligned}&|\psi_\alpha(t)-\lambda_\alpha t|\\ &\leq t[\sum|c|(a^2\hat{\xi}\hat{A}_3+2\epsilon/\pi\hat{\rho})(1+aA_3+|\hat{c}|\hat{t})+|\lambda|(aA_3+|\hat{c}|\hat{t})].\end{aligned} \quad (5.15)$$

Now ψ_α satisfies (5.14) if

$$\sum|c|(a^2\hat{\xi}\hat{A}_3+2\epsilon/\pi\hat{\rho})(1+aA_3+|\hat{c}|\hat{t})+|\lambda|(aA_3+|\hat{c}|\hat{t})<\inf_{\alpha\alpha}|\lambda|. \quad (5.16)$$

We have assumed, of course, that none of the λ_α's is zero. Conditions (3.4) and (5.14) are not independent, since (5.14) implies $|\varphi_\alpha(t)|\leqslant(\gamma+|\lambda_\alpha|)t$.

The $1/y$ factor is no longer troublesome:

$$\begin{aligned}1/y&\leqslant\hat{\xi},\quad t\in\bar{\Omega}\\ &\leqslant\frac{1}{\hat{\rho}^2}\left[\frac{\hat{\omega}+\epsilon}{\inf_\alpha|\lambda_\alpha|-\gamma}\right]^2\equiv\xi_\Omega,\quad t\in\Omega.\end{aligned} \quad (5.17)$$

If we return to the argument following Eq. (4.21), we see that one new term is added to $y-\bar{y}$, viz.,

$$\frac{2\hat{t}\hat{c}}{\hat{t}-\tau}\frac{\tau^2}{\pi}P\int_0^1\frac{\rho(t)\Delta(t)dt}{t(t-\tau)}. \tag{5.18}$$

The pole in (5.18) is cancelled by a corresponding second-order zero in the coefficient $1/y$. We define some notation

$$\eta_\Omega=\sup_{\tau\in\Omega}\left|\frac{\bar{x}}{y\bar{y}}\frac{\hat{t}\hat{c}}{\hat{t}-\tau}\right|,\quad \hat{\eta}=\sup_{\tau\in\bar{\Omega}}\left|\frac{\hat{t}\hat{c}}{\hat{t}-\tau}\right|. \tag{5.19}$$

In place of (4.26) there is the inequality

$$\begin{aligned}||f|^2-|\bar{f}|^2|\leqslant 2a&\begin{bmatrix}\hat{\xi}\\ \xi_\Omega\end{bmatrix}\tau\sup|\Delta|\\ &+2\begin{bmatrix}a^2\hat{\xi}^2\tau\hat{\eta}\\ \eta_\Omega\end{bmatrix}(A_4H[\Delta]+A_2\sup|\Delta|)\\ &+2\begin{bmatrix}a^2\hat{\xi}^2\tau\\ \xi_\Omega(\hat{\omega}+\epsilon)/\hat{\rho}^2\end{bmatrix}[(A_4H[\Delta]+A_2\sup|\Delta|)\\ &\times(1+bA_1+aA_2)+aA_5\sup|\Delta|],\end{aligned} \tag{5.20}$$

where

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{cases}\alpha, & \tau\in\bar{\Omega}\\ \beta, & \tau\in\Omega.\end{cases} \tag{5.21}$$

Equation (5.20) may be summarized by stating that

$$||f|^2-|\bar{f}|^2|\leq\begin{bmatrix}\hat{R}_1\tau\\ R_{1\Omega}\end{bmatrix}\sup|\Delta|+\begin{bmatrix}\hat{R}_2\tau\\ R_{2\Omega}\end{bmatrix}H[\Delta], \tag{5.22}$$

where $\hat{R}_i$ is proportional to a, and $R_{i\Omega}$ is not. When we introduce (5.22) in (4.20) we do the integration in two parts: $\int=\int_\Omega+\int_{\bar{\Omega}}$. The integral over $\bar{\Omega}$ is made small by choosing a small, while the Ω integral is made small by taking Ω itself to be a short interval. Referring to (4.20), we find

$$\begin{aligned}\sup|B-\bar{B}|\leq&\sum|c|(\hat{R}_1\sup|\Delta|+\hat{R}_2H[\Delta])\hat{A}_6\\ &+\sum|c|[R_{1\Omega}\sup|\Delta|+R_{2\Omega}H[\Delta]]2\epsilon\sup_\Omega\rho/\pi.\end{aligned} \tag{5.23}$$

