Equivalence Principle for Massive Bodies Including Rotational Energy and Radiation Pressure

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The Einstein gravitational theory and the Brans-Dicke gravitational theory are investigated for the gravitational-to-inertial mass ratio (m_g/m_i) they yield for massive systems. We examine models for stars in equilibrium but undergoing rotation or containing electromagnetic radiation. General conditions on the space-time metric of many mass sources are obtained which lead to $m_g/m_i=1$ for the massive systems. It is found that $m_g/m_i\neq 1$ for a nonrotating star in the Brans-Dicke theory. The violation is of order 10^{-6} for the Sun. In the Brans-Dicke theory there are also contributions to $m_g/m_i\neq 1$ due to the radiation content of a star. This violation is of order (radiation energy)/(stellar mass), yielding a correction of about 10^{-8} for the Sun.

I. INTRODUCTION

In previous papers,[1,2] hereafter referred to as I and II, an investigation of the gravitational-to-inertial mass ratio m_g/m_i of massive systems in gravitational theories was begun. The essential result of II was that a two-body orbiting system or a gas sphere of mass elements has a m_g/m_i ratio of one for Einstein's theory but not in the scalar-tensor theory of Brans and Dicke (BD).[3] In I and also in another paper[4] we discussed possible experimental tests of m_g/m_i for astronomical objects.

In this paper we add the following results to this investigation:

(a) The m_g/m_i ratio for a rotating gas sphere is obtained for a general gravitational theory which is expressible in a space-time geometrical form. The rotation does not change m_g/m_i in either the Einstein or BD theory.

(b) The m_g/m_i ratio for a gas sphere maintained in equilibrium partially by electromagnetic radiation pressure is obtained. The BD gravitational theory predicts that m_g/m_i differs from 1 by a term of order (total radiation energy/total mass) of the gas sphere, whereas Einstein's theory gives no violation of $m_g/m_i=1$ due to radiation pressure. This violation of the equivalence principle for the Sun in the BD theory is of order 10^{-8} if a solar temperature of 10^7 °K is assumed.

(c) The active gravitational mass[5] for massive bodies is determined for the general gravitational theory. Einstein's theory predicts an active gravitational mass which agrees with the Newtonian energy of the system ($M_a=m+T+V$); the BD theory yields a gravitational mass which is not the Newtonian energy.

(d) In the Appendix, the metric component g_{00} is calculated in the BD theory for the case of many mass sources, complete to second order in the mass strengths. g_{00} is required to this accuracy to make the conclusions above concerning the m_g/m_i ratio of massive bodies in the BD theory.

II. ROTATING GAS SPHERE

As in II, we write the most general space-time metric produced by many mass elements m_i:

$$g_{00}=1-2\sum_i\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}+2\beta\left(\sum_i\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\right)^2+2\alpha'\sum_{ij}\frac{m_im_j}{|\mathbf{r}_i-\mathbf{r}_j|}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_i|}+\frac{1}{|\mathbf{r}-\mathbf{r}_j|}\right)+\chi\sum_i\frac{m_i(\mathbf{r}-\mathbf{r}_i)\cdot\mathbf{a}_i}{|\mathbf{r}-\mathbf{r}_i|}-4\alpha''\sum_i\frac{m_iv_i^2}{|\mathbf{r}-\mathbf{r}_i|}+\alpha'''\sum_i\frac{m_i[(\mathbf{r}-\mathbf{r}_i)\cdot\mathbf{v}_i]^2}{|\mathbf{r}-\mathbf{r}_i|^3}+\cdots,\quad(1)$$

$$g_{0k}=4\Delta\sum_i\frac{m_iv_i^k}{|\mathbf{r}-\mathbf{r}_i|}+4\Delta'\sum_i\frac{m_i(\mathbf{r}-\mathbf{r}_i)\cdot\mathbf{v}_i(r-r_i)^k}{|\mathbf{r}-\mathbf{r}_i|^3}+\cdots,\quad(2)$$

and

$$g_{kk'}=-\left(1+2\gamma\sum_i\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\right)\delta_{kk'}+\cdots.\quad(3)$$

The coefficients γ, β, Δ, Δ', α', α'', α''', and χ are dimensionless, to be determined by a gravitational theory's field equations. In Einstein's theory all are equal to 1 except $\Delta'=0$.

