Exact Consequences of Broken $O(4)$ Symmetry for Regge Trajectories. I. $M=0$*

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We obtain the complete set of constraints imposed by broken $O(4)$ symmetry on Regge daughter sequences having $M=0$. The constraints determine the pth derivative of all the daughters in terms of a finite number of constants. Similar results are obtained for daughter residues.

I. INTRODUCTION

PARTIAL-WAVE expansions of scattering amplitudes are expansions in representations of the compact little group of the total four-momentum. In the physical region, and c.m. frame, the total four-momentum has only a time component, and the compact little group is $O(3)$. However, at zero four-momentum the compact little group is $O(4)$.[1,2] The consequences of the enlarged group at vanishing four-momentum are most evident after a Sommerfeld-Watson transformation of the expansion has been performed. Since $O(3)$ is a subgroup of $O(4)$, each Toller pole in the $O(4)$ complex "angular momentum" plane corresponds to an infinite sequence of Regge poles—a parent and daughters—spaced at integer intervals below the parent Regge pole.[3]

$O(4)$ symmetry can be broken in two ways. First, the total four-momentum of a process can vanish on the mass shell only if the initial and final particles have pairwise equal masses. However, the breaking of $O(4)$ symmetry by unequal mass cannot eliminate the requirement that Regge poles occur in integer-spaced daughter sequences. The general principles of quantum theory require that if a daughter sequence is present in a pairwise equal-mass process, then it will be present in an unequal-mass process having the same internal quantum numbers.[4] The second type of breaking occurs when $P^2=t$ is nonzero. Away from $t=0$, the Regge daughters no longer need have integer spacing and it is this type of breaking which we principally wish to study in this paper. For technical reasons we do this by analyzing an unequal-mass process, so that both types of breaking are actually present. Accordingly, we emphasize again that the pattern of deviation from integer spacing we find in an unequal-mass process must be the same as that found in an equal-mass process.

In the present study we will consider spinless external particles. This has the consequence, for both equal-mass[1] and unequal-mass[5–7] processes, that the $O(4)$ quantum number M of Regge daughter sequences must be zero. In the conclusion we comment on the modifications of our work which arise for other integer M, and we hope to devote a future paper to the topic. Deviations from integer spacing for $t\neq 0$ are expressed by "mass formulas," which specify the derivatives of daughter trajectories at $t=0$ in terms of a few constants. For $M=0$ sequences, formulas are known for the first[2,7,8] and second[8] derivatives. These are equivalent to

$$\begin{aligned}\alpha^{(1)}(k)&=A_0{}^1+A_1{}^1k(2\alpha_0-k+1)\,,\\ \tfrac{1}{2}\alpha^{(2)}(k)&=A_0{}^2+A_1{}^2k(2\alpha_0-k+1)+A_2{}^2k(k-1)\\ &\quad\times(2\alpha_0-k+1)(2\alpha_0-k)+2A_0{}^1A_1{}^1k+2(A_1{}^1)^2\\ &\quad\times k^2(2\alpha_0-k+1)\,.\end{aligned}\qquad(1)$$

Here the A's are parameters which contain the dynamics, α_0 is the $t=0$ intercept of the parent, and $k=0,1,\cdots$ is an index which labels the parent ($k=0$) and daughters ($k>0$). The physics behind such formulas is most clearly displayed in the work of Domokos.[2,9] He uses the Bethe-Salpeter equation as the basis for his discussion. At $P_\mu=0$ there is $O(4)$ symmetry, so he uses the $O(4)$ symmetric states as a basis and treats P_μ as a perturbation. When he uses perturbation theory and the Wigner-Eckart theorem for $O(4)$, he obtains the formula for $\alpha^{(1)}(k)$. In this formula the A's are related to reduced matrix elements, and $k(2\alpha_0-k+1)$ to $O(4)$ Clebsch-Gordan coefficients. The reason the answer is so simple is that P_μ transforms like a low-rank $O(4)$ irreducible tensor. The formula for $\alpha^{(2)}(k)$ is also comprehensible from this point of view: In second-order perturbation theory one expects new matrix elements as well as products of first-order elements. The task of generalizing Eq. (1) to all derivatives is therefore equivalent to devising an algorithm which displays the general term of the perturbation expansion, as it appears after use of the Wigner-Eckart theorem.

It is this generalization, and the analogous one for the reduced residues, that we present in this paper. The reason we do not emphasize the residue formulas is

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[1] D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).

[2] G. Domokos, Phys. Rev. **159**, 1387 (1967).

[3] For special choices of external spin, helicity, and $O(4)$ quantum number M, every other daughter may be missing.

[4] This point is elaborated at the end of the Introduction.

[5] J. C. Taylor, Nucl. Phys. **B3**, 504 (1967).

[6] J. B. Bronzan and C. E. Jones, Phys. Rev. Letters **21**, 564 (1968).

[7] P. Di Vecchia and F. Drago, Phys. Letters **27B**, 387 (1968).

[8] J. B. Bronzan, C. E. Jones, and P. K. Kuo, Phys. Rev. **175**, 2200 (1968).

