Fermion Avatars of the Sugawara Model*

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We argue that the expression for the energy-momentum tensor as a quadratic function of currents given in the Sugawara model must be reinterpreted as the limit of the corresponding expression, with the currents given a small separation minus its vacuum expectation value as the separation goes to zero, if the expression is to have any chance at all of making sense in a quantum field theory. We calculate this limit for a wide class of free Dirac theories with the usual currents in both two and four dimensions. In many (though not all) cases, we find that the limit yields the usual energy-momentum tensor. In calculating the limit in four dimensions, we have to multiply by the square of the separation to ensure that the limit exists; this is a reflection of the divergence of the Schwinger terms in the equal-time commutators of currents.

I. INTRODUCTION

THE Sugawara[1] model is the outstanding realization of the pure current dynamics envisioned by Dashen and Sharp.[2] It is a theory in which the energy-momentum tensor is given as a quadratic function of vector and axial-vector currents; thus, the equal-time commutation algebra of currents plays the same role in determining the dynamics of the theory as does the equal-time commutation algebra of fields in conventional canonical field theory.

In Sec. II, we review the defining equations of the Sugawara model and the construction, by Bardakci and Halpern,[3] of a Lagrangian field theory in which the currents obey these equations. We demonstrate that in classical field theory, the Bardakci-Halpern construction is reversible. In other words, not only can a Sugawara model be constructed from every Bardakci-Halpern field theory by appropriately defining the currents as functions of the fields, but a Bardakci-Halpern field theory can be constructed from every Sugawara model by defining the fields as certain path integrals of the currents.

In Sec. III, we argue that the situation may be considerably more complicated in quantum field theory. The fundamental reason for the difficulty is that the formal Sugawara expression for the energy-momentum tensor is guaranteed to be divergent, since the product of any local field with itself must have an infinite vacuum expectation value. This is not a new problem; it is the same as the one that arises in defining the currents themselves as quadratic functions of fields in ordinary quantum electrodynamics. We suggest that the solution used in quantum electrodynamics should be used here; the currents should be separated by a small amount, the vacuum expectation value should be subtracted, and the separation should then be sent to zero. In other words, the formal Sugawara prescription for constructing the energy-momentum tensor should be reinterpreted as a limit.

Of course, such a reinterpretation means that many supposed theorems about the Sugawara model, obtained by naive manipulation of equal-time commutators, are endangered. To demonstrate this, we "prove" four false "theorems" about the Sugawara model. These theorems would be true if the limiting procedure were not necessary. One of these theorems goes so far as to assert that the model cannot be Lorentz-invariant. Another asserts that fermion fields can never be realized in this model. We believe that this shows that we are not being fussbudgets in insisting on the necessity of the limiting procedure.

In Sec. IV, we consider a wide class of theories based upon free Dirac fields, in both two and four dimensions. We construct the currents in the usual way as normal-ordered bilinear forms in the fields and calculate explicitly the Sugawara limit. In many cases (though not all) we find that the limit is the conventional energy-momentum tensor for free Dirac fields. In many of the cases, this is true despite the fact that the space components of the currents do not commute at equal time, and naive arguments (of the sort discussed above) would lead one to believe that it would be impossible for the Sugawara expression to yield the correct energy-momentum tensor.

II. SUGAWARA MODEL AND THE BARDAKCI-HALPERN REALIZATION

A. Sugawara Model

In practical cases, the Sugawara model is of interest only when the underlying group is either $SU(2)$, $SU(3)$, chiral $SU(2)\times SU(2)$, or chiral $SU(3)\times SU(3)$. However, it is hardly more difficult to discuss the model for a general compact Lie group G, as we shall do here, than to discuss these particular cases. The fundamental dynamical variables are a set of currents $j_\mu{}^a$, where a

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[1] H. Sugawara, Phys. Rev. **170**, 1659 (1968); C. Sommerfield, *ibid.* **176**, 2019 (1968).
[2] R. F. Dashen and D. H. Sharp, Phys. Rev. **165**, 1857 (1968).
[3] K. Bardakci and M. B. Halpern, Phys. Rev. **172**, 1542 (1968).

[4] Here, and in all subsequent equations, Greek indices run from 0 to 3, Latin indices from the middle of the alphabet run from 1 to 3, repeated indices are summed over, and the signature of the metric tensor is $(+---)$.

runs from 1 to dim$G$. These obey the following equal-time commutation relations[4]:

$$[j_0^a(\mathbf{x}),j_0^b(\mathbf{y})]=ic^{abc}j_0^c(\mathbf{x})\delta^3(\mathbf{x}-\mathbf{y}), \quad (2.1a)$$

where c^{abc} are the structure constants of G;

$$[j_0^a(\mathbf{x}),j_i^b(\mathbf{y})]=ic^{abc}j_i^c(\mathbf{x})\delta^3(\mathbf{x}-\mathbf{y}) +ic\delta^{ab}\partial_i^x\delta^3(\mathbf{x}-\mathbf{y}), \quad (2.1b)$$

where c is some constant; and

$$[j_i^a(\mathbf{x}),j_j^b(\mathbf{y})]=0. \quad (2.1c)$$

The dynamics of the model is determined by the energy-momentum tensor

$$T_{\mu\nu}=(2c)^{-1}(\{j_\mu{}^a,j_\nu{}^a\}_+-g_{\mu\nu}j_\lambda{}^a j^{a\lambda}), \quad (2.2)$$

where the sum over repeated indices is implied.