Now consider a massive body consisting of many mass elements plus a distant mass m_E. The acceleration of

[1] K. Nordtvedt, Jr., Phys. Rev. **169**, 1014 (1968).
[2] K. Nordtvedt, Jr., Phys. Rev. **169**, 1017 (1968).
[3] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).
[4] K. Nordtvedt, Jr., Phys. Rev. **170**, 1186 (1968).
[5] Active gravitational mass of a system is defined as the strength of the mass parameter which gives the Newtonian gravitational potential about the system.

each mass element toward m_E is given by [adapting Eq. (41) of paper II]

$$\mathbf{a}_i=\mathbf{g}_E\Big\{1+(4\Delta-2\beta-2\gamma-\tfrac{1}{2}\chi)\sum_j\frac{m_j}{r_{ij}}+\gamma v_i^2+[4\Delta'+\tfrac{1}{2}(\chi-\alpha')+\gamma+\beta]\sum_j\frac{m_j}{r_{ij}^3}r_{ij\parallel}^2-(2\gamma+2)v_{i\parallel}^2\Big\}+[4\Delta'+\tfrac{1}{2}(\chi-\alpha')+\gamma+\beta]g_E\sum_j\frac{m_j}{r_{ij}^3}r_{ij\parallel}\mathbf{r}_{ij\perp}-(2\gamma+2)g_E v_{i\parallel}\mathbf{v}_{i\perp}. \quad (4)$$

$\mathbf{g}_E$ is the Newtonian acceleration toward the external body m_E. $\parallel$ and $\perp$ refer to the direction $\mathbf{g}_E$. Summing (4) over all the mass elements of the massive body, each weighted by their mass, gives the acceleration of the whole body

$$\mathbf{a}=\mathbf{g}_E\Big\{1+(4\Delta-2\beta-2\gamma-\tfrac{1}{2}\chi)\sum_{ij}\frac{m_im_j}{Mr_{ij}}+\frac{\gamma}{M}\sum_i m_iv_i^2+[4\Delta'+\tfrac{1}{2}(\chi-\alpha')+\gamma+\beta]\sum_{ij}\frac{m_im_j}{Mr_{ij}^3}r_{ij\parallel}^2-\frac{(2\gamma+2)}{M}\sum_i m_iv_{i\parallel}^2\Big\}-\frac{(2\gamma+2)}{M}g_E\sum_i m_iv_{i\parallel}\mathbf{v}_{i\perp}+[4\Delta'+\tfrac{1}{2}(\chi-\alpha')+\gamma+\beta]\frac{g_E}{M}\sum_i\frac{m_i}{r_{ij}^3}r_{ij\parallel}\mathbf{r}_{ij\perp}. \quad (5)$$

To simplfy (5), several virial conditions are employed:

$$\sum_i m_iv_i^2(t)+\sum_i m_iv_i^2(r)=\tfrac{1}{2}\sum_{ij}\frac{m_im_j}{r_{ij}}, \quad (6a)$$

$$\sum_i m_iv_i^2(t)_{\parallel}+\sum_i m_iv_i^2(r)_{\parallel}=\tfrac{1}{2}\sum_{ij}\frac{m_im_j}{r_{ij}^3}r_{ij\parallel}^2, \quad (6b)$$

$$\sum_i m_iv_i(r)_{\parallel}v_i(r)_{\perp}=\tfrac{1}{2}\sum_{ij}\frac{m_im_j}{r_{ij}^3}r_{ij\parallel}r_{ij\perp}. \quad (6c)$$