Toward the goal of majorizing $\sup|\psi-\bar{\psi}|$, we note that the right-hand side of (4.31) is to be replaced by the right side of (5.23) multiplied by a. Then from (4.19) and (5.23), the modification of (4.31), (4.30), and (5.9), we get the desired bound of $\sup|\psi-\bar{\psi}|$, which replaces (4.32):

$$\begin{aligned}&\sup|\psi-\bar{\psi}|\\ &\quad\leq[\text{right-hand side of (5.23)}]\times[1+aA_7]\\ &\qquad+A_7[|\lambda|+\sum|c|(a^2\hat{\xi}\hat{A}_3+2\epsilon/\pi\hat{\rho})]\sup|\Delta|.\end{aligned} \tag{5.24}$$

For estimating $H[\psi-\bar{\psi}]$, we see that (4.35) is to be replaced by

$$\begin{aligned}X\leq|\lambda|\sup|\Delta|+\sum|c|\{&\sup|\Delta|(a^2\hat{\xi}\hat{A}_3+2\epsilon/\pi\hat{\rho})\\ &+a\hat{A}_6(\hat{R}_1\sup|\Delta|+\hat{R}_2H[\Delta])\\ &+(2a\epsilon/\pi)\sup_\Omega\rho(R_{1\Omega}\sup|\Delta|+R_{2\Omega}H[\Delta])\}.\end{aligned} \tag{5.25}$$

For the first term on the right of (4.34), one obtains

$$\begin{aligned}&|G(t,t')-\bar{G}(t,t')|\\ &\quad\leq\sum|c|[(\hat{R}_1\sup|\Delta|+\hat{R}_2H[\Delta])\hat{A}_7\\ &\qquad+(R_{1\Omega}\sup|\Delta|+R_{2\Omega}H[\Delta])(2\epsilon/\pi)\sup_\Omega\rho].\end{aligned} \tag{5.26}$$

When (5.25) and (5.26) are introduced in (4.34), one has the required bound on $H[\psi-\bar{\psi}]$:

$$\begin{aligned}H[\psi-\bar{\psi}]\leq&[\text{right-hand side of (5.26)}]\\ &+A_7[\text{right-hand side of (5.25)}].\end{aligned} \tag{5.27}$$

As in Sec. 4, (5.24) and (5.27) yield a sufficient condition that A be a contraction mapping.

Now let us show that the various parameters can be chosen so that all our conditions for a unique iterative solution in the set K are met. The relevant parameters having to do with the physical model are $|\lambda|=\sup_\alpha|\lambda_\alpha|$, $\inf_\alpha|\lambda_\alpha|$, and $\hat{c}$. The parameters having to do with the mathematical technique are a, b, and ϵ. We write

$$\begin{gathered}\epsilon=\alpha_0x^3,\quad a=\alpha_1x,\quad b=\alpha_2x,\\ |\lambda|=\alpha_3x,\quad \inf_\alpha|\lambda_\alpha|=\alpha_4x,\quad |\hat{c}|=\alpha_5x^3,\quad \gamma=\alpha_6x.\end{gathered} \tag{5.28}$$

The inequalities (5.5), (5.11), (5.13), and (5.16), which ensure $A(K)\subset K$, can be written as

$$\begin{aligned}\alpha_5/\alpha_0+O(x)&<1,\\ \alpha_3+O(x)&\leqslant\alpha_1,\alpha_2,\\ O(x)&\leqslant\alpha_4,\end{aligned} \tag{5.29}$$

where $O(x)$ stands for an expression which vanishes as x when the α_i are held fixed. In deriving (5.29) we have used the results of Appendix B, namely, that $\xi_\Omega,\eta_\Omega=O(x^{-2})$ and $\hat{\xi},\hat{\eta}=O(1)$. The inequalities (5.29) can certainly be satisfied for small x and small α_5/α_0, provided we choose $\alpha_3<\alpha_1,\alpha_2$, and $\alpha_4>0$. Note that α_3/α_4 is fixed by the model, and $\alpha_6<\alpha_4\leqslant\alpha_3$. As in Sec. 4 we have $\beta=O(x)$ in (4.39), so the operator A is a contraction mapping when x is sufficiently small.