[9] G. Domokos and P. Suranyi, Budapest report (unpublished).

that they pertain only to a particular case: elastic scattering of unequal-mass spinless particles. However, the trajectory formulas are valid for $M=0$ daughter sequences wherever they occur. Since we deal with elastic scattering of unequal-mass spinless particles, it is perhaps useful to review how daughters crop up in unequal-mass processes, and how we are sure they correspond to an $M=0$ Toller pole in the $O(4)$ complex angular momentum plane. Daughters are required in unequal-mass scattering because when masses are unequal the c.m. three-momentum $p^2=[t-(m+\mu)^2]\times[t-(m-\mu)^2]/4t$ is singular at $t=0$. This has the consequence that the contribution of a single Regge pole to the full amplitude is not analytic at $t=0$, in violation of Mandelstam analyticity.[10] However, if a daughter sequence, integer-spaced at $t=0$, is added, analyticity can be restored. Indeed, Eqs. (1) have been derived both by group theory[2] and by demanding analyticity in unequal-mass processes.[7,8] It is by fully reducing all the constraints imposed by analyticity at $t=0$ that we generalize Eqs. (1). As to how we know that we are dealing with an $M=0$ Toller pole, we cite the papers which demonstrate this by means of factorization.[5–7] The argument of these papers is that analyticity requires daughters both in unequal-mass (UU) processes and in processes with unequal-masses on one side and equal-masses on the other (UE). Factorization then requires daughters in equal-mass processes, with definite residue ratio; for spinless processes, the daughters add up to the contribution of an $M=0$ Toller pole.

Before becoming involved in the argument, it is useful to display the generalization we find. It can be written most compactly in the form

$$\alpha(k,t)=\alpha_0-k+\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\frac{\partial^n}{\partial\alpha_0{}^n}\left[\sum_{q=1}^{\infty}t^q\sum_{i=0}^{q}A_i{}^q\times\frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}\right]^{n+1}, \quad (2)$$

where the $A_i{}^q$ are parameters which can be functions of α_0, but not of k.[10a] This formula generates equations for all the derivatives when each side is expanded in power series in t. To expand the right side, we define

$$h(k,\alpha_0,q)=\sum_{i=0}^{q}A_i{}^q\frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}, \quad (3)$$

and

$$\bar{h}(k,\alpha_0,q,n)=\sum_{\substack{m_1,\cdots,m_n\\(n)}}{}^{\prime\prime}\prod_{i=1}^{q}\frac{1}{m_i!}[h(k,\alpha_0,i)]^{m_i}, \quad (4)$$

where

$$\sum_{\substack{m_1,\cdots,m_q\\(n)}}{}^{\prime\prime}$$

means sum over all sets of q non-negative integers $\{m_i\}$ subject to the restrictions

$$\sum_{i=1}^{q}m_i=n\,,\quad \sum_{i=1}^{q}im_i=q\,. \quad (5)$$

Then, when the left side of Eq. (2) is expanded by Taylor's theorem, and the right side by the multinomial theorem, we obtain

$$\frac{1}{p!}\alpha^{(p)}(k)=\sum_{n=0}^{p-1}\frac{\partial^n}{\partial\alpha_0{}^n}\bar{h}(k,\alpha_0,p,n+1)\,. \quad (6)$$

Equation (6) yields results equivalent to Eq. (1) for $p=1$ and 2. The analog of Eq. (2) for residues is given in Eq. (46a).

II. ANALYTICITY CONDITIONS

A. Step I

We present the derivation of Eq. (2) in three steps. In the first step we write down all the analyticity conditions and perform a first reduction of them. In steps II and III further manipulations are made, resulting in the end in necessary and sufficient conditions for analyticity in a compact form like Eq. (2).

The scattering we study is the elastic, unequal-mass, spinless process $m+\mu\to m+\mu$ in the t channel. The contribution of a Regge pole at $\alpha(t)$ to the full amplitude has the form $\beta(t)Q_{-\alpha(t)-1}(-z_t)$, where various factors have been absorbed in the residue. At $t=0$, $z_t=1$, so this term is not analytic at $t=0$, and we must introduce daughters to restore analyticity.[5–8,10] The daughter trajectories are labelled by an index $k=0,1,\cdots$, and $\alpha(k,t=0)=\alpha_0-k$. When Q is written as a hypergeometric function, the contribution of the kth daughter to the full amplitude is

$$F(k,t,u)=u^{\alpha(k,t)}\sum_{n=0}^{\infty}C(k,n,t)u^{-n},$$

$$C(k,n,t)=\frac{\gamma(k,t)}{t^k}\frac{(4p^2)^n}{n!}\left[\frac{\Gamma(\alpha(k,t)+1)}{\Gamma(\alpha(k,t)+1-n)}\right]^2\times\frac{\Gamma(2\alpha(k,t)+1-n)}{\Gamma(2\alpha(k,t)+1)}. \quad (7)$$

Here p is the t-channel c.m. momentum, and $\gamma(k,t)$ is the reduced residue of the kth daughter with a factor t^{-k} removed. Singular reduced residues are necessary to restore analyticity, and once t^{-k} is removed we can restore analyticity with $\gamma(k,t)$ analytic at $t=0$. We next