(r) refers to rotational motion and (t) refers to thermal motion. Let $T(\text{rot})$ be the rotational kinetic energy of the massive body and θ be the angle between $\mathbf{g}_E$ and the body's rotation axis; then

$$\sum_i m_iv_i^2(r)=2T(\text{rot}), \quad (7a)$$

$$\sum_i m_iv_i^2(r)_{\parallel}=\sin^2\theta\, T(\text{rot}), \quad (7b)$$

$$\sum_i m_iv_i(r)_{\parallel}v_i(r)_{\perp}=\sin\theta\cos\theta\, T(\text{rot}). \quad (7c)$$

Equation (7c) is valid if the perpendicular direction is in the plane defined by $\mathbf{g}_E$ and the body rotation axis. Otherwise (7c) gives 0. Also, we have the average over the thermal motion:

$$\sum_i m_iv_i^2(t)_{\parallel}=\tfrac{1}{3}\sum_i m_iv_i^2(t). \quad (8)$$

Finally, (5) gives

$$\mathbf{a}=\mathbf{g}_E\Big\{1+(\tfrac{1}{2}(8\Delta-4\beta-3\gamma-\chi)+\tfrac{1}{6}(8\Delta'+2\beta+\chi-\alpha'-2))\sum_{ij}\frac{m_im_j}{Mr_{ij}}+\frac{(8\Delta'+2\beta+\chi-\alpha'-2)(\sin^2\theta-\tfrac{2}{3})T(\text{rot})}{M}\Big\}+\Big[\frac{(8\Delta'+2\beta+\chi-\alpha'-2)g_E\sin\theta\cos\theta\, T(\text{rot})}{M}\Big]\boldsymbol{\xi}. \quad (9)$$

$\boldsymbol{\xi}$ is the unit vector perpendicular to $\mathbf{g}_E$ but in the plane of $\mathbf{g}_E$ and the body rotation axis. For the m_g/m_i ratio of the rotating massive body to be one leads to two conditions on the metric coefficients:

$$8\Delta-4\beta-3\gamma-\chi=0, \quad (10a)$$

$$8\Delta'+2\beta+\chi-\alpha'-2=0. \quad (10b)$$

Using the results of the Appendix to this paper, it is seen that (10b) is fulfilled for both the Einstein and BD theories. In fact we will denote theories with (10b) not satisfied as pathological in that they predict a noncentral $1/R^2$ acceleration toward distant mass.

Equation (10a) does not vanish in the BD theory,

$$(8\Delta-4\beta-3\gamma-\chi)_{\text{BD}}=-1/(2+\omega), \quad (11)$$

so (9) yields

$$\mathbf{a}_{\text{BD}}=\mathbf{g}_E\Big(1-\frac{1}{4+2\omega}\sum_{ij}\frac{m_im_j}{Mr_{ij}}\Big). \quad (12)$$

Using Dicke's suggested value of $\omega=6$, the Sun will fall anomalously slow in a gravitational field, the correction amounting to about $\frac{1}{8}$ of (Sun's gravitational self-energy/Sun's mass).

III. GAS SPHERE WITH RADIATION PRESSURE

More realistically, stars have some of the internal pressure needed to balance the gravitational attraction of the matter supplied by electromagnetic radiation pressure. The pressure is related to the radiation energy density by

$$p(\mathbf{x})=\tfrac{1}{3}u(\mathbf{x}).$$

The modified virial relation for equilibrium now reads

$$\sum_i m_iv_i^2+\int u(\mathbf{x})d\bar{V}=\tfrac{1}{2}\sum_{ij}\frac{m_im_j}{r_{ij}}. \quad (13)$$

Also, the presence of radiation in a body will affect the rate at which the body falls because of the perturbation (deflection and frequency shift) of that radiation by the external mass m_E.

