Spin and Orbital Motions of a Particle in a Homogeneous Magnetic Field*

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(Received 13 December 1968)

The spin and orbital motions of a particle in a homogeneous magnetic field are examined in a classical approximation. Application is made to the determination of the g-factor anomalies of the electron and muon. It is found that the *experimentally observed difference frequency* between the spin and orbital motions depends on the component of particle velocity parallel to the magnetic field, and is proportional to $(1-v_z^2/c^2)^{1/2}$, where the magnetic field defines the z axis.

I. INTRODUCTION

INTEREST in the electron and muon g-factor anomalies has led to detailed theoretical study of the spin and orbital motions of a charged lepton in electromagnetic fields. In the analysis of the electron experiments at Michigan[1,2] and of the muon experiments at CERN[3,4] and Berkeley,[5] a classical approximation may be used[6,7]; in planned electron experiments involving transitions from one quantum state to another, such approximations are insufficient. The work presented here is strictly classical, and applies to the Michigan electron experiments completed to date, as well as to all muon experiments completed or presently planned. A case of great experimental interest is that of motion in nearly homogeneous pure magnetic fields. Here we discuss, in a very simple manner, some aspects of motion in a strictly homogeneous magnetic field. In particular, we do not find it necessary to introduce a specific relativistic generalization of the particle spin.

II. RELATION OF $\frac{1}{2}g-1$ TO SPIN AND ORBITAL MOTION

The case where the lepton orbital motion is confined to a plane perpendicular to the magnetic field $\mathbf{B}$ is well understood. Denoting by ω_c the orbital (or cyclotron) frequency, and by ω_s the spin precession frequency of the spin component transverse to $\mathbf{B}$, we have[8]

$$\omega_c = eB/m\gamma_\perp \equiv \omega_0/\gamma_\perp\,, \tag{1}$$

$$\omega_s = \omega_0(\tfrac{1}{2}g + 1/\gamma_\perp - 1)\,, \tag{2}$$

where, as usual, e is the lepton charge, m its mass, and g its gyromagnetic ratio $g\approx 2(1+\alpha/2\pi)$. The subscript in $\gamma_\perp$ indicates that the orbital motion is perpendicular to $\mathbf{B}$. Equation (2) is derived in the Appendix. The *difference* frequency, which may be measured directly, is then

$$\omega_D \equiv \omega_s - \omega_c = (\tfrac{1}{2}g-1)\omega_0\,. \tag{3}$$

This remarkable (and well-known) result implies that although ω_s and ω_c both depend on the speed of the lepton when the motion is relativistic, the *difference* frequency is rigorously independent of the speed.

For helical motion in a homogeneous $\mathbf{B}$ field, Eq. (3) breaks down. We can most easily analyze this case by imagining that we are simply viewing *planar motion* from a frame of reference drifting parallel to the $\mathbf{B}$ field, with velocity $-\beta_z$ (where $\mathbf{B}$ defines the z axis; we note that $\mathbf{B}$ is the same in both the original frame and the drift frame). We can further imagine that in the original frame where the lepton motion is planar, we have a clock with hands rotating at ω_c, ω_s, and ω_D. Indeed, the lepton itself may be said to constitute such a clock. In the drifting frame all three frequencies are time-dilated by a factor $\gamma_z \equiv (1-\beta_z^2)^{-1/2}$, so that, in particular,

$$\omega_D = (\tfrac{1}{2}g-1)\omega_0/\gamma_z\,. \tag{3$'$}$$

Equation (3′) is exact, and applies to all classical motion in homogeneous $\mathbf{B}$ fields. A partial check of this simple derivation is contained in the generalization

* Research sponsored by the Air Force Office of Scientific Research, Office of Aerospace Research, U. S. Air Force, under AFOSR Contract No. F44620-68-C-0075.
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[5] G. Schrank, G. R. Henry, and R. Swanson, University of California Radiation Laboratory Report No. UCRL-16469, 1965 (unpublished).

[6] A particularly lucid analysis in terms of a polarization four-vector has been given by V. Bargmann, Louis Michel, and V. L. Telegdi, Phys. Rev. Letters **2**, 435 (1959).

[7] That a classical analysis is sufficient for consideration of the expectation value of the spin was observed by F. Bloch, Phys. Rev. **70**, 460 (1946).