[10] D. Z. Freedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967); M. L. Goldberger and C. E. Jones, *ibid.* **150**, 1269 (1966).
[10a] The parameters $A_i{}^q$ may be taken independent of α_0 without loss of generality. See Ref. 13a.

expand $u^{\alpha(k,t)}$

$$u^{\alpha(k,t)}=u^{\alpha_0-k}\sum_{s=0}^{\infty}\frac{(\ln u)^s}{s!}[\alpha(k,t)-\alpha_0+k]^s. \quad (8)$$

When we sum over daughters, the full amplitude can be written

$$F(t,u)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{u^{\alpha_0-r}(\ln u)^s}{s!t^r}(4tp^2)^r\sum_{k=0}^{r}R(k,\alpha(k,t)) \times[\alpha(k,t)-\alpha_0+k]^s\frac{\Gamma(2\alpha(k,t)-r+k+1)}{(r-k)![\Gamma(\alpha(k,t)-r+k+1)]^2}, \quad (9)$$

where the super-reduced residue is

$$R(k,\alpha(k,t))=\frac{\gamma(k,t)(4tp^2)^{-k}[\Gamma(\alpha(k,t)+1)]^2}{\Gamma(2\alpha(k,t)+1)}. \quad (10)$$

Here we have used the fact that if $\alpha^{(1)}(k,0)\neq 0$ we can regard R as a function of k and $\alpha(k,t)$ rather than as a function of k and t. Equation (1) shows that in general $\alpha^{(1)}(k,0)\neq 0$.

Since $F(t,u)$ must be analytic at $t=0$ for all u, the analyticity conditions implied by Eq. (9) are[11]

$$\sum_{k=0}^{r}\frac{\partial^q}{\partial t^q}\left\{R(k,\alpha(k,t))[\alpha(k,t)-\alpha_0+k]^s \times\frac{\Gamma(2\alpha(k,t)-r+k+1)}{(r-k)![\Gamma(\alpha(k,t)-r+k+1)]^2}\right\}_{t=0}=0 \quad (q<r,\ 0\leqslant s). \quad (11)$$

We reduce these equations by first expanding the terms

$$R(k,\alpha(k,t))\frac{\Gamma(2\alpha(k,t)-r+k+1)}{(r-k)![\Gamma(\alpha(k,t)-r+k+\frac{1}{2})]^2} =\sum_{p=0}^{\infty}\frac{[\alpha(k,t)-\alpha_0+k]^p}{p!}\frac{\partial^p}{\partial\alpha(k,t)^p}\left\{R(k,\alpha(k,t)) \times\frac{\Gamma(2\alpha(k,t)-r+k+1)}{(r-k)![\Gamma(\alpha(k,t)-r+k+1)]^2}\right\}_{\alpha_0-k}. \quad (12)$$

It is convenient to define the symbols

$$R^{(w)}(k,\alpha_0-k)=\frac{\partial^w}{\partial\alpha(k,t)^w}R(k,\alpha(k,t))|_{\alpha_0-k},$$
$$M_{rk}(\alpha_0)=\Gamma(2\alpha_0-r-k+1)/(r-k)!. \quad (13)$$

Using Leibnitz's formula for the derivative of a product, we can evaluate the right side of Eq. (12), and substitute into Eq. (11). The analyticity conditions now take the form

$$\sum_{k=0}^{r}\sum_{p=0}^{\infty}\sum_{w=0}^{p}\frac{R^{(w)}(k,\alpha_0-k)}{w!(p-w)!}\left\{\frac{\partial^{p-w}}{\partial\alpha_0^{p-w}}\frac{M_{rk}(\alpha_0)}{[\Gamma(\alpha_0-r+1)]^2}\right\} \times\frac{\partial^q}{\partial t^q}\{[\alpha(k,t)-\alpha_0+k]^{p+s}\}_{t=0}=0 \quad (q<r,\ 0\leqslant s). \quad (14)$$

We next evaluate the t derivative in Eq. (14). Using the multinomial theorem, we obtain

$$\frac{\partial^q}{\partial t^q}\{[\alpha(k,t)-\alpha_0+k]^{p+s}\}_{t=0}=\frac{\partial^q}{\partial t^q}\left\{\left[\sum_{i=1}^{q}\frac{t^i}{i!}\alpha^{(i)}(k)\right]^{p+s}\right\}_{t=0}$$
$$=q!(p+s)!\sum_{\substack{m_1,\cdots,m_q\\(p+s)}}''\prod_{i=1}^{q}\frac{1}{m_i!}\left[\frac{\alpha^{(i)}(k)}{i!}\right]^{m_i}$$
$$\equiv f(k,\alpha_0,q,p+s). \quad (15)$$

The symbol $\sum''$ is explained in the Introduction. We have written f as a function of α_0, even though Eq. (15) states that it is a function of the derivatives $\alpha^{(i)}(k)$. In doing this we anticipate that the result of our study will be to determine these derivatives in terms of α_0 and parameters, as in Eq. (2). In view of Eq. (5) we have the useful result

$$f(k,\alpha_0,q,p+s)=0 \quad (q<p+s). \quad (16)$$

A variant of Leibnitz's formula is

$$x^{(n)}y=\sum_{u=0}^{n}\frac{(-1)^u n!}{u!(n-u)!}[xy^{(u)}]^{(n-u)}. \quad (17)$$