To forbid ghosts when CDD poles are present, we can take advantage of the inequality (5.14). By (5.14) one is assured that $N(\omega)$ has no finite zero, hence that the imaginary part of the integral appearing in $D(z)$ is definitely positive (or definitely negative) in either of the open half-planes $\mathrm{Im}z\gtrless0$. Then in order to forbid complex zeros we have only to choose the sign of the residue $\hat{c}_\alpha$ to be the same as that of $N_\alpha(\omega)$, i.e., the same as that of λ_α. Then $\mathrm{Im}D(z)$ is positive (negative) definite off the real axis. $D(z)$ is a Herglotz function.[9] Similarly, $\mathrm{Im}D(\omega+i0)$, $\omega>1$, is not zero, and the only remaining question concerns the real axis for $\omega\leq1$. We have assumed that the first CDD pole is a finite distance $\Delta\omega$ above threshold. If $|\hat{c}|$ is so small that

$$|\hat{c}/\Delta\omega|+bA_1+aA_2<1, \tag{5.30}$$

then $\mathrm{Re}D(\omega+i0)$ cannot vanish at threshold, $\omega=1$. Since we have already assumed (5.5) with $\epsilon<\Delta\omega$, (5.30) is fulfilled. For $\omega<1$ it is sufficient that

$$|\hat{c}/\Delta\omega|+aA_2<1. \tag{5.31}$$

Thus, $\mathrm{sgn}\hat{c}_\alpha=\mathrm{sgn}\lambda_\alpha$ is the only new assumption needed to prevent ghosts.

It is interesting to look at the phase-shift behavior implied by our choice of $\mathrm{sgn}\hat{c}_\alpha$, and to ask what happens with the opposite choice of sign. The integral in D, call it $I(z)$, is less than one in magnitude on the physical cut $|I(\omega+i0)|\leq bA_1+aA_2<1$, $\omega\geqslant 1$. Since

$$\tan\delta=-\mathrm{Im}D(\omega+i0)/\mathrm{Re}D(\omega+i0)=\rho N(\omega)/\mathrm{Re}D(\omega+i0),$$

it is easy to read off the qualitative phase-shift behavior using only positivity of $1+I$ and the fact that ρN and I vanish at infinity. Two cases are to be distinguished: (i) $\lambda_\alpha<0$ ("repulsive" Born term); (ii) $\lambda_\alpha>0$ ("attractive" Born term). In the repulsive case the phase shift is first negative and small, then changes sign at the energy of the CDD pole, then goes through the resonance, and finally approaches $\delta(\infty)=\pi$ from below. In the attractive case the phase is initially positive, then it goes through the resonance, then through π at the CDD pole, and finally to $\delta(\infty)=\pi$ from above. This is all the same as in the soluble models first discussed by Castillejo, Dalitz, and Dyson.[17]

If, on the other hand, we had chosen $\mathrm{sgn}\hat{c}_\alpha=-\mathrm{sgn}\lambda_\alpha$, the phase for case (i) would be like the negative of the previous case (ii), and case (ii) would be like the negative of case (i). Thus, there is a pseudoresonance with the phase going downward through $-\frac{1}{2}\pi$, and $\delta(\infty)=-\pi$. This is a situation usually presumed to violate causality, so it is natural to expect that it involves ghosts. One can definitely assert that there are ghosts, because of the following argument, which is partly due to Sugawara and Kanazawa.[10] Our D function can certainly be represented in the form[9,10]

$$D(z)=R(z)\mathfrak{D}(z)=\frac{P(z)}{\hat{\omega}-z}\exp\left[-\frac{z}{\pi}\int_1^\infty\frac{\delta(\omega)d\omega}{\omega(\omega-z)}\right], \tag{5.32}$$

where $P(z)$ is a polynomial. Furthermore, $\mathfrak{D}(\omega+i0)\sim\omega^{\delta(\infty)/\pi}$, $\omega\to\infty$. Since $D(z)\to 1$, $|z|\to\infty$, it follows that $P(z)$ is of second order if $\delta(\infty)=-\pi$. Since $\mathfrak{D}(z)$ has no zeros or poles on the first sheet, $D(z)$ has two zeros z_1,z_2. These zeros must be complex, with $z_1=z_2^*$, since real zeros have been ruled out through (5.14) and (5.30). The zeros give ghost poles of f if N does not vanish at the corresponding points. The following useful formula [Ref. 9, Eq. (III.8)], will show that zeros of N and D cannot coincide:

$$f(z)=B(z)-\frac{1}{\pi D(z)}\int_1^\infty\frac{\rho(\omega)N(\omega)B(\omega)d\omega}{\omega-z}. \tag{5.33}$$

