Methods of Treating Coulomb Interference Corrections in Potential Scattering*

STEPHEN RAE AND PETER THURNAUER
Department of Physics, University of Vermont, Burlington, Vermont 05401
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Three distinct, though related, formalisms, developed by Block, by West, and by Auvil, for dealing with Coulomb interference corrections to nuclear potential scattering problems are considered. Such formalisms are particularly relevant to the analysis of recent pion-helium scattering experiments, and their equivalence is examined. It is shown that to first order in the Coulomb coupling parameter n, the results for the scattering amplitudes given by West and by Auvil (as generalized in the present work) are equivalent but differ from those suggested by Block. On the other hand, to first order in n the results for the cross sections are all seen to be equal.

I. INTRODUCTION

THE possibility of performing experiments aimed at discovering the electromagnetic form factor of the charged pion through $\pi^{\pm}$-He scattering[1] and the recent realization of such experiments[2] have stimulated interest in finding an accurate description of Coulombic effects as they appear when in competition with nuclear forces. It is desirable that a suitable formalism for such a description, apart from allowing for deviations from pure Coulomb interactions, also avoid the rather involved Coulomb wave functions and employ the generally simpler (approximate or phenomenological) wave functions used to describe scattering from nuclear potentials. Such as approach, giving an expansion of the scattering amplitude in powers of the Coulomb coupling parameter, was first discussed by Schiff.[3] Unfortunately, the approximation given by Schiff yields terms that are logarithmically divergent, even though the cross section can be calculated in a useful manner to first order in the Coulomb coupling parameter by this method.[4]

In order to remove the divergences associated with Schiff's formalism, Block[5] has shown by a rather ingenious approach that if the scattering amplitude is multiplied by a suitably chosen, unobservable phase factor, one can then obtain finite results when the expansion in powers of the Coulomb parameter is made. Taking a different point of view, however, West[6] has been able to extend the original formalism of Schiff's by introducing integrating factors in such a manner that an apparently consistent and well-defined expansion of the scattering amplitude is also obtained. Finally, Auvil[7] has attempted a more systematic and rigorous extension of Block's approach which in essence consists of an examination of the phase shifts rather than the scattering amplitudes directly.

West's formalism applies very generally to nuclear (or, more precisely, finite-range) plus "Coulomb-like" potentials, where "Coulomb-like" refers to the behavior of the potential at infinity, being there the same as for a pure Coulomb potential. Elsewhere, a Coulomb-like potential has arbitrary behavior, though it is generally taken to be nonsingular. Block's derivation is based on a nuclear-plus-pure-Coulomb potential, but he generalizes his results to Coulomb-like potentials in an intuitive manner. Although Auvil's results are for a nuclear-plus-pure-Coulomb potential, they may be extended in a straightforward manner to include Coulomb-like potentials, as discussed below. Obviously, for the physical problem under study, namely, $\pi^{\pm}$-He scattering, the interest is in Coulomb-like rather than in pure Coulomb potentials.

Although the approaches and mathematical techniques used by the three authors whose work is being examined here appear quite distinct from one another, their results may, of course, be compared. It is the purpose of the present work to examine the various results for two possible situations: first, scattering from a nuclear-plus-pure-Coulomb potential and second, the physically more interesting case of scattering from a nuclear-plus-Coulomb-like potential. It is shown that, although, at least to first order in the Coulomb coupling parameter, West's and Auvil's results are in agreement, they differ somewhat from those given by Block for the scattering amplitude for both cases considered. However, it turns out that all results for the relevant cross sections are nevertheless equivalent to first order in the Coulomb parameter.

In Sec. II, we give a very brief review of the methods used respectively by the three authors and establish a common notation for discussing all the formalisms. We also extend Auvil's results to include the case of Coulomb-like potentials. Section III is devoted to examining the results obtained by the various approaches for scattering from a nuclear potential plus both pure Coulomb and Coulomb-like potentials. Section IV presents a short discussion of our results.