When we substitute Eq. (15) into Eq. (14) and use Eq. (17), the analyticity conditions take the form (after a rearrangement of indices)

$$\sum_{k=0}^{r}\sum_{p=0}^{\infty}\sum_{w=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^u}{p!w!u!}\frac{\partial^p}{\partial\alpha_0^p}\left\{\frac{M_{rk}(\alpha_0)}{[\Gamma(\alpha_0-r+1)]^2} \times\frac{\partial^u}{\partial\alpha_0^u}[R^{(w)}(k,\alpha_0-k)f(k,\alpha_0,q,w+u+p+s)]\right\}=0 \quad (q<r,\ 0\leqslant s). \quad (18)$$

Note, in view of Eq. (16), that all the sums really extend over finite ranges.

The sum over p is actually redundant in Eq. (18). To see this, define

$$g(\alpha_0,q,p+s,r)=\sum_{k=0}^{r}\sum_{w=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^u}{u!w!}\frac{M_{rk}(\alpha_0)}{[\Gamma(\alpha_0-r+1)]^2} \times\frac{\partial^u}{\partial\alpha_0^u}[R^{(w)}(k,\alpha_0-k)f(k,\alpha_0,q,w+p+u+s)]. \quad (19)$$

[11] Recall that $p^2t=\frac{1}{4}[t-(m+\mu)^2][t-(m-\mu)^2]$ is analytic and nonzero at $t=0$.

Equation (18) can be written

$$\sum_{p=0}^{n}\frac{1}{p!}\frac{\partial^p}{\partial\alpha_0{}^p}g(\alpha_0,q,q+p-n,r)=0,\quad(q<r,\ 0\leqslant n\leqslant q)\tag{20}$$

where we have taken Eq. (16) into account. The $n=0$ and $n=1$ equations are

$$g(\alpha_0,q,q,r)=0$$

and

$$g(\alpha_0,q,q-1,r)+\frac{\partial}{\partial\alpha_0}g(k,\alpha_0,q,q,r)=0.\tag{21}$$

Since the first equation is an identity in α_0, the second equation becomes

$$g(\alpha_0,q,q-1,r)=0.\tag{22}$$

Continuing in this way, the analyticity conditions are seen to be

$$g(\alpha_0,q,s,r)=0,\quad(q<r,\ 0\leqslant s\leqslant q)\tag{23}$$

where we have again noted Eq. (16). We can factor the irrelevant $[\Gamma(\alpha_0-r+1)]^{-2}$ out of Eq. (19), and put the analyticity conditions in the form

$$\sum_{k=0}^{r}M_{rk}(\alpha_0)\sum_{w=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^u}{w!u!}\frac{\partial^u}{\partial\alpha_0{}^u}\times[R^{(w)}(k,\alpha_0-k)f(k,\alpha_0,q,w+u+s)]=0\quad(q<r,\ 0\leqslant s\leqslant q).\tag{24}$$

B. Step II

The inverse of $M_{rk}(\alpha_0)$ is[12]

$$M_{ki}^{-1}(\alpha_0)=\frac{(-1)^{k-i}(2\alpha_0-k+1)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}.\tag{25}$$

The analyticity conditions, Eqs. (24), can be inverted to read

$$\sum_{w=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^u}{w!u!}\frac{\partial^u}{\partial\alpha_0{}^u}[R^{(w)}(k,\alpha_0-k)f(k,\alpha_0,q,u+w+s)]=q!\sum_{i=0}^{q}B_i{}^{(q,s)}M^{-1}{}_{ki}(\alpha_0),\quad(0\leqslant s\leqslant q),\tag{26}$$

where the $B_i{}^{(q,s)}$ are functions of α_0, but not of k. By definition, $f(k,\alpha_0,q,u+w+s)/q!$ is the coefficient of t^q in the expansion of $[\alpha(k,t)-\alpha_0+k]^{u+w+s}$. Hence, the analyticity conditions may be written

$$\sum_{w=0}^{\infty}\sum_{u=0}^{\infty}\frac{(-1)^u}{w!u!}\frac{\partial^u}{\partial\alpha_0{}^u}\times\{R^{(w)}(k,\alpha_0-k)[\alpha(k,t,\alpha_0)-\alpha_0+k]^{u+w+s}\}=\sum_{q=s}^{\infty}t^q\sum_{i=0}^{q}B_i{}^{(q,s)}M_{ki}^{-1}(\alpha_0)\quad(0\leqslant s).\tag{27}$$

Here we have displayed explicitly the dependence of $\alpha(k,t)$ on α_0, as discussed above. The summation over w may be evaluated to yield the analyticity conditions in the form

$$\sum_{u=0}^{\infty}\frac{(-1)^u}{u!}\frac{\partial^u}{\partial\alpha_0{}^u}\{R(k,\alpha(k,t,\alpha_0))[\alpha(k,t,\alpha_0)-\alpha_0+k]^{u+s}\}=\sum_{q=s}^{\infty}t^q\sum_{i=0}^{q}B_i{}^{(q,s)}M^{-1}{}_{ki}(\alpha_0)\quad(0\leqslant s).\tag{28}$$