The coefficient of $1/D$ in (5.33) is a Herglotz function, and hence it cannot vanish at $z=z_1,z_2$. The Herglotz property comes from the fact that both ρN and B have definite signs. To check the sign of B, note that as in (5.8) we have

$$|B_\alpha(t)-\lambda_\alpha t|\leq t\sum|c|(a^2\xi\hat{A}_3+2\epsilon/\pi\hat{\rho}). \tag{5.34}$$

In (5.16) we have required that the right-hand side of (5.34) be less than $\inf_\alpha|\lambda_\alpha|$, so B as well as N has the same sign as the Born term.

For sufficiently small coupling constant and CDD residue, the argument of Ref. 1, Sec. 4, shows that our solution of the Low equation obeys Levinson's relation in the form

$$\delta(\infty)-\delta(1)=-\pi(n_b-n_c), \tag{5.35}$$

where n_b is the number of stable particles and n_c the number of poles of the *appropriate* D function. An appropriate D function is one which has zeros at the stable particle energies, and no other zeros, and which tends to one at infinity. Thus the D function we have used in this paper is not appropriate for Levinson's theorem, except in those channels where there is no stable particle. In a channel where there is one stable particle, the D function of Levinson's theorem contains one more zero and one more pole than the D function used in this paper.

In our example of a ghost-ridden amplitude there is an illegitimate Levinson relation of the form $\delta(\infty)-\delta(1)=-\pi(2-1)=-\pi$. The two ghost zeros count in the same way as stable states. This can be understood in that derivation of Levinson's relation which is based on examining the change in the phase increment when zeros and poles of the S matrix leave or enter the physical sheet as the interaction is turned on.

6. APPLICATION OF SCHAUDER'S THEOREM FOR EXISTENCE PROOF ONLY

If we ask only for existence of a solution irrespective of uniqueness or a means of calculation, then we can get by with weaker conditions than those of Sec. 5. According to (3.9), Schauder's theorem will guarantee a solution of the N/D equation if $A(K)\subset K$, and A is continuous in its action on K. The set K is the same as in (3.4) and (3.5), and the norm is $\|\varphi\|=\sup_{\alpha,t}|\varphi_\alpha(t)|$. As in Sec. 5 we shall make sure that $A(K)\subset K$ and that there are no ghosts by satisfying (5.5), (5.11), (5.13), and (5.16). To prove continuity of A we must show that if φ and $\bar{\varphi}$ are any two members of K, then $\sup|A\varphi-A\bar{\varphi}|<\epsilon$ when $\sup|\varphi-\bar{\varphi}|<\delta(\epsilon)$. [Continuity of A with respect to the distance (3.6) has already been shown. Our goal now is to prove it with the simpler

[17] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1956).

norm $\|\varphi\| = \sup|\varphi|$.] Take first the case without CDD poles. By recalling (4.19)–(4.22), (4.30), and (4.31) it becomes clear that the main job is to show that $\sup|y-\bar{y}|$ vanishes with $\sup|\Delta| = \sup|\varphi-\bar{\varphi}|$. We must, therefore, estimate the following integral in a way that avoids the $H[\Delta]$ term that we encountered before:

$$\left|\tau^2\frac{P}{\pi}\int_0^1 \frac{\rho(t)\Delta(t)dt}{t(t-\tau)}\right| \leq \left|\frac{\tau^2}{\pi}\int_0^1 \frac{dt\,\rho(t)}{t}\frac{\Delta(t)-\Delta(\tau)}{t-\tau}\right| + \|\Delta\|A_2. \quad (6.1)$$

The integral on the right-hand side of (6.1) is handled by breaking it into two parts, one over the interval $l = [\tau - \|\Delta\|, \tau + \|\Delta\|]$, and one over the remainder of the unit interval. To avoid awkwardness at the ends of the unit interval we first formally extend $L = [0,1]$ to $L_\epsilon = [-\epsilon, 1+\epsilon]$, where $\epsilon \geqslant a$. Because of (3.4), $l \subset L_\epsilon$ when $\tau \in L$. On $L_\epsilon - L$ we define

$$\begin{aligned} \rho(t) &= 0, & t &\in L_\epsilon - L \\ \varphi(t) &= \varphi(1), & 1 &\leq t \leq 1+\epsilon \\ &= \varphi(0), & -\epsilon &\leq t \leq 0. \end{aligned} \quad (6.2)$$