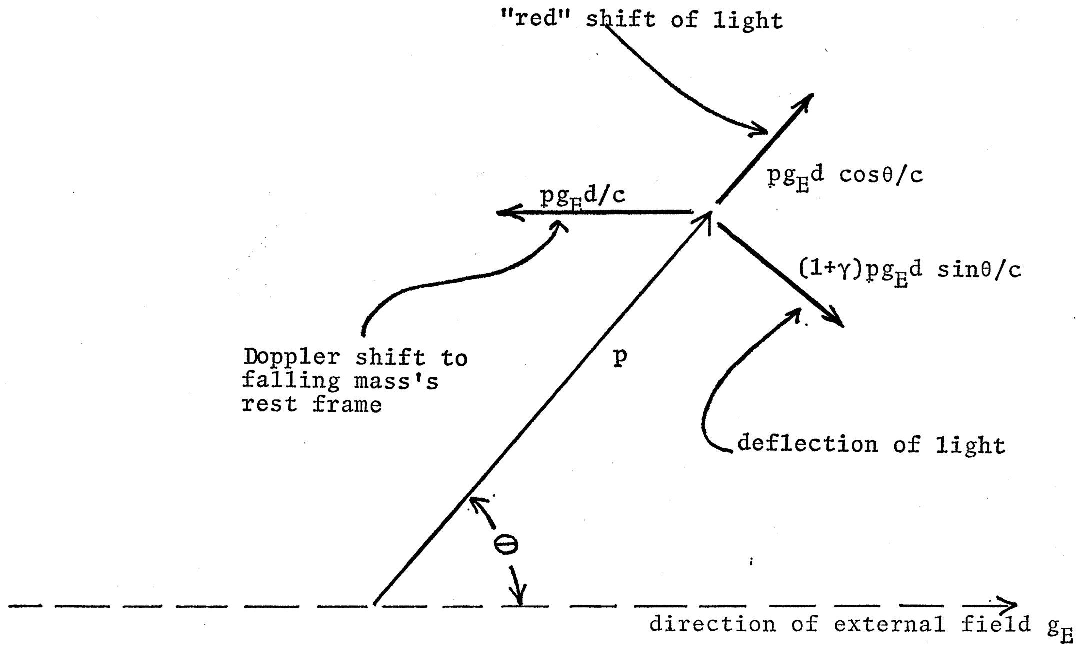

FIG. 1. A photon of momentum p has three contributions to its momentum change during travel of distance d.

Consider a pulse of radiation which has reflected off one mass element and travels a distance d before interacting with another mass element. Figure 1 shows the three changes in the momentum of the electromagnetic pulse relative to the *rest frame* of the massive body after the time $t=d/c$.

The total momentum parallel to g_E absorbed by the mass elements from the radiation is then

$$\delta p=(g_E dp/c)[(1+\gamma)\sin^2\theta+\cos^2\theta-1].$$

Dividing by the time elapsed between radiation impacts on matter, d/c, summing over all the radiation in the body, and making the directional average, yields an acceleration of the gas sphere due to the gravitational attraction of the radiation

$$\delta\mathbf{a}=\mathbf{g}_E\left(\tfrac{2}{3}\gamma\int u(\mathbf{x})\frac{dV}{M}\right). \tag{14}$$

However, the radiation modifies (13), the virial relation in the gas sphere. Also the radiation alters $\mathbf{a}(\text{int})_i$, the internal acceleration of the matter which enters into the result (5). The total acceleration of the sphere including (5) and (14) becomes

$$\mathbf{a}=\mathbf{g}_E\left[1+(\tfrac{1}{2}(8\Delta-4\beta-3\gamma-\chi)+\tfrac{1}{6}(8\Delta'+2\beta+\chi-\alpha'-2))\sum_{ij}\frac{m_i m_j}{Mr_{ij}}+(\tfrac{1}{3}(1-\gamma))\int u(x)\frac{dV}{M}\right]. \tag{15}$$

Assuming an average temperature of 10^7 °K for the Sun, (15) gives an anomalous value for m_g/m_i which is greater than 1 by about 10^{-8} in the BD theory if we set the parameter $\omega=6$.