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II. REVIEW OF FORMALISMS

The brief review which is presented in this section is given mainly to define various physical quantities of interest; complete details may be found in the relevant references.[5–7]

We are interested in nonrelativistic scattering from a central potential which consists of the sum of a short-range (strong) component, denoted here by $u(r)$, and a long-range, comparatively weak, Coulomb-like part, denoted by $v(r)$, where r is the distance of the position of the particle from the scattering center.[8] The solution $\psi(r,\theta)$ for scattering of a particle of mass m with relative wave number k by the potential $u(r)$ alone may be given by an expansion in Legendre polynomials with the familiar form[9]:

$$\psi(r,\theta)=\sum_{l=0}^{\infty}(2l+1)i^l e^{i\delta_l}R_l(r)P_l(\cos\theta), \tag{1}$$

where $R_l(r)$ is the regular solution to the radial part of the wave equation, with asymptotic behavior

$$R_l\sim(1/kr)\sin(kr-\tfrac{1}{2}l\pi+\delta_l). \tag{2}$$

On the other hand, the asymptotic form of the radial part of the pure Coulomb wave of angular momentum l is proportional to the expression[9]

$$(1/kr)\sin(kr-\tfrac{1}{2}l\pi-n\ln 2kr+\eta_l), \tag{3}$$

where the Coulomb coupling parameter is $n=Ze^2m/k$, Ze being the nuclear charge. The phase shifts for scattering from a pure Coulomb potential are defined to have the well-known form

$$\eta_l=\arg\Gamma(l+1+in). \tag{4}$$

It follows that the radial solution to the problem of scattering from the sum of the nuclear potential u and the pure Coulomb potential must have the asymptotic form

$$(1/kr)\sin(kr-\tfrac{1}{2}l\pi-n\ln 2kr+\eta_l+\delta_l+\Delta_l). \tag{5}$$

Finally, we associate phase shifts α_l with the asymptotic form of the partial-wave solution, $S_l(r)$, to the problem of scattering from the sum of the potentials $u+v$, in the following manner:

$$S_l(r)\sim(1/kr)\sin(kr-\tfrac{1}{2}l\pi-n\ln 2kr+\eta_l+\delta_l+\alpha_l). \tag{6}$$

The basic approach used by all the aforementioned authors in deriving a formalism suitable to the problem of interest is to consider the wave functions R_l as the unperturbed solutions on which the Coulomb or Coulomb-like potential acts as a perturbation. However, unless due caution is exercised in handling this procedure, difficulties arise from the fact that the asymptotic wave forms of the R_l as given by (2) can never by themselves yield asymptotic expressions of the form (5) or (6). Hence, as emphasized by West and also by Antoine,[4] in order to obtain a consistent approach, one must either introduce a cutoff Coulomb or Coulomb-like potential from the very beginning, or else the logarithmically distorted nature of the asymptotic waves must somehow be built directly into the formalism. The former device[10] is employed by Auvil; the latter approach is used by West and essentially also by Block.

We now examine briefly the work under discussion. We begin with Auvil's[7] approach, generalizing his results for pure Coulomb potentials in a very straightforward manner to the case of nuclear-plus-Coulomb-like potentials. We define the potential

$$\begin{aligned} v_R(r)&=v(r), \quad r<R\\ &=0, \quad\quad r>R \end{aligned} \tag{7}$$

and consider scattering from the now finite-range potential, $u+v_R$. Making use of standard techniques for short-range potential problems, one obtains

$$e^{i(\eta_l-n\ln 2kR+\alpha_l)}\sin(\eta_l-n\ln 2kR+\alpha_l)=-k\int_0^\infty r^2R_lS_Rv_Rdr, \tag{8}$$

where S_R can be shown to satisfy an integral equation of the familiar form:

$$S_R(r)=R_l(r)+\int_0^\infty G_l(r,r')v_R(r')S_R(r')dr'. \tag{9}$$

The function $G_l(r,r')$ is the Green's function appropriate to this scattering problem with the now finite-range potentials.[11]

We may now follow Auvil's procedure and expand the left- and right-hand sides of Eq. (8) in powers of n and compare the corresponding coefficients on each side of the equation for large values of R. The expansion of the right-hand side of (8) is achieved by substituting for S_R the results obtained by solving the integral equation (9) by repeated substitutions; on the left-hand side of Eq. (8), η_l and α_l are expanded in a Taylor's

[8] The potentials referred to throughout are the so-called reduced potentials, having value $2m$ times the usual definition of potential, where m is the reduced mass of the particle moving in the field of the potential.