The left side of Eq. (28) may be written

$$L_s=\sum_{u=0}^{\infty}\sum_{m=0}^{\infty}\delta_{mu}(-1)^u\frac{\partial^u}{\partial\alpha_0{}^u}\{R(k,\alpha(k,t,\alpha_0))\times[\alpha(k,t,\alpha_0)-\alpha_0+k]^{m+s}\}.\tag{29}$$

We use an integral representation of δ_{mu} and exchange summation and integration to obtain[13]

$$\begin{aligned}L_s&=\int_0^{2\pi}\frac{d\phi}{2\pi}\left[\sum_{u=0}^{\infty}\frac{(-1)^ue^{-i\phi u}}{u!}\frac{\partial^u}{\partial\alpha_0{}^u}\right]\{\sum_{m=0}^{\infty}R(k,\alpha(k,t,\alpha_0))e^{im\phi}[\alpha(k,t,\alpha_0)-\alpha_0+k]^{m+s}\}\\&=\int_0^{2\pi}\frac{d\phi}{2\pi}\frac{R(k,\alpha(k,t,\alpha_0-e^{-i\phi}))[\alpha(k,t,\alpha_0-e^{-i\phi})-\alpha_0+e^{-i\phi}+k]^s}{1-e^{i\phi}[\alpha(k,t,\alpha_0-e^{-i\phi})-\alpha_0+e^{-i\phi}+k]}\\&=\frac{1}{2\pi i}\oint\frac{dz\,R(k,\alpha(k,t,\alpha_0+z))[\alpha(k,t,\alpha_0+z)-\alpha_0-z+k]^s}{\alpha(k,t,\alpha_0+z)-\alpha_0+k},\end{aligned}\tag{30}$$

where $z=-e^{-i\phi}$, and the contour integral is counterclockwise around $|z|=1$. We assume that the $B_i{}^{(q,s)}$ in Eq. (28) are analytic functions of α_0. Later, we verify that this assumption is consistent, and in Sec. III we argue that it is physically reasonable. With this assumption, the numerator of the contour integral in Eq. (30)

[12] J. B. Bronzan, Phys. Rev. (to be published).

[13] This device was suggested by Professor K. A. Johnson.

can be seen to be an analytic function of z. If it were singular, the left side of Eq. (28) would be singular in α_0, which is inconsistent with the analyticity we have assumed for the right side of Eq. (28). At $t=0$ the denominator of the contour integral has precisely one simple zero at $z=0$. For t in some neighborhood of zero there will continue to be one zero inside the unit circle, at $z_0(k,t,\alpha_0)$ determined by

$$\alpha(k,\ t,\ \alpha_0+z_0(k,t,\alpha_0))=\alpha_0-k\,,\quad \lim_{t\to 0} z_0(k,t,\alpha_0)=0\,. \tag{31}$$

Hence, the analyticity conditions may be written

$$\frac{R(k,\ \alpha_0-k)[-z_0(k,t,\alpha_0)]^s}{\partial\alpha(k,t,\alpha_0)/\partial\alpha_0|_{\alpha_0+z_0(k,t,\alpha_0)}} =\sum_{q=s}^{\infty} t^q \sum_{i=0}^{q} B_i^{(q,s)} M^{-1}{}_{ki}(\alpha_0) \quad (0\leqslant s)\,. \tag{32}$$

We select the $s=0$ and $s=1$ equations from Eq. (32) and use them to determine the coefficients of the power-series expansion of $-z_0(k,t,\alpha_0)$ in t. This power-series expansion has no constant term because of Eq. (31). The expressions for the coefficients are complicated, but their dependence of k can be displayed because of the following lemma, which we prove in the Appendix

$$\frac{M^{-1}{}_{ki}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)}=\sum_{q=0}^{i} C_{q,p}(\alpha_0)\frac{M^{-1}{}_{k,p+q}(\alpha_0)}{M^{-1}{}_{kp}(\alpha_0)}, \tag{33}$$

where the $C_{q,p}(\alpha_0)$ are independent of k and polynomial in α_0. Then the power series for $-z_0(k,t,\alpha_0)$ has a determined dependence on k:

$$-z_0(k,t,\alpha_0)=\sum_{q=1}^{\infty} t^q \sum_{i=0}^{q} A_i{}^q(-1)^i \frac{M^{-1}{}_{ki}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)}, \tag{34}$$

where the $A_i{}^q$ are independent of k, but not of α_0. Multiplication of Eq, (34) by itself s times, and repeated use of Eq. (33), establishes a similar structure for $[-z_0(k,t,\alpha_0)_2]^s$:

$$[-z_0(k,t,\alpha_0)]^s=\sum_{q=s}^{\infty} t^q \sum_{i=0}^{q} A_i{}^{(q,s)}(-1)^i \frac{M^{-1}{}_{ki}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)}. \tag{35}$$

Equations (33) and (35) imply that Eq. (32) is true for all s if it is true for $s=0$.