IV. ACTIVE GRAVITATIONAL MASS

The rate at which a test particle accelerates toward a massive body will define the active gravitational mass of the body:

$$d^2\mathbf{x}/dt^2\equiv M_a\mathbf{R}/R^3. \tag{16}$$

For the test particle at rest the geodetic equation reduces to

$$\frac{d^2x^k}{dt^2}=-\Gamma_{00}{}^k\cong\frac{d}{dt}g_{0k}-\frac{1}{2}\frac{d}{dx^k}g_{00}. \tag{17}$$

Using the general metric expansions (1) and (2), the first term in (17) gives

$$\frac{dg_{0k}}{dt}=\frac{R_k}{R^3}\left[(4\Delta+4\Delta')\left(\sum_{ij}\frac{m_i m_j}{r_{ij}{}^3}r_{ij\parallel}{}^2-\sum_i m_i v_{i\parallel}{}^2\right)-4\Delta'\left(\sum_{ij}\frac{m_i m_j}{r_{ij}}-\sum_i m_i v_i{}^2\right)\right]. \tag{18}$$

$\mathbf{R}$ is the vector between the massive body and the test particle. We have kept only the acceleration component parallel to $\mathbf{R}$, as assuming a spherical symmetric sphere leaves $\mathbf{R}$ as the only direction in the problem. We have used for the internal particle accelerations, needed in evaluating dg_{0k}/dt

$$\mathbf{a}(\mathrm{int})_i=\sum_j m_j \frac{\mathbf{r}_{ji}}{r_{ij}^3}.$$

The ∇g_{00} term is straightforward to evaluate. The total acceleration of a test particle is then

$$d^2\mathbf{x}/dt^2=M\mathbf{R}/R^3,$$

with

$$M=\sum_i m_i+(2\alpha''-\tfrac{1}{6}\alpha'''-\tfrac{4}{3}\Delta+\tfrac{4}{3}\Delta')\sum_i m_i v_i^2 - (\alpha'+\tfrac{1}{6}\chi-\tfrac{2}{3}\Delta+\tfrac{2}{3}\Delta')\sum \frac{m_i m_j}{r_{ij}}. \quad (19)$$

To obtain (19) we have performed the directional averages

$$\sum_i \frac{m_i m_j}{r_{ij}^3} r_{ij\parallel}^2=\tfrac{1}{3}\sum_i \frac{m_i m_j}{r_{ij}}, \quad (20a)$$

$$\sum_i m_i v_{i\parallel}^2=\tfrac{1}{3}\sum_i m_i v_i^2. \quad (20b)$$

In order that (19) yield a M which gives the Newtonian energy of the collection of mass elements, we need two conditions on the metric coefficients:

$$2\alpha'-\tfrac{1}{6}\alpha'''-\tfrac{4}{3}\Delta+\tfrac{4}{3}\Delta'=\tfrac{1}{2}, \quad (21a)$$

$$\alpha'+\tfrac{1}{6}\chi-\tfrac{2}{3}\Delta+\tfrac{2}{3}\Delta'=\tfrac{1}{2}. \quad (21b)$$

The condition on the metric such that M be the Newtonian energy *only for systems at equilibrium* (where the virial theorem can be applied relating kinetic to potential energy) is less restrictive:

$$\alpha'-\alpha''+\tfrac{1}{12}\alpha'''+\tfrac{1}{6}\chi=\tfrac{1}{4}. \quad (22)$$