[9] See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Co., Inc., New York, 1955), 2nd ed., Secs. 19 and 20.

[10] We note that Antoine's work (Ref. 4) also makes use of a cutoff potential; however, Antoine's procedure is based on an expansion of the scattering amplitude, rather than the phase shifts, in powers of the Coulomb coupling parameter. This leads to certain apparent divergences, which can presumably be shown to vanish for specific choices for the potentials involved.

[11] See, for example, M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964), pp. 304–306.

series about $n=0$. Thus, one has to first order in n

$$n[\eta_l'-\ln 2kR+\alpha_l']$$
$$=-k\int_0^R r^2R_l^2v_R dr$$
$$=-k\int_0^R r^2(R_l^2v_R-j_l^2v_0)-k\int_0^R r^2j_l^2v_0dr, \quad (10)$$

where the prime denotes differentiation with respect to n, and v_0 is the pure Coulomb potential,

$$v_0=2nk/r. \quad (11)$$

But Auvil has shown that[12]

$$\lim_{R\to\infty}\left(-k\int_0^R r^2j_l^2v_0dr\right)=n[\eta_l'-\ln 2kR]. \quad (12)$$

Thus, we can identify

$$n\alpha_l'=-\int_0^\infty kr^2(R_l^2v-j_l^2v_0)dr, \quad (13)$$

where we may allow $R\to\infty$, since the integral is well-behaved in this limit.[13] One may now proceed to examine higher orders in n, at each stage presumably obtaining finite, physically observable results.

Let us next consider the approach used by West.[6] This formalism makes use of basically simple identities containing Wronskians of solutions of the Schrödinger equation for various potentials. By manipulating such identities, West is able to obtain a very general integral equation [Eq. (47) of Ref. 6] involving solutions to scattering problems with quite arbitrary potentials. If one applies this equation to the reaction of interest, and the (assumed) known regular and irregular solutions of the nuclear problem as well as the unknown solution of the complete problem are inserted into West's equation, one obtains an exact integral equation involving S_l. Since the asymptotic behavior of the complete solution is given by (5) or (6) and as the integral equation may presumably be solved by repeated substitutions, one is able to obtain ultimately an expression for the complete scattering amplitude in powers of the Coulomb coupling parameter n.

There arise in this procedure, however, integrals of the type

$$\int_0^\infty v(r)e^{in\ln 2kr}dr. \quad (14)$$

Such integrals are not convergent if the usual Riemannian definition of an infinite integral is understood for evaluating (14). Accordingly, a prescription for obtaining finite results for such integrals is introduced by West that amounts to giving n a small positive imaginary part, which is to be set equal to zero after the (now well-defined) integration has been performed.[14] Thus, by the use of this prescription, all integrals become well-defined, and one has a well-behaved and apparently consistent scheme for expressing the scattering amplitude for the complete reaction as an expansion in powers of the Coulomb coupling constant.

The result obtained by West for the amplitude for scattering from a nuclear-plus-Coulomb-like potential to first order in n is

$$f^{(W)}(\theta)=-\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\times\int_0^\infty vr^2e^{2i\delta_l}R_l^2(2kr)^{2in}dr, \quad (15)$$

where the integral is of course defined through West's prescription. The expression for the scattering amplitude can be put into a more convenient form [Eq. (64) of Ref. 6]. Equation (15) also applies to the case of a nuclear-plus-pure-Coulomb potential if v is replaced by v_0. For reference, we write down the pure Coulomb result here, also in more convenient form than Eq. (15).