Sugawara showed that these equations define a (formally) consistent relativistic field theory. The equations of motion are

$$\partial^\mu j_\mu{}^a=0, \quad (2.3a)$$

and

$$\partial_\mu j_\nu{}^a-\partial_\nu j_\mu{}^a=(c^{abc}/2c)\{j_\mu{}^b,j_\nu{}^c\}_+. \quad (2.3b)$$

B. Lagrangian Realization

Bardakci and Halpern were able to find a Lagrangian field theory possessing a set of currents that obeyed the equations of the Sugawara model. This field theory can be constructed in the following way: Let $\mathbf{T}^a$ be a set of Hermitian matrices forming a faithful representation of the Lie algebra of G; that is to say,

$$[\mathbf{T}^a,\mathbf{T}^b]=ic^{abc}\mathbf{T}^c. \quad (2.4)$$

If the structure constants have been defined in a standard (Cartan) basis, this implies that

$$\mathrm{Tr}\mathbf{T}^a\mathbf{T}^b=\lambda\delta^{ab}, \quad (2.5)$$

with λ a constant depending on the representation. Let φ^a be a set of scalar fields. Define

$$\mathbf{U}=\exp i\mathbf{T}^a\varphi^a. \quad (2.6)$$

The Lagrangian for the theory is then, according to Bardakci and Halpern,

$$\mathfrak{L}=(c/2\lambda)\,\mathrm{Tr}\partial_\mu\mathbf{U}^\dagger\partial^\mu\mathbf{U}. \quad (2.7)$$

(Although we shall not do so here, it is straightforward to show that this function of the φ's depends only on G and not on the particular representation chosen.)

This Lagrangian is obviously invariant under the transformation

$$\mathbf{U}\to\mathbf{V}\mathbf{U}, \quad (2.8)$$

where $\mathbf{V}$ is the representative of any fixed element of G. (Note that this transformation would assume an extremely complicated nonlinear form if written directly as a transformation of the φ's.) The associated infinitesimal transformations are

$$\delta^a\mathbf{U}=i\mathbf{T}^a\mathbf{U}, \quad (2.9)$$

and the corresponding conserved currents are

$$j_\mu{}^a=(c/i\lambda)\,\mathrm{Tr}\mathbf{T}^a\mathbf{U}\partial_\mu\mathbf{U}^\dagger. \quad (2.10)$$

Since these currents are obtained from the infinitesimal transformations (2.8), it follows that their fourth components must be the generators of the corresponding local transformations

$$[j_0^a(\mathbf{x}),\mathbf{U}(\mathbf{y})]=\mathbf{T}^a\mathbf{U}(\mathbf{x})\delta^3(\mathbf{x}-\mathbf{y}), \quad (2.11)$$

from which the commutators (2.1a) and (2.1b) follow as immediate consequences. The commutator (2.1c) is trivially valid, since the space components of the currents involve only the space derivatives of the φ's.

To show that the energy-momentum tensor has the proper form, we use a formula from general relativity: If the action integral I for any field theory is written in generally covariant form, then

$$T_{\mu\nu}=2\delta I/\delta g^{\mu\nu}. \quad (2.12)$$

Now, it follows from Eq. (2.5) that

$$\mathbf{U}\partial_\mu\mathbf{U}^\dagger=-(\partial_\mu\mathbf{U})\mathbf{U}^\dagger =\lambda^{-1}\mathbf{T}^a\,\mathrm{Tr}\mathbf{T}^a\mathbf{U}\partial_\mu\mathbf{U}^\dagger. \quad (2.13)$$

Thus,

$$\mathfrak{L}=(1/2c)j_\mu{}^a j^{\mu a}, \quad (2.14)$$

and the covariant form of the action integral is

$$I=(1/2c)\int d^4x(\sqrt{-g})g^{\mu\nu}j_\mu{}^a j_\nu{}^a, \quad (2.15)$$

from which Eq. (2.2) follows directly.

(It should be remarked that it is possible to define another set of currents by using right multiplication

$$\mathbf{U}\to\mathbf{U}\mathbf{V}, \quad (2.16)$$

instead of left multiplication in Eq. (2.8). We could have used these currents as well as our original ones to obtain a realization of the Sugawara model. However, we cannot use the full set of currents to obtain a realization of the Sugawara model for $G\times G$ as one might think, since the full set of currents has q-number Schwinger terms in some of the space-time commutators.)

C. Uniqueness of the Realization in the Classical Case

All of the arguments of the preceding sections are, of course, purely formal: They involve the products of field operators with themselves at the same space-time point, and we know that in a quantum field theory, such objects must be divergent. (In particular, they must have infinite vacuum expectation values unless the fields themselves vanish.) However, it is possible to consider classical Sugawara models in which the currents are c-number functions of space and time, and the commutators are replaced by Poisson brackets. In this case, the operations of the preceding sections are all legitimate; further, it is possible to show that the Bardakci-Halpern construction described above is unique in the sense that every classical Sugawara model can be obtained from such a field theory. In other words, the set of classical Sugawara models (defined by equations of motion and Poisson brackets for vector currents) is the same as the set of classical Bardakci-Halpern field theories (defined by equations of motion and Poisson brackets for scalar fields).