Equations (21a) and (21b) are satisfied for Einstein's theory, but not for the BD theory. Using the results of the Appendix,

$$\Delta=\alpha''=(3+2\omega)/(4+2\omega),$$

leads to

$$M_{\mathrm{BD}}=\sum_i m_i-\left(\frac{3\omega+8}{3\omega+6}\right)\sum_{ij}\frac{m_i m_j}{2r_{ij}} +\left(\frac{3\omega+4}{3\omega+6}\right)\sum_i \tfrac{1}{2}m_i v_i^2. \quad (23)$$

Using the virial theorem, (23) becomes

$$M_{\mathrm{BD}}=\sum_i m_i-\frac{1}{4}\left(\frac{\omega+4}{\omega+2}\right)\sum_{ij}\frac{m_i m_j}{r_{ij}}, \quad (24)$$

which is not the Newtonian energy.

For completeness we give the active gravitational mass of a rotating massive sphere otherwise in equilibrium:

$$M=\sum_i m_i+(\alpha''-\alpha'-\tfrac{1}{12}\alpha'''-\tfrac{1}{6}\chi)\sum_{ij}\frac{m_i m_j}{r_{ij}} +\tfrac{1}{2}(\chi-\alpha''')(\sin^2\theta-\tfrac{2}{3})T(\mathrm{rot}). \quad (25)$$

As before, $T(\mathrm{rot})$ is the rotational kinetic energy, and θ is the angle between the rotation axis and the direction $\mathbf{R}$. $\chi=\alpha'''=1$ for the BD theory as well as Einstein's theory.

V. DISCUSSION

In Sec. II it was shown that the gravitational-to-inertial mass ratio of a massive body in the BD theory is

$$\frac{M_g}{M_i}=1-\frac{1}{\omega+2}\sum_{ij}\frac{m_i m_j/2r_{ij}}{M} \quad (26)$$

Brans[6] has previously calculated the inertial mass of such a body and obtained just the Newtonian mass

$$M_i=M_N=\sum_j m_j+\tfrac{1}{2}\sum_j m_j v_j^2-\tfrac{1}{2}\sum_{ij}\frac{m_i m_j}{r_{ij}}.$$

So we can conclude from (26) that the passive gravitational mass of a massive body is

$$M_{g(p)}=M_N-\frac{1}{\omega+2}\sum_{ij}\frac{m_i m_j}{2r_{ij}}. \quad (27)$$

(The passive gravitational mass is the strength with which a body couples to a gravitational field; $\mathbf{F}=M_{g(p)}\mathbf{g}$.) But (24) gives the result that a massive body's active gravitational mass is identical to (27). Therefore Newton's law of action and reaction is valid for the interaction of two massive bodies in the BD theory:

$$\mathbf{F}_{12}=M_{g(p)}^{(1)}M_{g(a)}^{(2)}\mathbf{R}_{21}/R_{21}^3,$$
$$\mathbf{F}_{21}=M_{g(p)}^{(2)}M_{g(a)}^{(1)}\mathbf{R}_{12}/R_{12}^3;$$

so

$$\mathbf{F}_{21}=-\mathbf{F}_{12}.$$

Where in the scalar-tensor theory of BD is the equivalence principle violated when considering massive bodies? The nonlinearity of the BD theory in the mass sources is in agreement with Einstein's theory. As derived in the Appendix both β and α', which yield the g_{00} metric terms which are nonlinear in mass sources, are 1 in agreement with Einstein's theory. χ and α''' are 1 in both theories. These coefficients are simply the retardation corrections to the Newtonian potential expected of a theory which fulfills Lorentz invariance.

The BD metric has $\gamma=(1+\omega)/(2+\omega)$, which is now well known and affects light propagation experiments and a planet's perihelion advance. But we have also derived the metric coefficients

$$\Delta=\alpha''=(3+2\omega)/(4+2\omega).$$

[6] C. Brans, Phys. Rev. **125**, 2194 (1962).