Consider the integral in (15) with v_0 substituted for v. Adding and subtracting the sum of integrals,

$$\int_0^\infty v_0r^2(2kr)^{2in}j_l^2dr+\int_{1/2k}^\infty v_0r^2(2kr)^{2in}\left[\frac{e^{2i\delta_l}-1}{2(kr)^2}\right]dr \quad (16)$$

to the integral under consideration, one obtains

$$\int_0^\infty v_0r^2(2kr)^{2in}\left\{R_l^2e^{2i\delta_l}-j_l^2-\left[\epsilon\left(r-\frac{1}{2k}\right)\right]\frac{e^{2i\delta_l}-1}{2(kr)^2}\right\}dr+\int j_l^2v_0r^2(2kr)^{2in}dr+\frac{e^{2i\delta_l}-1}{2k^2}\int_{1/2k}^\infty v_0(2kr)^{2in}dr, \quad (17)$$

where $\epsilon(r-1/2k)$ is the Heaviside step function. The last integral may be evaluated by West's prescription and yields a factor ik. Substituting into (15), one obtains[6]

$$f_{CN}^{(W)}(\theta)=f_N-\int_0^\infty j_0(qr)v_0r^2(2kr)^{2in}dr-\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\int_0^\infty v_0r^2\left\{R_l^2e^{2i\delta_l}-j_l^2-\left[\epsilon\left(r-\frac{1}{2k}\right)\right]\frac{e^{2i\delta_l}-1}{2(kr)^2}\right\}dr, \quad (18)$$

[12] Reference 7, Appendix A.

[13] More precisely, we assume that the Coulomb-like potential v approaches v_0 rapidly enough as $r\to\infty$, so that the integral in (13) is indeed well defined.

[14] The use of integrating factors is a not uncommon mathematical device in physics, and as discussed in Ref. 6, the prescription introduced by West is closely related to the definition of the Dirac δ function.

where f_N is the nuclear scattering amplitude

$$f_N=\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l}-1)P_l(\cos\theta) \tag{19}$$

and $q=2k\sin^2(\frac{1}{2}\theta)$. The second term on the right-hand side of (18) is the Born Coulomb amplitude and yields, as is well known, the correct magnitude for the exact Coulomb scattering amplitude. Thus, if we denote this integral as $f_{C,B}$, Eq. (18) becomes

$$f_{CN}^{(W)}(\theta)=f_N-f_{C,B}-\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\int_0^{\infty}v_0r^2 \times\left\{R_l^2e^{2i\delta_l}-j_l^2-\left[\epsilon\left(r-\frac{1}{2k}\right)\right]\frac{e^{2i\delta_l}-1}{2(kr)^2}\right\}dr. \tag{20}$$

Lastly, we outline very briefly Block's[5] procedure. Block begins with Schiff's[3] analysis, but he notes that only the squared modulus of the scattering amplitude is observable, so that modifying the amplitude by a phase factor leads to no physical consequences. Choosing such a phase factor to be $\exp(2i\eta_0)$ and expanding the phase shifts in powers of n [see the discussion preceding Eq. (10) above], Block obtains, unlike Schiff, finite results for the amplitude to first order in n. The derivation is based on scattering from a nuclear-plus-pure-Coulomb potential, but the results are generalized in an intuitive manner essentially by replacing v_0 by v throughout.

III. COMPARISON OF RESULTS

A. Results for Nuclear-Plus-Pure-Coulomb Potential

We now compare to first order in the Coulomb coupling parameter the results obtained by the three methods discussed in Sec. II for the case of scattering from the potential $u+v_0$. West's result for this situation is given by Eq. (20) above.