The proof is simple: Let x be any point, and let P be any smooth path going from the origin to x. Define

$$\mathbf{U}(P,x)=P\exp\frac{i\lambda}{C}\int_P \mathbf{T}^a j_\mu{}^a dx^\mu, \tag{2.17}$$

where the P symbol indicates that the terms in the exponential are to be ordered such that the matrices which are later on the path are on the left. If we make a small variation in P, keeping the end points fixed, it is easy to see that the corresponding change in $\mathbf{U}$ is proportional to

$$\partial_\mu j_\nu{}^a-\partial_\nu j_\mu{}^a-(c^{abc}/c)j_\mu{}^b j_\nu{}^c. \tag{2.18}$$

(The last term arises from the path ordering.) But Eq. (2.3b) says that this object vanishes. Thus $\mathbf{U}$ is independent of the path chosen and depends only on x. Once we have $\mathbf{U}(x)$, we can define $\varphi(x)$ by Eq. (2.6). Further, differentiation of Eq. (2.17) leads to Eq. (2.10). Thus, starting from the currents of the Sugawara model, one can define a set of scalar fields such that the currents are expressed as functions of the fields in the same manner as in the Lagrangian model.

The only question that remains to be settled is whether the equations of motion and Poisson brackets of these scalar fields are the same as those given in the Lagrangian model. However, since the fields are given by Eq. (2.17) as functions of the Sugawara currents, their equations of motion and Poisson brackets are certainly uniquely determined by those of the currents. Also, we have already shown that the Poisson brackets and equations of motion of the Lagrangian model are consistent with those of the Sugawara model. Thus, the answer to the question must be yes, and we have shown the desired uniqueness.

We emphasize that even were this result true in the quantum case (we shall show it is not), this would *not* necessarily mean that the Sugawara theory would be incapable of accommodating Fermi particles. We will give a simple argument to demonstrate this: Ultimately all physics experiments are done, by moving things with our hands and observing the results with our eyes. Since both of these processes are electromagnetic in character, all physics experiments can be reduced to measuring the response of the electromagnetic current to prior applications of that current. In other words, all we measure in practice, are the Green's functions for the electromagnetic current. Nevertheless, despite this handicap, we have no difficulty in detecting the existence and determining the properties of Fermi particles. Obviously, the vector nature of the current is not of crucial importance in this argument; therefore, if we had the complete set of Green's functions for the scalar fields that enter into the Bardakci-Halpern Lagrangian, we might well find, in calculating the results of certain thought experiments, that the theory contained Fermi particles. For the details of just what properties of the Green's functions are relevant, see any book on experimental physics.

In more formal language, we would say that there may exist fermion states in the Hilbert space of solutions to the Sugawara-Bardakci-Halpern theory, without there necessarily being fermion interpolating fields. One *can* decide whether or not fermion states exist, for example, from a complete knowledge of the spectral function of the vacuum expectation value of the time ordered product of two currents. Thus, for example, the threshold behavior of the spectral function of the two-point function for a 1^- current

$$\rho(p^2)(g^{\mu\nu}p^2-p^\mu p^\nu)=\sum_n \langle 0|j^\mu|n\rangle\langle n|j^\nu|0\rangle\delta(n-p), \tag{2.19}$$

depends on the spins of the intermediate states. Near the threshold for production of a particle-antiparticle pair (of mass m), $\rho(p^2)\approx(p^2-4m^2)^{1/2}$ or $(p^2-4m^2)^{3/2}$, if the particle is a fermion or a boson.

III. SINGULARITIES OF THE SUGAWARA ENERGY-MOMENTUM TENSOR

A. Products of Currents

We have stated that the products that occur in the fundamental Sugawara equation (2.2) are necessarily divergent in a quantum field theory. How, then, are they to be interpreted? A clue is offered by ordinary quantum electrodynamics, where currents are defined as bilinear forms in underlying fields. There, the answer to the problem is to separate the two fields by a small amount, subtract the vacuum expectation value, and allow the separation to go to zero. This suggests that the same stratagem be used to define products of currents themselves; that is to say, that an expression like

$$j_\mu(x)j_\nu(x)$$

should be interpreted as a limit

$$\lim_{\epsilon\to 0} j_\mu(x+\tfrac{1}{2}\epsilon)j_\nu(x-\tfrac{1}{2}\epsilon)-\langle j_\mu(x+\tfrac{1}{2}\epsilon)j_\nu(x-\tfrac{1}{2}\epsilon)\rangle_0. \tag{3.1}$$

However, we know in quantum electrodynamics, that even this procedure does not suffice to make the electric current well defined, except for the trivial case when the electron charge vanishes. It is also necessary to multiply the expression (3.1) by an ϵ-dependent constant, which goes to zero as ϵ vanishes, at least in perturbation theory. Formally, this arises because the true electromagnetic current is a bilinear function in the unrenormalized fields, multiplied by a finite constant. However, only the separated product of *renormalized* fields is well defined in perturbation theory. Thus, as we allow the separation to go to zero, we must simultaneously "undo the renormalization."[5] A similar procedure might also be necessary in the Sugawara model. If it is, we will refer to the phenomenon as "renormalization of the Sugawara constant c," in analogy with the electrodynamic case. As a guess, we might expect c renormalization to be necessary in theories in which the Schwinger terms in the current commutators are divergent.

With these interpretations of products of currents, we are in a position to determine whether, in any given field theory the Sugawara model expression gives the correct energy-momentum tensor. All we have to do is to calculate the matrix elements of the limit and see if it reproduces the matrix elements of the energy-momentum tensor. For a solvable model, this is a direct calculation. For a nonsolvable but renormalizable theory, the calculation can be done order-by-order in perturbation theory; this is an onerous, but well-defined task. For nonrenormalizable theories, the best we can do is use formal arguments based on the interaction picture; this is notoriously unreliable, but it may be suggestive. In this paper, we will restrict ourselves to solvable models; nevertheless, despite the triviality of these theories, we will obtain some surprises.