However, these are not new differences between the BD and Einstein theory in a theoretical sense. They result from a Lorentz transformation applied to the γ metric term. The linear metric of a static mass in the BD theory is

$$g_{00}=1-2m/r, \quad g_{kk'}=(-1-2\gamma m/r)\delta_{kk'}.$$

Applying a Lorentz transformation,

$$x^k=\frac{1}{(1-v^2)^{1/2}}(x'^k-v^kt'),$$

$$t=\frac{1}{(1-v^2)^{1/2}}(t'-\mathbf{v}\cdot\mathbf{x}'),$$

yields the metric of a mass moving at velocity $\mathbf{v}$:

$$g_{00}'=1-2\frac{m}{r}\left(\frac{1+\gamma v^2}{1-v^2}\right),$$

$$g_{kk'}'=-1-2\frac{m}{r}\left(\frac{\gamma+v^2}{1-v^2}\right),$$

$$g_{0k}'=2(m/r)(1+\gamma)v^k.$$

Approximating the above to the necessary order, we get

$$\Delta=\alpha''=\tfrac{1}{2}(1+\gamma)=(3+2\omega)/(4+2\omega).$$

To the order to which we have investigated this problem, there is only a single fundamental difference between the Einstein and BD theory: $\gamma\neq 1$ in the BD theory.

The γ metric term affects a planet's perihelion advance by generating new forces on moving bodies of the form

$$\gamma v^2\nabla(m/r)$$

and

$$\gamma d/dt[m/r(\mathbf{v})].$$

It is not surprising in retrospect then that massive bodies in the BD theory have an m_g/m_i which depends on γ. Such bodies have internal kinetic energy which participates in maintaining system equilibrium. The coupling of moving particles to a scalar field differs from their coupling to a tensor field. This is the essential cause of the different result above for the BD theory.

A remarkable thing is that in an atom where kinetic energy balances electrical energy to maintain equilibrium, the γ metric term affects electrical forces in such a manner as to compensate for the γ dependence of the kinetic-energy contribution to m_g/m_i. Hence, the BD theory is in agreement with the "Eotvos" experiments in which different solid substances of laboratory size have an m_g/m_i ratio which is 1 to a part in 10^{11}. The details of these calculations will be presented in a future paper.

Though the active gravitational mass of a massive system may be of theoretical interest, it appears essentially impossible to experimentally determine whether a system's gravitational mass is equal to its Newtonian mass. On the other hand, as we have shown in Refs. 1 and 4, the ratio of a body's passive gravitational mass to inertial mass may be experimentally measurable in the near future.

APPENDIX

Here we calculate the metric component g_{00} to second order in mass strengths in the scalar-tensor theory of BD. We use isotropic coordinates. Their field equations are

$$\phi^i{}_{|i}=8\pi/(3+2\omega)T, \qquad \text{(A1)}$$

$$R_{ij}=-\frac{8\pi}{\phi}\left(T_{ij}-\frac{1+\omega}{3+2\omega}Tg_{ij}\right)-\frac{\phi_{i|j}}{\phi}-\omega\frac{\phi_i\phi_j}{\phi^2}, \qquad \text{(A2)}$$

with R_{ij} the Ricci curvature tensor, ϕ the scalar field, T_{ij} the stress-energy tensor of matter, $T=T_{ij}g^{ij}$. ω is a dimensionless parameter which Dicke has suggested is of order 6 or greater.

We assume many individual mass sources m_i. To sufficient accuracy (A1) is solved to yield

$$\phi=\phi_\infty+\frac{2}{3+2\omega}\sum_i\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}. \qquad \text{(A3)}$$

m_i are the mass sources. Writing (A2) for $i=j=0$ (time), we have to sufficient accuracy

$$R_{00}=-\frac{8\pi}{\phi}\left(T_{00}-\frac{1+\omega}{3+2\omega}Tg_{00}\right)-\frac{\phi_{0|0}}{\phi_\infty}. \qquad \text{(A4)}$$