Let us first attempt to put Auvil's result into the form of (20). Consider the expression

$$e^{2i(\delta_l+\eta_l+\Delta_l)}-1=(e^{2i\delta_l}-1)+(e^{2i\eta_l}-1) +e^{2i\delta_l}[e^{2i(\Delta_l+\eta_l)}-1]-(e^{2i\eta_l}-1). \tag{21}$$

If we use Auvil's result [Eq. (27) of Ref. 7; see also Eq. (13) above] that

$$n\Delta_l'=-\int_0^{\infty}kr^2v_0(R_l^2-j_l^2)dr \tag{22}$$

and make use also of (12), we may expand the last two terms on the right-hand side of (21) in powers of n to obtain their value to first order as:

$$2i(e^{2i\delta_l}-1)\lim_{R\to\infty}\left[-\int_0^R kr^2j_l^2v_0dr+n\ln 2kR\right] -2ie^{2i\delta_l}\lim_{R\to\infty}\int_0^R kr^2v_0(R_l^2-j_l^2)dr. \tag{23}$$

Adding to and subtracting from (23) an expression of the form

$$2i\lim_{R\to\infty}\int_0^R kr^2v_0j_l^2dr \tag{24}$$

yields

$$-2i\lim_{R\to\infty}\left[\int_0^R kr^2v_0(R_l^2e^{2i\delta_l}-j_l^2)dr +n(e^{2i\delta_l}-1)\ln 2kR\right]. \tag{25}$$

But we may write

$$n\ln 2kR=\frac{1}{2k}\int_{1/2k}^R v_0dr, \tag{26}$$

so that (21) becomes to first order in n:

$$e^{2i(\delta_l+\eta_l+\Delta_l)}-1=(e^{2i\delta_l}-1)+(e^{2i\eta_l}-1)-2i\int_0^{\infty}kr^2v_0 \times\left\{R_l^2e^{2i\delta_l}-j_l^2+\left[\epsilon\left(r-\frac{1}{2k}\right)\right]\frac{e^{2i\delta_l}-1}{2(kr)^2}\right\}dr. \tag{27}$$

The total scattering amplitude is given by

$$f_{CN}(\theta)=\frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)[e^{2i(\delta_l+\eta_l+\Delta_l)}-1]. \tag{28}$$

When (27) is substituted into (28) one obtains Auvil's result

$$f_{CN}^{(A)}(\theta)=f_N+f_C-\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\int_0^{\infty}r^2v_0 \times\left\{R_l^2e^{2i\delta_l}-j_l^2-\left[\epsilon\left(r-\frac{1}{2k}\right)\right]\frac{e^{2i\delta_l}-1}{2(kr)^2}\right\}dr, \tag{29}$$

where the Coulomb amplitude is

$$f_C=-e^{i\{2\eta_0-n\ln[\sin^2(\theta/2)]\}}f_{C,B}. \tag{30}$$

From (20), (29), and (30) we deduce that to first order in n, Auvil's and West's results are completely equivalent.

Consider next Block's results for the scattering amplitude [Eqs. (12), (13), and (16) of Ref. 5], which may be written as

$$f_{CN}^{(B)}(\theta)=f_N-f_{C,B}-\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\times\left[e^{2i\delta_l}\int_0^\infty v_0r^2(R_l^2-j_l^2)dr+(e^{2i\delta_l}-1)\int_0^\infty v_0r^2(j_l^2-j_0^2)dr\right].\quad(31)$$

Adding and subtracting a term similar to (24) as well as an integral involving the product of v_0 and the Heaviside step function and then collecting terms, one obtains

$$f_{CN}^{(B)}(\theta)=f_{CN}^{(W)}(\theta)+\frac{2i}{k}f_N\int_0^\infty v_0[\sin^2kr-\tfrac{1}{2}\epsilon(r-1/2k)]dr\quad(32)$$

to first order in n. Although Block's result for the total scattering amplitude is not quite equivalent to that given by West and by Auvil to first order in n, the fact that the difference involves a pure imaginary multiple of the nuclear amplitude and is of order n means that the corresponding results for the cross section will nevertheless be the same to first order in n.[4]

B. Results for Nuclear-Plus-Coulomb-Like Potential

We now examine the results of the three formalisms for scattering from a potential of the form $u+v$. West's result is given explicitly in his paper [Eq. (64) of Ref. 6].