An important question to be answered in any model is whether the limiting procedure gives the proper energy-momentum tensor when only spacelike separations are taken, or whether it is also necessary to average over timelike separations. If the latter is the case, then many of the consequences of the Sugawara model, obtained by naive manipulation of the equal time commutators, are endangered; indeed, the whole concept of a dynamics determined by current commutation relations disappears.

B. Four False Theorems

We emphasize that taking this care with limits is essential if we are to avoid a hopeless muddle of contradictions. To demonstrate this, we will give four arguments which would be valid were it not necessary to define the energy-momentum tensor as a limit. However, we will show in Sec. IV, by constructing counterexamples, that all four arguments are false. In each case, the false conclusion is italicized.

(1) It is possible to show, from Lorentz invariance and positive definiteness of the inner product, that the equal-time commutator of T^{00} with T^{0i} and of T^{0i} with T^{jk} must contain terms proportional to *three* derivatives of a delta function.[6] However, it follows directly from the defining equations of the Sugawara model, Eqs. (2.1) and (2.2), that no such terms can occur. *Therefore, the Sugawara model must violate Lorentz invariance or positivity of the metric.*

(2) Let us suppose we can find a theory of the Sugawara form involving a spinor field ψ. Let us consider M^{0i}, the generator of Lorentz transformations in the ith direction. At time $t=0$,

$$M^{0i}=\int x^i T^{00} d^3x. \tag{3.2}$$

On the other hand, from the known transformation properties of spinor fields

$$[M^{0i},\psi(0)]=\tfrac{1}{2}i\gamma^i\gamma^0\psi(0). \tag{3.3}$$

This is clearly consistent with the Sugawara form for the energy-momentum tensor only if the commutators of the currents and ψ involve gradients of δ functions. *Thus, the Sugawara model can only accommodate Fermi fields if the currents themselves involve gradients of those fields.* In particular, when the currents are Noether currents arising from an internal symmetry transformation, the Sugawara theory cannot accommodate fermion fields.[7]

(3) Let us explore the possibility of relaxing the connection between the Schwinger terms and the coefficient c appearing in the expression for the energy-momentum tensor. To be specific, let us replace Eq. (2.1b) by

$$[j_0{}^a(\mathbf{x}),j_i{}^b(\mathbf{y})]=ic^{abc}j_i{}^c(\mathbf{x})\delta^3(\mathbf{x}-\mathbf{y}) +ic^b\delta^{ab}\partial_i{}^x\delta^3(\mathbf{x}-\mathbf{y}), \tag{3.4}$$

(no sum on b) while retaining Eq. (2.2). Then direct calculation shows that Eq. (2.3a) is replaced by

$$\partial^0 j_0{}^a+(c^a/c)\partial^i j_i{}^a=0. \tag{3.5}$$

For this to be Lorentz-invariant, c^a *must equal* c, *for every* a.

(4) In the same spirit, let us attempt to weaken the space-space commutators by replacing Eq. (2.1c) by

$$[j_i{}^a(\mathbf{x}),j_j{}^b(\mathbf{y})]=C_{ij}{}^{ab}(\mathbf{x})\delta^3(\mathbf{x}-\mathbf{y}), \tag{3.6}$$

where

$$C_{ij}{}^{ab}=-C_{ji}{}^{ba}. \tag{3.7}$$

Again, direct calculation shows that Eq. (2.3b) is replaced by

$$\partial_0 j_i{}^a-\partial_i j_0{}^a-(2c)^{-1}c^{abc}\{j_0{}^b,j_i{}^c\}_+=i\{C_{ij}{}^{ab},j^{bj}\}_+, \tag{3.8a}$$

and

$$\partial_i j_j{}^a-\partial_j j_i{}^a-(2c)^{-1}c^{abc}\{j_i{}^b,j_j{}^c\}_+=i\{j_0{}^b,C_{ij}{}^{ba}\}_+. \tag{3.8b}$$

[5] The same thing happens in the Thirring model; see K. Johnson, Nuovo Cimento **20**, 773 (1961).

[6] D. Boulware and S. Deser, J. Math. Phys. **8**, 1468 (1967).

[7] We emphasize that this "result" remains true in a theory of one space and one time dimension. Thus we are in sharp disagreement with C. Callan, R. Dashen, and D. Sharp, Phys. Rev. **165**, 1883 (1968), as well as C. Sommerfield, *ibid.* **176**, 2019 (1968), who claim that the Thirring model is equivalent to a Sugawara theory without any concern about the singular nature of the products.

The left-hand sides of these equations form the components of an antisymmetric tensor; therefore, the right-hand sides must also.[8] In particular, this implies that *the space-space commutators cannot be of the form suggested by the quark model (except in two dimensions).*

IV. FREE-FERMION MODELS

A. Definitions: Fundamental Formulas

The models we shall investigate will all be based upon a set of free Dirac fields of identical mass in either two or four dimensions. We will take the usual definition of the currents

$$j_\mu{}^a = :\bar{\psi}\gamma_\mu \mathbf{T}^a\psi:, \tag{4.1}$$

where the colon indicates normal ordering, and the T's are a set of Hermitian matrices obeying

$$[\mathbf{T}^a,\mathbf{T}^b]=ic^{abc}\mathbf{T}^c. \tag{4.2}$$

The equal-time commutators of these currents may readily be calculated. The time-time commutators are of the standard form, Eq. (2.1a). The space-time commutators are of the form given in Eq. (3.4), with

$$c^a=\pi^{-1}\,\mathrm{Tr}(\mathbf{T}^a)^2 \tag{4.3}$$

in two dimensions, and with c^a divergent in four dimensions. The space-space commutators are similar to those of the quark model; we will not need their explicit form in our work. The symmetric energy-momentum tensor for this system is given by

$$\Theta_{\mu\nu}=\tfrac{1}{4}i:(\bar{\psi}\gamma_\mu\partial_\nu\psi+\bar{\psi}\gamma_\nu\partial_\mu\psi-\partial_\nu\bar{\psi}\gamma_\mu\psi-\partial_\mu\bar{\psi}\gamma_\nu\psi):. \tag{4.4}$$