From (6.2) it follows that $|\varphi(t)-\varphi(\tau)| \leq b|t-\tau|^\mu$, $t \in L_\epsilon$, $\tau \in L$. This leads to the estimate

$$\left|\frac{\tau^2}{\pi}\int_0^1 \frac{dt\,\rho(t)}{t}\frac{\Delta(\tau)-\Delta(t)}{\tau-t}\right| \leq 2b\sup_\tau\left|\frac{\tau^2}{\pi}\int_l \frac{dt\,\rho(t)}{t}\frac{1}{|t-\tau|^{1-\mu}}\right| + 2\|\Delta\|\sup_\tau\left|\frac{\tau^2}{\pi}\int_{L_\epsilon - l}\frac{dt\,\rho(t)}{t(t-\tau)}\right|. \quad (6.3)$$

The integral in the second term on the right is bounded as a function of $\|\Delta\|$, while the first integral is less than a constant times $\|\Delta\|^\mu$. To prove this bound on the first integral we write $\delta = \|\Delta\|$ and treat separately the two cases $2\delta \leq \tau \leq 1$ and $0 \leq \tau < 2\delta$. If we apply $\rho(t) = O(t^\nu)$, then in the first case the integral is less than a constant times the following function:

$$\frac{\tau^2}{(\tau-\delta)^{1-\nu}}\int_{\tau-\delta}^{\tau+\delta}\frac{dt}{|t-\tau|^{1-\mu}} = \frac{\tau^2}{(\tau-\delta)^{1-\nu}}\left(\frac{2\delta^\mu}{\mu}\right) = O(\delta^\mu). \quad (6.4)$$

Here we use the fact that $\tau^2(1-\delta)^{\nu-1}$ has a positive derivative for $\tau \geqslant 2\delta$, and hence has a maximum value of $(1-\delta)^{\nu-1}$. For the second case we make the change of variable $t = \tau u$ to see that the integral is bounded by a constant times

$$\tau^{1+\mu+\nu}\int_{1-\Delta/\tau}^{1+\Delta/\tau}\frac{du}{|u|^{1-\nu}}\frac{1}{|u-1|^{1-\mu}}. \quad (6.5)$$

If $\nu+\mu<1$, this is less than

$$(2\delta)^{1+\mu+\nu}\int_{-\infty}^{\infty}\frac{du}{|u|^{1-\nu}}\frac{1}{|u-1|^{1-\mu}} = O(\delta^{1+\mu+\nu}). \quad (6.6)$$

For $\nu+\mu \geqslant 1$, it is more convenient not to change the variable. For $\nu+\mu>1$ we have, immediately,

$$\tau^2\int_{\tau-\delta}^{\tau+\delta}\frac{dt}{|t|^{1-\nu}}\frac{1}{|t-\tau|^{1-\mu}} = O(\tau^2) = O(\delta^2). \quad (6.7)$$

If $\nu+\mu=1$, then τ^ϵ times the integral in (6.7) is bounded, and (6.7) as a whole is $O(\delta^{2-\epsilon})$. Hence (6.3) is $O(\|\Delta\|^\mu)$, as was claimed.

When these results are used with (4.19)–(4.22), (4.30), and (4.31), we find that there is an $M>0$ such that

$$\|\psi-\bar{\psi}\| \leq M\|\varphi-\bar{\varphi}\|^\mu. \quad (6.8)$$

Hence A is continuous.

When there is a CDD pole the approach is the same, except for the modifications already carried out in Sec. 5. Again using (6.1) and (6.3) to majorize the principal-value integrals, we get a linear combination of $\|\Delta\|$ and $\|\Delta\|^\mu$ wherever there was a linear combination of $\sup|\Delta|$ and $H[\Delta]$ in the work of Sec. 5 [for instance, in (5.20), (5.23), and (5.24)]. Once more we have an inequality (6.8). The proof that K is convex and closed is elementary (cf. Ref. 1).

To summarize, Schauder's theorem guarantees that the N/D equation has at least one solution provided we impose our sufficient conditions for $A(K) \subset K$. (Continuity of A was proved without further assumptions.) If also $\text{sgn}\hat{c}_\alpha = \text{sgn}\lambda_\alpha$, the solution of the N/D equation gives a ghost-free solution of the Low equation.