Auvil's result as generalized in Sec. II may now be manipulated in a manner analogous to the procedure used in Sec. III A above except that now Δ_l is to be replaced by α_l. Thus, beginning with an expression similar to (21), we have a first-order term corresponding to (23) of the form

$$2i(e^{2i\delta_l}-1)\lim_{R\to\infty}\left[-\int_0^R kr^2j_l^2v_0dr+n\ln 2kR\right]-2ie^{2i\delta_l}\lim_{R\to\infty}\int_0^R kr^2(R_l^2v-j_l^2v_0)dr.\quad(33)$$

Adding and subtracting from (33) an expression of the type

$$2i\lim_{R\to\infty}\left[\int_0^R kr^2v\left(j_l^2+\frac{e^{2i\delta_l}-1}{2(kr)^2}\right)dr\right],\quad(34)$$

one obtains

$$-2i\int_0^\infty kr^2v\left(e^{2i\delta_l}R_l^2-j_l^2-\frac{e^{2i\delta_l}-1}{2(kr)^2}\right)dr-2i\int_0^\infty kr^2j_l^2(v-v_0)dr+2i(e^{2i\delta_l}-1)\times\lim_{R\to\infty}\left[n\ln 2kR-\frac{1}{2k}\int_0^R vdr\right].\quad(35)$$

But one may write

$$\int_0^R vdr=\int_0^{1/2k}vdr+\int_{1/2k}^R(v-v_0)dr+2nk\ln 2kR.\quad(36)$$

The two integrals on the right-hand side of (36) are finite and real in the limit $R\to\infty$, so we may denote their sum in this limit by $nk\gamma$, where γ is a real constant. Putting these various results together, one obtains finally to first order in n the generalized result of Auvil's

$$e^{2i(\delta_l+\eta_l+\alpha_l)}-1=(1-in\gamma)(e^{2i\delta_l}-1)+(e^{2i\eta_l}-1)-2i\int_0^\infty kr^2v\left[e^{2i\delta_l}R_l^2-j_l^2-\frac{e^{2i\delta_l}-1}{2(kr)^2}\right]dr-2i\int_0^\infty kr^2j_l^2(v-v_0)dr.\quad(37)$$

This result is completely well defined and is now easily seen by use of (28) and (30) to be equivalent to first order in n to West's result[15] for the scattering amplitude.

Block's results for the scattering amplitude may also be put in the standard form in a manner completely analogous to that used in obtaining Eq. (32). For the case of a Coulomb-like potential one finds, as might be expected,

$$f^{(B)}(\theta)=f^{(W)}(\theta)+\frac{2i}{k}f_N\int_0^\infty v(\sin^2kr-\tfrac{1}{2})dr,\quad(38)$$

where $f^{(W)}(\theta)$ refers to West's result for scattering from a nuclear-plus-Coulomb-like potential. Thus, the results for the cross section are again equivalent to order n.

IV. DISCUSSION

We have examined the results of the three methods for treating Coulomb interference corrections and have

[15] Note that there is apparently a misprint after Eq. (62) of Ref. 6. The sum of the first two integrals on the right-hand side of Eq. (62) of that paper is real to first order in n, so that if their sum is denoted by $nk\gamma$, one obtains a term $f_n(1-in\gamma)$ on the right-hand side of Eq. (63) in Ref. 6, agreeing with the corresponding term of Eq. (37) in the present work.

found that while West's and Auvil's results agree to order n, the results for the scattering amplitude differ with those given by Block. This difference, which is not really unexpected, has no observable physical consequences to order n, and the cross sections to this order are all equivalent.

One may, in principle, compare the various formalisms to higher orders in n. However, only Auvil's approach yields higher-order terms in a systematic and straightforward manner, and comparison of these results with the others is very involved. In this sense, Auvil's formalism can be considered the most useful in practice, if it is in fact correct, as we suspect it to be. In any case, one may attempt to test the various approaches by considering a particular example, such as scattering from a charged hard sphere.