We will calculate the Sugawara expression for the energy-momentum tensor and see if it agrees with (4.4). Following the limiting procedure described in Sec. III, we define

$$T_{\mu\nu}=\lim_{\epsilon\to 0}\left\{\frac{1}{2c}[j_\mu{}^a(x+\tfrac{1}{2}\epsilon)j_\nu{}^a(x-\tfrac{1}{2}\epsilon)+j_\nu{}^a(x+\tfrac{1}{2}\epsilon)\times j_\mu{}^a(x-\tfrac{1}{2}\epsilon)-g_{\mu\nu}j^{\lambda a}(x+\tfrac{1}{2}\epsilon)j_\lambda{}^a(x-\tfrac{1}{2}\epsilon)] -\text{vacuum expectation value}\right\}. \tag{4.5}$$

As discussed in Sec. III in two dimensions, we expect c to be a constant. In four dimensions, we expect it to depend on ϵ and to diverge as ϵ goes to zero. To study the limit, it is convenient to turn (4.5) into a normal-ordered polynomial using Wick's theorem. Since the original ordering is ordinary ordering, not time ordering, the relevant contraction is not the Feynman function but

$$S^{(+)}(x-y)=\psi(x)\bar{\psi}(y)-:\psi(x)\bar{\psi}(y):=\langle\psi(x)\bar{\psi}(y)\rangle_0. \tag{4.6}$$

Applying Wick's theorem, we find

$$T_{\mu\nu}=\lim_{\epsilon\to 0}\frac{1}{2c}[Q_{\mu\nu}(x,\epsilon)+j_{\mu\nu}(x,\epsilon)+j_{\nu\mu}(x,\epsilon)+j_{\mu\nu}(x,-\epsilon)+j_{\nu\mu}(x,-\epsilon)-g_{\mu\nu}(j_\lambda{}^\lambda(x,\epsilon)+j_\lambda{}^\lambda(x,-\epsilon))], \tag{4.7}$$

where

$$Q_{\mu\nu}(x,\epsilon)=:[j_\mu{}^a(x+\tfrac{1}{2}\epsilon)j_\nu{}^a(x-\tfrac{1}{2}\epsilon)+j_\nu{}^a(x-\tfrac{1}{2}\epsilon)\times j_\mu{}^a(x+\tfrac{1}{2}\epsilon)-g_{\mu\nu}j_\lambda{}^a(x-\tfrac{1}{2}\epsilon)j^{a\lambda}(x+\tfrac{1}{2}\epsilon)]: \tag{4.8}$$

is a quartic, normal-ordered polynomial, and

$$j_{\mu\nu}(x,\epsilon)=:\bar{\psi}(x+\tfrac{1}{2}\epsilon)\gamma_\mu\mathbf{T}^aS^{(+)}(\epsilon)\gamma_\nu\mathbf{T}^a\psi(x-\tfrac{1}{2}\epsilon):, \tag{4.9}$$

is a quadratic polynomial.

Note that since the quartic term is fully normal ordered, it can have no singularities as ϵ goes to zero; therefore, we may replace it in the limit by its value at zero. Also, it is manifestly a tensor regardless of the direction of ϵ.

We will now apply these formulas to a sequence of special models.

B. Single Massless Fermi Field in Two Dimensions

In this case, there is only one current. We choose T to be one; the Schwinger constant by Eq. (4.3) is π^{-1}. First we will show that the quartic term Eq. (4.8) vanishes. By the remarks of the preceding paragraph, it suffices to show that Q_{00} vanishes. Choosing a basis for the Dirac γ's in which

$$\gamma_0=\sigma_z \tag{4.10a}$$

and

$$\gamma_1=i\sigma_y, \tag{4.10b}$$

and indicating by subscripts the two components of the Dirac field, we find

$$Q_{00}=:(j_0)^2+(j_1)^2:=:[(\bar{\psi}_1\psi_1-\bar{\psi}_2\psi_2)^2+(\bar{\psi}_1\psi_2-\bar{\psi}_2\psi_1)^2]:. \tag{4.11}$$

Because of the antisymmetry of the normal-ordered product, any term with two ψ_1's, two ψ_2's, etc., automatically vanishes; thus,

$$Q_{00}=-2:[\bar{\psi}_1\psi_1\bar{\psi}_2\psi_2+\bar{\psi}_1\psi_2\bar{\psi}_2\psi_1]:. \tag{4.12}$$

But these two terms differ from each other by an odd permutation; thus, Q_{00} and hence all components of Q vanishes.