We see that Eq. (34) follows from Eq. (32) for $s=0$ and $s=1$, and hence is necessary for analyticity. On the other hand, Eq. (32) for $s=0$ and Eq. (34) imply Eq. (32) for $s\geq 1$. Therefore, the analyticity conditions are equivalent to Eq. (32) for $s=0$ and Eq. (34)

$$\frac{R(k,\ \alpha_0-k)}{\partial\alpha(k,t,\alpha_0)/\partial\alpha_0|_{\alpha_0+z_0(k,t,\alpha_0)}}=\sum_{q=0}^{\infty} t^q \sum_{i=0}^{q} B_i{}^q M^{-1}{}_{ki}(\alpha_0)\,,$$

$$-z_0(k,t,\alpha_0)=\sum_{q=1}^{\infty} t^q \sum_{i=0}^{q} A_i{}^q(-1)^i \times\frac{M^{-1}{}_{ki}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)}. \tag{36}$$

We remark that at this point we have separated the constraint on the trajectories from the constraint also involving the residues. This separation is a significant check on the consistency of Mandelstam analyticity with the notions of Regge theory.

Because the $C_{q,p}(\alpha_0)$ in Eq. (33) are analytic in α_0, the $B_i{}^{(q,s)}$ in Eq. (28) are analytic in α_0 if the $A_i{}^q$ and $B_i{}^q$ are. This establishes the consistency mentioned following Eq. (30). We shall return in Sec. III to the physical meaning of the assumption of analyticity for the $A_i{}^q$ and $B_i{}^q$.

C. Step III

To continue our analysis, let us consider the sums

$$S=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\alpha_0{}^n}\left[[-z_0(k,t,\alpha_0)]^n \times\frac{R(k,\ \alpha_0-k)}{\partial\alpha(k,t,\alpha_0)/\partial\alpha_0|_{\alpha_0+z_0(k,t,\alpha_0)}}\right],$$

$$\tilde{S}=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\alpha_0{}^n}\left\{[-z_0(k,t,\alpha_0)]^{n+1} \times\left[1+\frac{\partial z_0(k,t,\alpha_0)}{\partial\alpha_0}\right]\right\}. \tag{37}$$

The manipulations of Eqs. (29) and (30) permit us to write S and $\tilde{S}$ as integrals

$$S=\frac{1}{2\pi i}\oint\frac{dz\,R(k,\ \alpha_0-k+z)}{[z+z_0(k,\ t,\ \alpha_0+z)]\partial\alpha(k,t,\alpha_0)/\partial\alpha_0|_{\alpha_0+z+z_0(k,t,\alpha_0+z)}},$$

$$\tilde{S}=\frac{1}{2\pi i}\oint\frac{dz[-z_0(k,\ t,\ \alpha_0+z)][1+\partial z_0(k,t,\alpha_0)/\partial\alpha_0|_{\alpha_0+z}]}{z+z_0(k,\ t,\ \alpha_0+z)}. \tag{38}$$

Since the right sides of Eqs. (36) are analytic functions of α_0, the only singularities of these integrands come from the vanishing of $z+z_0(k,\ t,\ \alpha_0+z)$. z_0 vanishes at $t=0$, so in some neighborhood of $t=0$ there will be only one simple pole inside the unit circle at $z_1(k,t,\alpha_0)$, determined from

$$z_1(k,t,\alpha_0)+z_0(k,\,t,\,\alpha_0+z_1(k,t,\alpha_0))=0\,,\qquad \lim_{t\to0} z_1(k,t,\alpha_0)=0\,. \tag{39}$$

Then the sums become

$$S=\frac{R(k,\,\alpha_0-k+z_1(k,t,\alpha_0))}{[1+\partial z_0(k,t,\alpha_0)/\partial\alpha_0]_{\alpha_0+z_1(k,t,\alpha_0)}][\partial\alpha(k,t,\alpha_0)/\partial\alpha_0|_{\alpha_0}]}\,,$$
$$\bar{S}=z_1(k,t,\alpha_0)\,. \tag{40}$$

Equation (31) is an identity in α_0. We evaluate it at $\alpha_0\to\alpha_0+z_1$, and find that

$$\alpha(k,\,t,\,\alpha_0+z_1+z_0(k,\,t,\,\alpha_0+z_1))-\alpha_0+k=z_1\,, \tag{41}$$

for any z_1. We now let z_1 be determined by Eq. (39):

$$z_1(k,t,\alpha_0)=\alpha(k,t,\alpha_0)-\alpha_0+k\,. \tag{42}$$

Also, if we differentiate Eq. (41) with respect to α_0, we find that

$$\left[1+\frac{\partial z_0(k,t,\alpha_0)}{\partial\alpha_0}\bigg|_{\alpha_0+z_1}\right]\times\frac{\partial\alpha(k,t,\alpha_0)}{\partial\alpha_0}\bigg|_{\alpha_0+z_1+z_0(k,t,\alpha_0+z_1)}=1\,. \tag{43}$$