7. CONCLUSIONS AND OUTLOOK

Let us summarize our theorems. The crossing matrix is arbitrary, and the cutoff obeys the conditions (4.7) which imply $\rho(\omega) = O(\omega^{-\nu})$, $\nu>0$, $\omega\to\infty$. Let K be the set of all real functions on the unit interval satisfying (3.4), (3.5), and (5.14), where a, b, γ, and μ are some fixed numbers. Let the N/D equation be written as in (2.5), with at most one CDD pole at $\hat{\omega}>1$. Suppose that the coupling constant, the CDD residue $\hat{c}$ and the constants a and b are chosen to satisfy inequalities (5.5), (5.11), (5.13), and (5.16). [The essential structure of these inequalities is given by (5.29). They involve a parameter ϵ, which can be taken arbitrarily small.] Let $\hat{c}_\alpha$ and λ_α have the same sign.

Theorem 1. Under the circumstances just described, there is at least one solution N of the N/D equation, lying in the set K. The corresponding $f = N/D$ satisfies the Low equation. If $\hat{c} \neq 0$, then $f(\hat{\omega}) = 0$.

Theorem 2. Assume the hypotheses of Theorem 1, and also that the parameters in Eqs. (5.24) and (5.27) are such that the transformation $\psi = A\varphi$ is a contraction mapping; i.e.,

$$\sup|\psi-\bar{\psi}|+H[\psi-\bar{\psi}] \leq \beta(\sup|\varphi-\bar{\varphi}|+H[\varphi-\bar{\varphi}]), \quad \beta<1.$$

Then there is a unique solution N of the N/D equation in the set K, and it is obtained by an iteration $\varphi_n = A\varphi_{n-1}$ beginning with an arbitrary element φ_0 of K. The error at the nth iteration is bounded as in (3.3). $f=N/D$ satisfies the Low equation, and $f(\hat{\omega})=0$ if $\hat{c}\neq 0$.

Theorem 1 is from Sec. 6, Theorem 2 from Secs. 4 and 5. Both may be extended in an obvious way to the case of any finite number of CDD poles.

The range of coupling constants allowed in Theorem 1 is considerably bigger than that of the corresponding theorem in Ref. 1. A numerical evaluation of the allowed range has been carried out; it is reported in Ref. 22. Our solutions seem to be of a rather dull type. The N function does not differ greatly from the Born term, and the latter is required (probably) to be a good deal smaller than the empirically determined Born term of pion-nucleon scattering. In view of this situation, the work of the present paper (and of Refs. 1 and 4), is at best a preliminary exercise in technique, a warm-up for more penetrating work on nonlinear S-matrix equations.

What are the prospects for doing better? On the one hand, some recent work of Atkinson[18] has been very encouraging. By an application of Schauder's principle, he shows that there exists a neutral $\pi-\pi$ scattering amplitude which satisfies crossing, unitarity, and an unsubtracted Mandelstam representation. In its essentials the proof is like that of Ref. 1; i.e., it uses a space of Hölder continuous functions with a supremum norm, and works directly with the dispersion relations and estimates of singular integrals. It is gratifying that the elaborations necessary for the relativistic problem do not stand in the way of a proof. On the other hand, Atkinson's solution is still of the "small" nonresonant type, not likely to be related to observed scattering.

In order to get into the region of strong coupling, dynamical resonances, and large nonlinearities, we think that the essential step will be first to find an approximate solution of the equations at full coupling strength. The approximation might, in practice, be very rough. It could be something like Chew's $N-N^*$ saturation of the Low equation.[19] Once an approximation is known, then one might be able to apply the fixed-point theorems more advantageously. If φ_0 is a proposed approximation, then we have a fixed-point problem $\varphi_1 = A(\varphi_0+\varphi_1)-\varphi_0$ for the difference φ_1 between an exact solution and the approximation. Alternatively, there exist iteration procedures like the Newton-Kantorovich method,[6,20] which will produce exact solutions φ from φ_0 if φ_0 is close enough to φ. Amatuni[6] has outlined a method of finding φ_0 and φ for equations of the Shirkov or Low type. By making a systematic approximation of the left-cut term by poles, he is able to reduce the problem to solution of nonlinear algebraic equations. If difficulties concerning ghosts can be overcome, then Amatuni's method has the advantage of facing squarely the strong-coupling situation.

