Eigenphase Shifts and Double-Humped Resonances

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(Received 4 December 1968)

Using eigenphase shifts, it is shown how a resonance can be single-humped in some reactions and double-humped in others, as appears to happen for the A_2.

THE interplay of several scattering channels can lead to resonance phenomena too complicated to describe with Breit-Wigner forms. Eigenphase shifts, on the other hand, seem a more suitable tool for describing such intrinsically multichannel phenomena as resonances. Just a few eigenphase shifts suffice to explain quite complicated Argand diagrams. The model is rather like the Ptolemaic theory of planetary motion. Quite different eigenphase shift behaviors can explain experimental data, and only by knowing the dynamical origin of a resonance can one choose between them.

In the following it will be shown how a resonance can appear to have two peaks in some reactions and only a single peak in others. This is a possible explanation of the A_2, which appears as a wide, two-peaked object in what is essentially the reaction $\pi\rho \rightarrow \pi\rho$, and as a much narrower single-peaked resonance in the reaction $\pi\rho \rightarrow K_1^0 K_1^0$.[1] It will not be necessary to postulate two separate resonances, as is done by Lassila and Ruuskanen.[2] The description of the A_2 in $\pi\rho \rightarrow \pi\rho$ given here is somewhat similar to that of Coulter and Shaw.[3] The resonance itself is very inelastic, and the resonance position coincides with a dip in the elasticity. These authors, however, do not attempt to describe the A_2 in the reaction $\pi\rho \rightarrow K_1^0 K_1^0$.

Before proceeding, it is appropriate to review briefly the theory of eigenphase shifts. The S-matrix elements connecting n two-body channels of the same spin, parity, and internal quantum numbers constitute, by unitarity, a unitary matrix. Since it is unitary, its eigenvalues have unit norm. They are $e^{2i\delta_k}$, where the δ_k are the eigenphase shifts. The corresponding eigenvectors are e^k:

$$e^k = (e_1^k, \cdots, e_n^k), \quad k=1, \cdots, n.$$

These eigenvectors are orthogonal and complete

$$\sum_{i=1}^{i=n} e_i^l e_i^{k*} = \delta^{lk}, \quad \sum_{k=1}^{k=n} e_i^k e_j^{k*} = \delta_{ij}. \tag{1}$$

Introducing the Hermitian matrices E^k, $E_{ij}^k = e_i^k e_j^{k*}$, which by (1) satisfy

$$E^k E^l = \delta^{kl} E^l, \quad \sum_{k=1}^{k=n} E^k = I, \tag{2}$$

one has

$$S = \sum_{k=1}^{k=n} e^{2i\delta_k} E^k. \tag{3}$$

By (2) this is easily seen to be a unitary matrix. The scattering matrix is defined by

$$T = \frac{S-I}{2i} = \sum_{k=1}^{k=n} \sin\delta_k \, e^{i\delta_k} E^k. \tag{4}$$

At a resonance, one, and only one, of the eigenphase shifts rises through $\frac{1}{2}\pi$. If it rises steeply enough, T must have a simple pole near the physical axis in the complex energy plane. The converse is also true.[4]

As a specific instance of this formalism, consider a two-channel problem. S is then a 2×2 matrix and has eigenvalues, $e^{2i\delta_1}$ and $e^{2i\delta_2}$. Let the corresponding eigenvectors be

$$e^1 = (\tfrac{3}{5}, \tfrac{4}{5}), \quad e^2 = (\tfrac{4}{5}, -\tfrac{3}{5}).$$

Then the matrices E^k are

$$E^1 = \begin{pmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{pmatrix}, \quad E^2 = \begin{pmatrix} 16/25 & -12/25 \\ -12/25 & 9/25 \end{pmatrix}. \tag{5}$$

It follows by (3) that the S-matrix elements for the reactions $1 \rightarrow 1$ and $1 \rightarrow 2$ are

$$S_{11} = (9/25)e^{2i\delta_1} + (16/25)e^{2i\delta_2},$$
$$S_{12} = (12/25)(e^{2i\delta_1} - e^{2i\delta_2}). \tag{6}$$

The T-matrix elements for these reactions are

$$T_{11} = (9/25)\sin\delta_1 \, e^{i\delta_1} + (16/25)\sin\delta_2 \, e^{i\delta_2},$$
$$T_{12} = (12/25)(\sin\delta_1 \, e^{i\delta_1} - \sin\delta_2 \, e^{i\delta_2}). \tag{7}$$

Now, identifying $1 \rightarrow 1$ with the reaction $\pi\rho \rightarrow \pi\rho$, and $1 \rightarrow 2$ with the reaction $\pi\rho \rightarrow K_1^0 K_1^0$, it is easy to contrive an eigenphase-shift behavior which gives the resonance behavior characteristic of the A_2 in these reactions. This behavior is illustrated in Fig. 1. The resonant eigenphase shift δ_1 rises through $\frac{1}{2}\pi$ at the

[1] G. Chikovani *et al.*, Phys. Letters **25B**, 44 (1967); D. Crennell, U. Karshon, K. W. Lai, J. M. Scarr, and I. O. Skillicorn, Phys. Rev. Letters **20**, 1318 (1967).
[2] K. Lassila and P. Ruuskanen, Phys. Rev. Letters **19**, 204 (1967).
[3] P. W. Coulter and G. L. Shaw, Phys. Rev. Letters **21**, 634 (1968).