We let z_1 be determined by Eq. (39), and Eq. (43) becomes

$$\left[1+\frac{\partial z_0(k,t,\alpha_0)}{\partial\alpha_0}\bigg|_{\alpha_0+z_1(k,t,\alpha_0)}\right]\frac{\partial\alpha(k,t,\alpha_0)}{\partial\alpha_0}\bigg|_{\alpha_0}=1\,. \tag{44}$$

Collecting our results, we have

$$S=R(k,\alpha(k,t))\,,\quad \bar{S}=\alpha(k,t,\alpha_0)-\alpha_0+k\,. \tag{45}$$

We can also evaluate S and $\bar{S}$ from Eq. (36). After a slight regrouping of terms in the expression for $\bar{S}$, we find that

$$R(k,t)=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\alpha_0{}^n}\left\{[\sum_{r=0}^{\infty}t^r\sum_{j=0}^{r}B_j{}^rM^{-1}{}_{kj}(\alpha_0)]\times\left[\sum_{q=1}^{\infty}t^q\sum_{i=0}^{q}A_i{}^q\frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}\right]^n\right\}\,, \tag{46a}$$

$$\alpha(k,t)=\alpha_0-k+\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\frac{\partial^n}{\partial\alpha_0{}^n}\times\left[\sum_{q=1}^{\infty}t^q\sum_{i=0}^{q}A_i{}^q\frac{k!\Gamma(2\alpha_0-k+2)}{(k-i)!\Gamma(2\alpha_0-k-i+2)}\right]^{n+1}\,. \tag{46b}$$

In writing these equations we have dropped the explicit dependence on α_0 on the left, written R as a function of t rather than $\alpha(k,t)$, and evaluated the ratio $M^{-1}{}_{ki}(\alpha_0)/M^{-1}{}_{k0}(\alpha_0)$.

Equations (46) are implied by the analyticity conditions, Eq. (36), but in fact Eqs. (46) also imply the analyticity conditions, and are equivalent to them. To see this, we put Eqs. (46) in a more condensed form. We define

$$G(k,t,\alpha_0)=\sum_{q=0}^{\infty}t^q\sum_{i=0}^{q}B_i{}^qM^{-1}{}_{ki}(\alpha_0)\,,$$
$$H(k,t,\alpha_0)=\sum_{q=1}^{\infty}t^q\sum_{i=0}^{q}A_i{}^q(-1)^i\frac{M^{-1}{}_{ki}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)}\,. \tag{47}$$

Thus, Eqs. (46) may be written

$$R(k,\alpha(k,t,\alpha_0))=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\alpha_0{}^n}G(k,t,\alpha_0)[H(k,t,\alpha_0)]^n\,,$$
$$\alpha(k,t,\alpha_0)-\alpha_0+k=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^n}{\partial\alpha_0{}^n}[H(k,t,\alpha_0)]^{n+1}\,. \tag{48}$$

Using the method of Eqs. (29) and (30) on the right, we find that

$$R(k,\alpha(k,t,\alpha_0))=\frac{G(k,\,t,\,\alpha_0+z_2(k,t,\alpha_0))}{1-\partial H(k,t,\alpha_0)/\partial\alpha_0|_{\alpha_0+z_2(k,t,\alpha_0)}}\,, \tag{49}$$
$$\alpha(k,t,\alpha_0)-\alpha_0+k=z_2(k,t,\alpha_0)\,,$$

where

$$z_2(k,t,\alpha_0)=H(k,\,t,\,\alpha_0+z_2(k,t,\alpha_0))\,,\qquad \lim_{t\to0}z_2(k,t,\alpha_0)=0\,. \tag{50}$$

These formal representations allow us to compute the left side of Eq. (28), again by the methods of Eqs. (29) and (30). When the calculation is finished, the right side of Eq. (28) emerges. Consequently, Eqs. (46) are equivalent to the analyticity conditions, and express the exact constraint imposed by broken $O(4)$ symmetry.

III. COMMENTS

In commenting on our results, we confine our attention to the trajectory constraint, Eq. (46b). First, we point out the physical meaning of the assumption of analyticity of the $A_i{}^q$ and α_0. Suppose one of the $A_i{}^q$ is singular at a point $\alpha_0{}^*$. It follows from Eq. (6) that as $\alpha_0\to\alpha_0{}^*$ (from at least some direction), either $\alpha^{(p)}(k)$ diverges for sufficiently high p or the $\alpha^{(p)}(k)$ increase more rapidly with p than $(p!)$. In either case, the radius of convergence of the Taylor's series for $\alpha(k,t)$ shrinks to zero as $\alpha_0\to\alpha_0{}^*$. Of course, to carry out the derivation of Sec. II, we have to start at an α_0 such that $|\alpha_0-\alpha_0{}^*|>1$, and continue the result to $\alpha_0{}^*$. We see that $\alpha(k,t)$ has a singularity which moves to $t=0$ as $\alpha_0\to\alpha_0{}^*$. We know that Regge trajectories can have singularities at arbitrary points only when two trajectories intersect there, so the singularity of A_i^q at α_0^* is to be associated with the intersection of Toller poles at $t=0$. Except in this special physical case the A_i^q are analytic. We stress that by the analytic continuation mentioned above, our formulas are valid arbitrarily close to α_0^*, and not just when $|\alpha_0-\alpha_0^*|>1$.