For the Chew-Low model[7] there are several proposals for φ_0 to be found in the literature (cf. footnote 5, Ref. 1), and these proposals are supposed to be the prototypes of bootstrap dynamics. It is only reasonable to ask whether such a φ_0 lies close to an actual solution φ. If the Newton-Kantorovich iteration beginning with φ_0 does not converge to a solution, then one would be hard pressed, we think, to attach any physical significance to φ_0.[21] If the iteration does converge, then we are interested in knowing whether the exact solution is of "pure bootstrap" type[5], i.e., whether it obeys the unsubtracted Low equation and Levinson's relation in the form $\delta(\infty)-\delta(1)=-\pi n_b$. It is not a pure bootstrap solution if the Huang-Mueller theorem[5] is true for the cutoff employed. It might involve an elementary nucleon $[\delta_{11}(\infty)-\delta_{11}(1)=-\pi(n_b-n_c)=-\pi(1-1)=0]$ with a dynamical N^* resonance $[\delta_{33}(\infty)-\delta_{33}(1)=-\pi(0-0)=0]$.

We think that a rigorous mathematical study of relatively simple cases (Low or Shirkov equations) in the strong-coupling domain will be a valuable, even necessary, complement to efforts directed toward the full relativistic equations.[18]

Finally, we mention that the Low equation written as a dispersion relation for the inverse amplitude lends itself to a neat existence proof in the non-CDD case.[22] The requirements of crossing symmetry lead to serious complications in the inverse amplitude formulation if CDD poles are present.

ACKNOWLEDGMENTS

One of us (R. Warnock) owes many thanks to Professor L. Van Hove and Professor J. Prentki for the hospitality of the Theoretical Study Division at CERN.

APPENDIX A

We shall prove that the metric space K defined in (3.4)–(3.6) is complete. The vector index α causes no

[18] D. Atkinson, Nucl. Phys. **B7**, 375 (1968); **B8**, 377 (1968).

[19] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).

[20] The Newton-Kantorovich method is a Banach space generalization of the familiar Newton method of finding a root of $f(x)=0$; i.e., $x_1=x_0-f(x_0)/f'(x_0), \cdots, x_{n+1}=x_n-f(x_n)/f'(x_n)$. For a thorough review of the N-K method and similar things, see *Nonlinear Integral Equations*, edited by P. M. Anselone (University of Wisconsin Press, Madison, Wis., 1964). See also L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces* (Pergamon Press, Inc., Oxford, England, 1964). For a discussion of the Newton-Kantorovich method as it applies to the Low equation, see R. L. Warnock in *Lectures in Theoretical Physics*, edited by K. T. Mahanthappa *et al.* (Gordon and Breach, Science Publishers, Inc., New York, 1969).

[21] An alternative opinion could be that φ_0 relates to physics, while exact solutions to the Low equation do not. This view would point toward modification of the equation.

[22] H. McDaniel and R. L. Warnock, Nuovo Cimento (to be published).

essential difficulty in the proof, so it will be ignored. Let $\{\varphi_n(t)\}$ be any Cauchy sequence in K. That is, for any $\epsilon>0$ there exists an integer $N(\epsilon)$ such that

$$\delta(\varphi_n,\varphi_m)<\epsilon,\quad n,m>N(\epsilon). \tag{A1}$$

Since $H[\psi]\geqslant 0$ for any ψ,

$$\sup|\varphi_n-\varphi_m|<\epsilon,\quad n,m>N(\epsilon). \tag{A2}$$

There exists in K a function $\varphi(t)$ such that

$$\sup|\varphi_n-\varphi|<\epsilon,\quad n>N_1(\epsilon). \tag{A3}$$

To show that φ exists we apply the Bolzano-Weierstrass theorem: Every bounded sequence on the real line has a convergent subsequence. Thus, for every t there is a subsequence $\{\varphi_{n_i}\}$ of $\{\varphi_n\}$ which has a limit $\varphi(t)$. Now

$$|\varphi_n(t)-\varphi(t)|\leq|\varphi_n(t)-\varphi_{n_i}(t)|+|\varphi_{n_i}(t)-\varphi(t)|. \tag{A4}$$