[4] A resonance may correspond to a zero of the phase shift δ proper, $S = \eta e^{2i\delta}$, if the elasticity η is less than $\frac{1}{2}$. A zero of an eigenphase shift, however, does not correspond to a resonance. In fact, it has no particular physical significance.

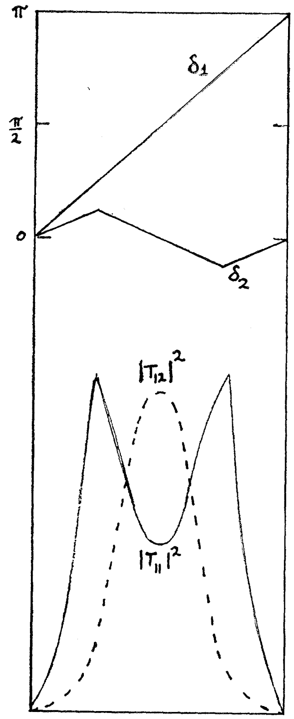

FIG. 1. Eigenphase shifts and cross sections for the reactions $1 \to 1$ (solid) and $1 \to 2$ (dashed). For simplicity the eigenphase shifts have been chosen to consist of line segments. The cusps of δ are not essential for the qualitative behavior (see Ref. 2) of the cross sections.

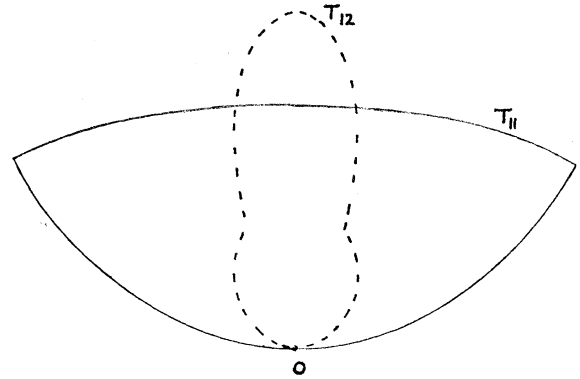

FIG. 2. Trajectories of T_{11} (solid) and T_{12} (dashed) in the complex plane. The origin is at zero.

resonance energy. The other, δ_2, passes down through zero at this energy. There is only one resonance pole required: that corresponding to the rise of δ_1 through $\frac{1}{2}\pi$.[4]

The trajectories of both T_{11} and T_{12} in the complex plane are illustrated in Fig. 2. To understand these trajectories qualitatively, it is helpful to rotate this figure by 90° and consider it as a plot of the trajectories of S_{11} and S_{12}. One sees from (6) that at the resonance energy the two terms in S_{11} interfere destructively, producing a dip, but interfere constructively in S_{12}. The cross sections for the reactions $1 \to 1$ and $1 \to 2$ are proportional to $|T_{11}|^2$ and $|T_{12}|^2$. The magnitudes of T_{11} and T_{12} can be read off Fig. 2. The result is plotted on Fig. 1. Experiments indicate that the peak in $\pi\rho \to K_1^0 K_1^0$ actually lies somewhat above the dip in $\pi\rho \to \pi\rho$. There is certainly enough freedom in the eigenphase-shift model to account also for this; the eigenphase-shift behavior indicated on Fig. 1 has been chosen, in part, for its simplicity.

One can easily see by (4) and (5) that there are two peaks also in the reaction $2 \to 2$ but only a single peak in $2 \to 1$. This means that in a K beam the experimental results should reverse, at least if K exchange is the predominant production mechanism of the A_2. Two peaks should be observed in the decay $A_2 \to K_1^0 K_1^0$ and a single peak in $A_2 \to \pi\rho$.

In fact, if there are more than two channels, but only two important eigenphase shifts which behave as indicated on Fig. 1, then the resonance should be double-peaked in all elastic reactions. Let i be the channel index; by (3) the S-matrix element for the elastic reaction $i \to i$ is

$$S_{ii} = (e_i^1)^2 e^{2i\delta_1} + (e_i^2)^2 e^{2i\delta_2} + (\text{unimportant terms}).$$

Since both $(e_i^1)^2$ and $(e_i^2)^2$ are positive, a destructive interference occurs when one eigenphase shift rises through $\frac{1}{2}\pi$ and the other falls through zero. For an inelastic reaction, $i \to j$; on the other hand,

$$S_{ij} = e_i^1 e_j^1 e^{2i\delta_1} + e_i^2 e_j^2 e^{2i\delta_2} + (\text{unimportant terms}),$$

and since the relative sign of $e_i^1 e_j^1$ and $e_i^2 e_j^2$ is unknown, the interference can be constructive or destructive. In inelastic reactions the resonance can appear in both forms, double-humped or single-humped.

It is not clear whether to expect the other members of the $SU(3)$ multiplet to which the A_2 belongs to also exhibit splitting. This depends on whether the behavior of the eigenphase shift δ_2 indicated on Fig. 1 is accidental or has a fundamental dynamical origin. This question, and in fact the entire mechanism discussed here, can only be tested with a dynamical model, for instance, in a multichannel ND^{-1} calculation.

I wish to acknowledge the many suggestions and criticisms of Professor Gordon Kane.