We now turn to the quadratic term. To evaluate this we need the explicit expression for the contraction function for a massless two-dimensional Dirac field:

$$S^{(+)}(\epsilon)=-(i/2\pi)(\gamma\cdot\epsilon/\epsilon^2). \tag{4.13}$$

[8] One can show that this can happen only when

$$(1/2c)\{\Lambda^{0l}[C_{ab}{}^{ij}],j_b{}^j\}_+=0,$$

where Λ^{0l} is defined for any operator $\mathcal{O}$ by

$$[M^{0l},\mathcal{O}(0)]=i\Lambda^{0l}[\mathcal{O}].$$

Now the expression (4.7) for the quadratic term is a sum of terms of the form

$$\mathrm{Tr}[F(\epsilon)S^{(+)}(\epsilon)+F(-\epsilon)S^{(+)}(-\epsilon)], \qquad (4.14)$$

with F a spinor function. If we expand F in power series

$$F(\epsilon)=F(0)+\epsilon^\mu\partial_\mu F+\cdots, \qquad (4.15)$$

and average ϵ over an orthonormal pair of vectors, one spacelike and one timelike, we find

$$\lim_{\epsilon\to 0}\mathrm{Tr}[F(\epsilon)S^{(+)}(\epsilon)+F(-\epsilon)S^{(+)}(-\epsilon)] = \frac{-i}{2\pi}\mathrm{Tr}\gamma\cdot\partial F. \qquad (4.16)$$

Applying this to Eqs. (4.7) and (4.9), we find that

$$T_{\mu\nu}=(-i/8\pi c):[\partial_\lambda\bar\psi\gamma_\mu\gamma^\lambda\gamma_\nu\psi-\bar\psi\gamma_\mu\gamma^\lambda\gamma_\nu\partial_\lambda\psi +\partial_\lambda\bar\psi\gamma_\nu\gamma^\lambda\gamma_\mu\psi-\bar\psi\gamma_\nu\gamma^\lambda\gamma_\mu\partial_\lambda\psi-g_{\mu\nu}\partial_\lambda\bar\psi\gamma_\rho\gamma^\lambda\gamma^\rho\psi +g_{\mu\nu}\bar\psi\gamma_\rho\gamma^\lambda\gamma^\rho\partial_\lambda\psi]:. \qquad (4.17)$$

Permuting the γ's and using the Dirac equation

$$\partial_\lambda\gamma^\lambda\psi=0, \qquad (4.18)$$

we find

$$T_{\mu\nu}=i/4\pi c:[\bar\psi\gamma_\mu\partial_\nu\psi+\bar\psi\partial_\nu\gamma_\mu\psi-\partial_\mu\bar\psi\gamma_\nu\psi-\partial_\nu\bar\psi\gamma_\mu\psi]: =(1/\pi c)\Theta_{\mu\nu}. \qquad (4.19)$$

This is precisely the desired result if we choose c to be π^{-1}. Notice that this model is a counterexample to the first and second false theorems "proved" in Sec. III.

C. Many Massless Fermi Fields in Two Dimensions

For a single field, the vanishing of the quartic term was a consequence of its symmetry in the Dirac indices. Clearly, for the same to happen for many fields, the quartic term must also be symmetric in internal indices. That is to say

$$(\mathbf{T}^a)_{\alpha\beta}(\mathbf{T}^a)_{\gamma\delta}=(\mathbf{T}^a)_{\alpha\delta}(\mathbf{T}^a)_{\gamma\beta}, \qquad (4.20)$$

where the Greek subscripts indicate matrix elements, and as always the sum over a is implied.

We will now show that this condition is not fulfilled for $SU(n)$ but is fulfilled for $U(n)$ if the normalization of the currents is chosen in a certain (unconventional) way. Let us choose as the generators of $SU(n)$ a set of n^2-1 traceless Hermitian matrices normalized such that

$$\mathrm{Tr}\mathbf{T}^a\mathbf{T}^b=\delta^{ab}. \qquad (4.21)$$

If we add to this set an extra matrix

$$\mathbf{T}^0=(1/\sqrt{n})\mathbf{I}, \qquad (4.22)$$

the full set of matrices now forms a basis for the space of all $n\times n$ matrices, with the inner product defined by the trace; that is to say for any matrices M and N

$$\mathrm{Tr}\mathbf{M}^\dagger\mathbf{T}^a\,\mathrm{Tr}\mathbf{T}^a N=\mathrm{Tr}\mathbf{M}^\dagger\mathbf{N}, \qquad (4.23)$$

where the sum now runs from zero to n^2-1. Differentiating with respect to M and N, we find

$$(\mathbf{T}^a)_{\alpha\beta}(\mathbf{T}^a)_{\gamma\delta}=\delta_{\alpha\delta}\delta_{\beta\gamma}; \qquad (4.24)$$

or, if we sum only over the traceless matrices

$$(\mathbf{T}^a)_{\alpha\beta}(\mathbf{T}^a)_{\gamma\delta}=\delta_{\alpha\delta}\delta_{\beta\gamma}-(1/n)\delta_{\alpha\beta}\delta_{\gamma\delta}. \qquad (4.25)$$

This does not have the desired symmetry properties. However, if we redefine T^0 with the unconventional normalization

$$\mathbf{T}^0=[(n+1)/n]^{1/2}\mathbf{I}, \qquad (4.26)$$

then the sum over all n^2 matrices is given by

$$(\mathbf{T}^a)_{\alpha\beta}(\mathbf{T}^a)_{\gamma\delta}=\delta_{\alpha\delta}\delta_{\beta\gamma}+\delta_{\alpha\beta}\delta_{\gamma\delta}, \qquad (4.27)$$

which does have the desired property. These matrices form a set of generators for $U(n)$, with Schwinger-term coefficients given by

$$c^a=1/\pi,\quad a\neq 0 \qquad (4.28a)$$

and

$$c^0=(n+1)/\pi. \qquad (4.28b)$$

The quadratic term has precisely the same structure as in the one-field model, except that between every pair of fields there occurs the matrix

$$\mathbf{T}^a\mathbf{T}^a=(n+1)\mathbf{I}. \qquad (4.29)$$

Thus, we will obtain the desired result if we choose

$$c=(n+1)/\pi. \qquad (4.30)$$

Note that these models are counterexamples to the third false theorem "proved" in Sec. III.