Take n so large that $|\varphi_n-\varphi_{n_i}|<\frac{1}{2}\epsilon$ when $n_i>n$. For each t take n_i so large that $|\varphi_{n_i}-\varphi|<\frac{1}{2}\epsilon$; thus, there is an $N_1(\epsilon)$ satisfying (A3). Furthermore, $\varphi\in K$. To check (3.4) note that

$$|\varphi|\leq|\varphi-\varphi_n|+|\varphi_n|<\epsilon+at \tag{A5}$$

for any positive ϵ; hence, $|\varphi|\leq at$. For (3.5) we have

$$\begin{aligned}|\varphi(t)-\varphi(t')|&\leqslant|\varphi(t)-\varphi_n(t)|\\&\quad+|\varphi(t')-\varphi_n(t')|+|\varphi_n(t)-\varphi_n(t')|\\&<2\epsilon+b|t-t'|^{\mu},\end{aligned} \tag{A6}$$

again for any ϵ. Therefore $|\varphi(t)-\varphi(t')|\leqslant b|t-t'|^{\mu}$.

Next, we show that for the same function $\varphi(t)$ there is an $N_2(\epsilon)$ so that

$$H[\varphi_n-\varphi]\leq\epsilon,\quad n>N_2(\epsilon). \tag{A7}$$

Again the Cauchy property (A1) implies that

$$H[\varphi_n-\varphi_m]<\epsilon,\quad n,m>N(\epsilon). \tag{A8}$$

That is, we have for all t and t' $(t\neq t')$

$$\frac{|\varphi_n(t)-\varphi_m(t)-\varphi_n(t')+\varphi_m(t')|}{|t-t'|^{\mu}}=\frac{|[\varphi(t)-\varphi_m(t)]-[\varphi(t')-\varphi_m(t')]-[\varphi(t)-\varphi_n(t)]+[\varphi(t')-\varphi_n(t')]|}{|t-t'|^{\mu}}<\epsilon,\quad n,m>N(\epsilon). \tag{A9}$$

For any fixed t, t' we can choose m to make $|\varphi(t)-\varphi_m(t)|$ and $|\varphi(t')-\varphi_m(t')|$ arbitrarily small. Hence, for all unequal t, t' we have

$$\frac{|[\varphi(t)-\varphi_n(t)]-[\varphi(t')-\varphi_n(t')]|}{|t-t'|^{\mu}}\leqslant\epsilon,\ n>N(\epsilon). \tag{A10}$$

That is to say, (A5) holds with $N_2=N$. Combining (A3) and (A5), we see that

$$\delta(\varphi_n,\varphi)<2\epsilon,\quad n>\max(N_1,N_2). \tag{A11}$$

Our Cauchy sequence $\{\varphi_n\}$ has a limit φ in K, so K is a complete metric space.

APPENDIX B

In this appendix we will discuss the functions $\hat{\xi}$, $\hat{\eta}$, ξ_Ω, and η_Ω in (5.17) and (5.19) as functions of x. Since $\hat{c}/\epsilon$ is a constant, it follows that $\hat{\xi}$ and $\hat{\eta}$ are constants. In β, ξ_Ω and η_Ω always occur with coefficient ϵ. It is clear that $\epsilon\xi_\Omega=O(x)$. Lastly, let $X\equiv\hat{c}/(\omega-\hat{\omega})$. From (5.4) we have

$$\eta_\Omega\leq\frac{1}{\hat{\rho}^2}\sup_\Omega|\alpha(t,x)|\equiv\frac{1}{\hat{\rho}^2}\left|\frac{\omega X}{[1+I(\omega)-X]^2+\rho^2\phi^2}\right|. \tag{B1}$$

It is convenient to break up Ω into a region θ and its complement, where

$$\theta=\{t|\,|\hat{c}/(\omega-\hat{\omega})|<3\}. \tag{B2}$$

As the reader may easily verify from (4.6) and (5.5), $t_*\in\theta\subset\Omega$. So in θ we have, from (5.14),

$$|\alpha|\leq 3(\hat{\omega}+\epsilon)/\hat{\rho}^2(\inf|\lambda_\alpha|-\gamma)^2. \tag{B3}$$

In $\Omega-\theta$, $X\geq 3$ and it is convenient to write

$$|\alpha|=\left|\frac{\omega}{X}\right|\Big/\left\{\left[\frac{1+I}{X}-1\right]^2+\frac{\rho^2\phi^2}{X^2}\right\}\leq 3(\hat{\omega}+\epsilon), \tag{B4}$$

since $1+I<2$ by (5.5). So from (B1)–(B4) we have $\epsilon\eta_\Omega=O(x)$.