In closing, we remark that although the original interest in this problem arose from the analysis of pion-helium scattering, numerous further applications of the above results are evident, and the basic approach suggested by the formalisms examined here will doubtless yield useful practical methods for analyzing the role of Coulomb corrections in future experiments.

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Crossing-Symmetric Regge Model and Isovector Meson Decays*

P. H. FRAMPTON
The Enrico Fermi Institute, The University of Chicago, Chicago, Illinois 60637
(Received 2 December 1968)

We investigate some features of a Regge model due to Veneziano. First, we state generally which subsidiary trajectories occur when one keeps only the leading-order term; second, by assuming that the A_{2L} meson is on a subsidiary trajectory, we derive an inequality concerning the $\pi\eta$ and $\pi\rho$ decay modes of A_{2L} and A_{2H} mesons; third, we study the decay widths for the first Regge recurrence of the ρ meson.

AN explicit model formula for the scattering amplitude, which makes use of the Euler Γ function, has been written down by Veneziano.[1] The formula has poles in all channels corresponding to zero-width particles lying on infinitely-rising linear Regge trajectories, and also has Regge asymptotic behavior in all channels when we average over the pole-containing part at high energy. Consequently the amplitude satisfies the superconvergence and finite-energy sum rules. The amplitude is what results when we write a simple product of poles in two channels multiplied by zeros to eliminate the double poles, and the fact that such a simple function is Regge-behaved suggests that the Regge-type power behavior stems from a sum over resonances in the direct channel—which is the converse of Schmid duality. An alternative to the sum of three double products of poles of the Veneziano representation is the single triple-product term of Virasoro.[2] The two representations are equivalent when a certain constraint is placed on the trajectories. In the present article we consider only the *double*-product model, and in this model we look at (i) the subsidiary trajectories which occur, (ii) decay of A_{2L} and A_{2H} mesons, and (iii) decay of the g meson (the spin-3 recurrence of the ρ meson).

(i) *Subsidiary trajectories in the model.* Keeping only the leading term in the model, the question of which subsidiary trajectories are present depends on whether or not we apply the constraint relation suggested by Veneziano.[1] It also depends on whether an equal-mass pair occurs in the initial or final state (as in an EE equal-to-equal mass process and in a UE unequal-to-equal mass process) or not (as in a UU unequal-to-unequal mass process). To investigate further we consider the s-channel process $\pi^a(p_a)+\eta(p_b)\rightarrow\pi^c(p_c)+\rho^d(p_d)$, which is UU in the s channel and UE in the t channel. We write the invariant amplitude in

$$T(stu)=\epsilon_{\alpha\beta\gamma\delta}\rho_\alpha p_{a\beta}p_{b\gamma}p_{c\delta}\epsilon^{acd}A(s,t,u) \tag{1}$$

as

$$A(stu)=\beta[B(1-\alpha_{A_2}(s),1-\alpha_\rho(t))+B(1-\alpha_\rho(t),1-\alpha_{A_2}(u))-B(1-\alpha_{A_2}(u),1-\alpha_{A_2}(s))], \tag{2}$$

where $B(x,y)$ is the Euler B function. We write the constant sum of the trajectory functions as

$$\alpha_{A_2}(s)+\alpha_\rho(t)+\alpha_{A_2}(u)=2+e, \tag{3}$$

so that $e=0$ is the Veneziano constraint. By inspection of Eq. (2) it is seen that the residues at t-channel poles are symmetric in s and u, and hence are even in $\cos\theta_t$ and correspond to odd-spin exchange only. For $e=0$ in Eq. (3), the poles at even positive $\alpha_0(t)$ do not occur; for $e\neq0$ they do. Consequently for $e=0$, only the even subsidiaries, spaced $K=2, 4, 6, \cdots$ units of angular

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