D. Massless Fermi Fields in Four Dimensions

In four dimensions, because the coefficient of the Schwinger term is divergent, we expect the Sugawara expression for the energy-momentum tensor to also diverge unless we renormalize c, that is to say, make c dependent on ϵ. This guess is verified by the form of the contraction function

$$S^{(+)}(\epsilon)\propto i\gamma\cdot\epsilon/\epsilon^4. \qquad (4.31)$$

This would clearly make the quadratic term, Eq. (4.9), diverge, unless we choose

$$c(\epsilon)\propto 1/\epsilon^2. \qquad (4.32)$$

If we do make such a choice and also average over an orthonormal quadruplet of vectors, the calculations are all identical to the previous ones, and we can obviously make the quadratic term yield the energy-momentum tensor provided

$$\mathbf{T}^a\mathbf{T}^a\propto\mathbf{I}. \qquad (4.33)$$

This condition can be fulfilled for a wide choice of $\mathbf{T}$'s. Schur's lemma guarantees that it holds whenever the fields form an irreducible representation of the group of the currents. It also holds in many cases where the representation is reducible; for example, in the cases discussed immediately above.

In sharp contrast to the situation in two dimensions, the quartic terms automatically vanish as ϵ goes to zero, since they are finite expressions multiplied by ϵ^2. Thus, there is no analog to Eq. (4.20); Eq. (4.33) is both necessary and sufficient for the limit of the Sugawara expression to reproduce the energy-momentum tensor.

These models are counterexamples to the fourth false theorem "proved" in Sec. III.

E. Massive Fermi Fields

We will now investigate how our conclusions are altered if the Fermi fields have nonzero mass. As before, we begin with a single Dirac field in two dimensions. For small ϵ, the contraction function is given by

$$S^{(+)}(\epsilon)\approx-(i/2\pi)(\gamma\cdot\epsilon/\epsilon^2)-(m/4\pi)(\ln\tfrac{1}{2}\epsilon^2m^2-\gamma_E)\,, \quad (4.34)$$

where γ_E is Euler's constant. At first glance, it might seem that the logarithmic terms in this expression would spoil the limit. However, it follows from Eqs. (4.7) and (4.9) that their coefficient in the expression for $T^{\mu\nu}$ is

$$:\bar{\psi}\gamma_\mu\gamma_\nu\psi:+:\bar{\psi}\gamma_\nu\gamma_\mu\psi:-g_{\mu\nu}:\bar{\psi}\gamma_\rho\gamma^\rho\psi:=0\,. \quad (4.35)$$

Of course, the quartic terms vanish just as before. Thus Eq. (4.17) for the Sugawara energy-momentum tensor is still valid. The Dirac equation for massless fields Eq. (4.19) is replaced by

$$(i\gamma\cdot\partial-m)\psi=0\,, \quad (4.36)$$

where m is the common mass of the fields. After some straightforward Dirac algebra, we find that

$$\Theta^{\mu\nu}=T^{\mu\nu}+\tfrac{1}{2}mg^{\mu\nu}:\bar{\psi}\psi:\,, \quad (4.37)$$

where, as always, $T^{\mu\nu}$ is the Sugawara expression for the energy-momentum tensor, and $\Theta^{\mu\nu}$ is the actual energy-momentum tensor.

Equation (4.37) is reminiscent of a suggestion of Bardakci, Frishman, and Halpern,[9] that, in the presence of symmetry breaking, the Sugawara energy-momentum tensor should be modified by adding a term proportional to the product of a scalar field and the metric tensor. Of course, in our case, since the current is vector, giving the field a mass does not break the symmetry.

It is easy to see that all of these conclusions hold without alteration for the case of many Dirac fields in two dimensions. If the masses of the fields are not identical, Eq. (4.37) is replaced by

$$\Theta^{\mu\nu}=T^{\mu\nu}+\tfrac{1}{2}g^{\mu\nu}:\bar{\psi}M\psi:\,, \quad (4.38)$$

where M is the mass matrix.

[9] K. Bardakci, Y. Frishman, and M. Halpern, Phys. Rev. **170**, 1353 (1968).

In four dimensions, the contraction function for small ϵ is given by

$$S^{(+)}(\epsilon)=[1/(2\pi)^3][(i\gamma\cdot\epsilon/\epsilon^4)-(m/2\epsilon^2)+O(\ln\epsilon)]\,. \quad (4.39)$$

Just as in two dimensions, we will restrict ourselves, for simplicity, to a single field; it is easy to check that everything we do is also applicable to the case of many fields. We choose $c(\epsilon)$ to be proportional to ϵ^{-2} and scaled such that

$$S^{(+)}(\epsilon)/2c(\epsilon)=-\tfrac{1}{2}(i\gamma\cdot\epsilon/\epsilon^2)+\tfrac{1}{4}m+O(\epsilon^2\ln\epsilon)\,. \quad (4.40)$$

Then, using the same techniques as before, and averaging over an orthonormal quadruplet, we find