We use the following results, derived elsewhere[2,3]:

$$\phi_\infty=(4+2\omega)/(3+2\omega), \qquad \text{(A5)}$$

$$\frac{1}{\phi}\simeq\frac{3+2\omega}{4+2\omega}\left(1-\frac{1}{2+\omega}\sum_i\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\right), \qquad \text{(A6)}$$

$$T_{00}=\rho g_{00}{}^2\left(\frac{dt}{ds}\right)^2\simeq\rho\left(1-2\sum\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}+v^2\right), \qquad \text{(A7)}$$

$$T=\rho. \qquad \text{(A8)}$$

Then

$$\phi_{0|0}=\frac{2}{3+2\omega}\left(\frac{d^2}{dt^2}\sum_i\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}-\Gamma_{00}{}^r\frac{d}{dx^r}\sum_i\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}\right). \qquad \text{(A9)}$$

But

$$\Gamma_{00}{}^r\cong-\frac{d}{dx^r}\sum\frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}. \qquad \text{(A10)}$$

So, finally, we have for the Ricci tensor component

$$R_{00}=-8\pi\rho\big(\tfrac{1}{2}-[(5+2\omega)/(4+2\omega)]\psi+[(3+2\omega)/(4+2\omega)]v^2\big)-[1/(2+\omega)][\ddot{\psi}+(\nabla\psi)^2], \quad \text{(A11)}$$

with

$$\psi=\sum_i \frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}.$$

Using the result obtained in the Appendix of II

$$g_{0k}=4\left(\frac{3+2\omega}{4+2\omega}\right)\sum_i \frac{m_i}{|\mathbf{r}-\mathbf{r}_i|}v_i^k, \quad \text{(A12)}$$

we obtain an expression for the Ricci tensor component which is of sufficient accuracy to calculate g_{00} to second order:

$$R_{00}=-\tfrac{1}{2}\nabla^2 g_{00}-[(\omega+3)/(\omega+2)]\ddot{\psi}+8\pi[(\omega+1)/(\omega+2)]\psi\rho+[(2\omega+3)/(\omega+2)](\nabla\psi)^2. \quad \text{(A13)}$$

Combining (A11) and (A13) yields

$$\nabla^2 g_{00}=8\pi\rho\big(1-[3/(\omega+2)]\psi+[(2\omega+3)/(\omega+2)]v^2\big)-2\ddot{\psi}+4(\nabla\psi)^2. \quad \text{(A14)}$$

In the product $\rho\psi$ the contribution to ψ from a particular mass is not to be included when evaluating the product at the location of that mass.

Equation (A14) is straightforwardly solved except for the point that in the first-order source term $8\pi\rho$ we must use the proper volume corrected to lowest order

$$dV=dV_\pi/(\sqrt{g})(dt/ds),$$

which yields

$$dV=dV_\pi\big(1-3[(1+\omega)/(2+\omega)]\psi-\tfrac{1}{2}v^2\big). \quad \text{(A15)}$$

We get then for g_{00}

$$g_{00}=1-2\psi+2\psi^2-\left(\frac{4\omega+6}{\omega+2}\right)\sum_i \frac{m_i v_i^2}{|\mathbf{r}-\mathbf{r}_i|}+2\sum_{ij}\frac{m_i m_j}{|\mathbf{r}_i-\mathbf{r}_j|}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_i|}+\frac{1}{|\mathbf{r}-\mathbf{r}_j|}\right)-\sum_i \frac{m_i(\mathbf{r}-\mathbf{r}_i)\cdot\mathbf{a}_i}{|\mathbf{r}-\mathbf{r}_i|}+\sum_i \frac{m_i[(\mathbf{r}-\mathbf{r}_i)\cdot\mathbf{v}_i]^2}{|\mathbf{r}-\mathbf{r}_i|^3}. \quad \text{(A16)}$$

Only one coefficient in (A16) differs from Einstein's theory values which are obtained by taking the limit $\omega\to\infty$.