Second, the possibility that the A_i^q are functions of α_0 has real consequences for the formula for the derivatives, Eq. (6). For the first and second derivatives, Eq. (2), it can be shown that the formula is essentially the same whether A_0^1 and A_1^1 depend on α_0 or not. This is demonstrated by redefining the A_i^2. However, for $\alpha^{(3)}(k)/6!$ there is a term

$$k[A_0^1+A_1^1(2\alpha_0-k+1)]^2\partial A_1^1/\partial\alpha_0$$

which cannot be eliminated by redefining the A_i^3. This means that $\alpha^{(3)}(k)$ depends upon five new parameters, A_i^3 and $\partial A_1^1/\partial\alpha_0$, as well as those present in the formulas for $\alpha^{(1)}(k)$ and $\alpha^{(2)}(k)$. This shows the conjecture made in Ref. 8 to be wrong. There it was postulated that $\alpha^{(p)}(k)$ depends on parameters present in $\alpha^{(p')}(k)$ $(p'<p)$ plus only $(p+1)$ new parameters.[13a]

The calculation of the analog of Eq. (46b) for other integer M can be accomplished using a generalization of the techniques developed in Ref. 12. For $M\geq 1$ one must use Reggeized helicity amplitudes, and there will be conspiring Regge trajectories of both natural and unnatural parities. The calculation of the zero-helicity amplitude is what we have presented in this paper. Factorization in the helicity indices demands that one of the parity sequences satisfy Eq. (46b), and in fact both of them will. However, conspiracy will not permit all the $(A_i^q)^+$ and $(A_i^q)^-$ for the two parity sequences to be independent.[7,12] It is only this relation between these parameters which can depend upon M, and which remains to be calculated. The generalization of our work to half-integer M is not so simple; one must start from the beginning.

[13a] *Note added in proof.* The statements made in this paragraph are incorrect. All dependence of $\alpha^{(3)}(k)$ on derivatives of the A's with respect to α_0 can be eliminated by redefining the A_i^3. In fact, it can be shown that no generality is lost if the A_i^q are taken to be independent of α_0 in Eq. (2). Thus the conjecture of Ref. 8 is correct. The author is indebted to Dr. Paul Fishbane for pointing out the error.

After this manuscript was written, we received a report by J. C. Taylor which derives a less explicit statement of the consequences of broken $O(4)$ symmetry.

APPENDIX

Using the notation $\Gamma(x+n)/\Gamma(x)=(x)_n$, we obtain

$$M^{-1}{}_{k,q+p}(\alpha_0)/M^{-1}{}_{k,p}(\alpha_0)=(-1)^q(k-p-q+1)_q \times(2\alpha_0+2-k-q-p)_q. \quad (A1)$$

Consider the linear combination

$$I(k)=\sum_{q=0}^{i} C_{q,p}(\alpha_0)(-1)^q(k-q-p+1)_q \times(2\alpha_0+2-k-p-q)_q. \quad (A2)$$

I is a polynomial of degree $2i$ in k. The $i+1$ constants $C_{q,p}(\alpha_0)$ are fixed by $C_{i,p}(\alpha_0)=1$ and the i linear inhomogeneous equations which come from demanding $I(k)=0$, $k=0, 1,\cdots, i-1$. These equations are

$$0=\sum_{q=0}^{i} C_{q,p}(\alpha_0)(-1)^q(k-q-p+1)_q \times(2\alpha_0+2-k-p-q)_q \quad (0\leqslant k\leqslant i-1). \quad (A3)$$

We compute

$$I(2\alpha_0+1-k)=\sum_{q=0}^{i} C_{q,p}(\alpha_0)(-1)^q(k-p-q+1)_q \times(2\alpha_0+2-k-p-q)_q=0 \quad (0\leqslant k\leqslant i-1). \quad (A4)$$

Hence, $I(k)$ is the unique polynomial which behaves like k^{2i} at infinity and vanishes at the $2i$ points $k=0$, $1,\cdots, i-1$; $2\alpha_0+1$, $2\alpha_0,\cdots$ $2\alpha_0-i+2$. This polynomial is $(-1)^i(k-i+1)_i(2\alpha_0+2-k-i)_i$. Thus,

$$I(k)=\frac{M^{-1}{}_{ki}(\alpha_0)}{M^{-1}{}_{k0}(\alpha_0)}=\sum_{q=0}^{i} C_{q,p}(\alpha_0)\frac{M^{-1}{}_{k,q+p}(\alpha_0)}{M^{-1}{}_{k,p}(\alpha_0)}. \quad (A5)$$

The left side of this equation is a polynomial in α_0. The only singularities $C_{q,p}(\alpha_0)$ can have are poles cancelled by the zeroes $(2\alpha_0+2-k-p-q)_q$. However, these zeros move with k, and $C_{q,p}(\alpha_0)$ is independent of k. Hence $C_{q,p}(\alpha_0)$ is a polynomial in α_0.