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{1}{2c}&[j_{\mu\nu}(x,\epsilon)+j_{\mu\nu}(x,-\epsilon)]\\ &=:(\tfrac{1}{2}m\bar{\psi}\gamma_\mu\gamma_\nu\psi+\tfrac{1}{8}i\bar{\psi}\gamma_\mu\gamma^\lambda\gamma_\nu\partial_\lambda\psi-\tfrac{1}{8}i\partial_\lambda\bar{\psi}\gamma_\mu\gamma^\lambda\gamma_\nu\psi):\\ &=:(\tfrac{1}{4}m\bar{\psi}\gamma_\mu\gamma_\nu\psi+\tfrac{1}{4}i\bar{\psi}\gamma_\mu\partial_\nu\psi-\tfrac{1}{4}i\partial_\mu\bar{\psi}\gamma_\nu\psi):\,, \quad (4.41)\end{aligned}$$

where we have used the Dirac equation at the last stage. Thus,

$$\Theta^{\mu\nu}=T^{\mu\nu}+mg^{\mu\nu}:\bar{\psi}\psi:. \quad (4.42)$$

This is a precise parallel to the two-dimensional equation (4.37). If there are many fields and their masses are different, then, for appropriate choice of $c(\epsilon)$

$$\Theta^{\mu\nu}=T^{\mu\nu}+g^{\mu\nu}:\bar{\psi}\mathbf{M}\psi:. \quad (4.43)$$

F. Axial-Vector Currents

In the preceding work, we have restricted ourselves to vector currents; however, in the models we have considered, it is also possible to define axial-vector currents:

$$j_\mu{}^5=:\bar{\psi}\mathbf{T}^a\gamma_\mu\gamma^5\psi:\,, \quad (4.44)$$

where γ^5 is normalized such that its square is one. If we study the expression $T_{\mu\nu}{}^5$ defined just as $T_{\mu\nu}$ was in Eq. (4.5), but with axial-vector currents replacing vector currents, it is easy to see that, in every term that survives the limiting procedure, the two γ^5's always occur with two γ_μ's between them, and may therefore be neglected. Hence, all of the results we have stated for $T_{\mu\nu}$ are equally true for $T_{\mu\nu}{}^5$, or for the average of these objects, $\frac{1}{2}(T_{\mu\nu}+T_{\mu\nu}{}^5)$.

V. CONCLUDING REMARKS

Since the operator products which occur in the defining equations of the Sugawara model can be defined only as a limit, in order to specify the theory completely, it is necessary to specify the nature of this limit—that is to say, to determine the singularity structure of the operator products for small separations. In the models we have investigated, we have been able to determine this singularity structure only because the models are based on a trivial exactly solvable canonical field theory. We have no idea what are reasonable hypotheses to make about the singularity structure in less trivial cases. We believe that this important unsolved problem must be faced if the program of defining dynamics exclusively in terms of currents is to be brought to a successful conclusion.

(Of course, exactly the same remarks could be made about conventional Lagrangian field theory; however, here, at least, perturbation theory can give us important clues.)

In particular, because we have found it necessary in all of our models to average over both timelike and spacelike directions, many results obtained by naive manipulation of the Sugawara equations are false in our models. The four false theorems of Sec. III are examples of this.

We find this extremely disquieting: The Sugawara model, which upon naive inspection appeared to be a set of logically connected propositions, each one necessarily following from its predecessors, has dissolved before our eyes into a collection of disconnected assertions, any one of which may or may not be true independently of the validity of the others, in any given theory.

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Interactions of Hadrons with Nuclei at High Energy. I*

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It is shown that under certain assumptions the Glauber multiple-scattering series can be summed explicitly to give a closed expression for the amplitude describing the interaction of a high-energy particle with a nucleus. The advantages of this description are discussed, and it is shown that available data are well described by the theory. Application of the theory to inelastic processes is given, and it is shown how quantities like σ_{A_1N} and $\sigma_{\Sigma N}$, total scattering cross sections which are not otherwise measurable, can be obtained.

I. INTRODUCTION

Recent high-precision work on proton-nucleus elastic scattering[1] has led to renewed interest in the problem of particle-nucleus interactions, both in the case of elastic[2] and inelastic[3] scattering. In this paper we wish to carry this work further along by making two points. First, we shall try to show why such interactions are potentially of great interest to particle physicists, and, second, we shall try to show that it is possible to formulate a theory of such interactions in a conceptually simple way, and that this theory agrees well with what data are available at the present time.

There are strong reasons why one could wish to understand particle-nuclear interactions. If one has a good theory for such processes, then one can use nuclear targets in high-energy experiments to obtain data which are not easily attainable by direct measurement, and in some cases data which are not directly attainable at all. In particular, there are three effects connected with nuclear targets which are potentially of great interest to particle physicists.

The first of these is the effect of *nuclear coherence*. If the cross section for a particular process to go on a hydrogen target is σ_H, then the cross section to go on a nucleus of atomic number A will be $\sigma_A = \sigma_H A^n$, where $n > 0$. Thus if we wished to examine rare production modes, which we frequently would like to do, we could look at production on a nucleus, and then use our theory to extract the production from a single nucleon.

The second useful effect is *nuclear rescattering*. One would frequently like to examine the interactions of very short-lived particles with nucleons. Unfortunately, most strongly interacting resonances have decay paths of the order of tens of Fermis, and thus cannot be made into beams for scattering experiments. However, if such a particle is produced inside of a nucleus, then it will have the chance to strike nucleons on its way out of the nucleus, and, once again, a good theory of particle-nucleus interactions would enable us to extract the

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