[8] We set $\hbar=c=1$, and define γ in the usual way: $\gamma\equiv(1-\beta^2)^{-1/2}$. The lepton spin direction in a frame where the electron is moving is defined to be the direction of the expectation value of the spin operator $\boldsymbol{\sigma}$ in the lepton rest frame, as suggested by C. G. Darwin, Proc. Roy. Soc. (London) **A124**, 425 (1929). See also E. P. Wigner, Rev. Mod. Phys. **29**, 255 (1957). The precession formulas of Ref. 6 are consistent with this definition; further, this definition is directly related to physically observed phenomena such as Møller and Bhabha scattering, and muon decay. In particular, if the spin direction is initially aligned with the particle velocity, then after one "$g-2$ cycle" the spin direction is again parallel to the velocity, and the scattering cross section or decay pattern begins a new cycle.

of Eq. (1), which should, of course, yield $\omega_c = \omega_0/\gamma$. Thus,

$$\omega_c = \omega_0/\gamma_\perp \gamma_z = \omega_0/\gamma\,, \tag{1'}$$

and we must have

$$\gamma = \gamma_\perp \gamma_z\,. \tag{4}$$

It is well-known, however, that two successive Lorentz transformations at a relative angle θ have an over-all γ given by

$$\gamma = \gamma_1 \gamma_2 (1 + \beta_1 \beta_2 \cos\theta)\,, \tag{5}$$

which, for our case ($\theta = \frac{1}{2}\pi$), yields Eq. (4) as expected.

Note that Eq. (3′) differs somewhat from a previously published formula, which for a homogeneous **B** field takes the form[9]

$$\omega_D = (\tfrac{1}{2}g - 1)\omega_0 \left(1 - \frac{\gamma}{\gamma+1}\beta_z^2\right). \tag{6a}$$

Equation (6a) arises from a time average of the vector equation

$$\boldsymbol{\omega}_D = (\tfrac{1}{2}g - 1)\omega_0 \left(\hat{z} - \frac{\gamma}{\gamma+1}\beta_z \boldsymbol{\beta}\right), \tag{6b}$$

which is the correct expression for the instantaneous angular velocity of spin relative to velocity in the laboratory frame. That is, the equation of motion of spin relative to velocity is simply

$$d\mathbf{S}/dt = -\boldsymbol{\omega}_D \times \mathbf{S}\,. \tag{7}$$

Solving this equation of motion, one finds that *measured quantities* (depending on $\mathbf{S}\cdot\boldsymbol{\beta}$) vary with the frequency ω_D as given by Eq. (3′), rather than by Eq. (6a).[10] Thus it appears that Eq. (6a) cannot be directly applied to lepton motion as it stands. The relation between Eqs. (3′) and (6a) is made clearer by expanding (3′) for small β_z:

$$\omega_D \approx (\tfrac{1}{2}g - 1)\omega_0 (1 - \tfrac{1}{2}\beta_z^4 - \tfrac{1}{8}\beta_z^4 - \cdots)\,. \tag{3''}$$

From this expansion it is clear that for $\beta_z \ll 1$ (the condition of experimental interest) Eqs. (3′) and (6a) approximately agree if the lepton is quite nonrelativistic. In the extremely relativistic region, Eq. (6a) yields a correction for motion along **B** which is about twice the correction of Eq. (3′).

III. CONCLUSIONS

With the present level of accuracy and the experimental configuration, the $\frac{1}{2}g - 1$ value for the muon, determined at CERN, is probably changed by a negligible amount. In the electron case, as done at Michigan, $\frac{1}{2}g - 1$ is reduced by roughly seven parts per million,[11] which is well within the experimental error of ± 26 ppm. In future, more accurate experiments, it may be more important that Eq. (3′) be used rather than Eq. (6a). Unfortunately, it may also be important to integrate the detailed equations of motion rather than simply to calculate the time average of Eq. (3′).[12]

Finally, we believe that the approach used here is of some intrinsic interest, since the desired answer is obtained without having to adopt *any* specific relativistic generalization of particle spin, such as a rank-2 tensor or a polarization four-vector.

ACKNOWLEDGMENTS

We wish to thank Dr. S. J. Brodsky and Professor W. K. H. Panofsky for stimulating discussions. We are also indebted to Professor G. W. Ford and Professor A. Rich for interesting and helpful communications regarding particle spin motion.

APPENDIX

For completeness, we derive Eq. (2). In order to avoid questions of relativistic generalizations of spin, we compute the spin precession in the instantaneous rest frame of the particle, in which $\mathbf{B}' = \gamma_\perp \mathbf{B}$, and

$$\omega_s' = \frac{eB'}{m}(\tfrac{1}{2}g) = \frac{e\gamma_\perp B}{m}(\tfrac{1}{2}g) = \tfrac{1}{2}g\gamma_\perp\omega_0\,. \tag{A1}$$

To get back to the laboratory frame, it is tempting simply to reduce ω_s' by the time-dilation factor $\gamma_\perp$, and in fact this is correct for neutral particles, such as a neutron or Λ. If, however, the particle is charged (and therefore accelerated), there is a Thomas precession term that must be added; for circular motion of frequency ω_c, the Thomas precession is given by

$$\Omega_{\rm Th} = \omega_c (1 - \gamma)\,. \tag{A2}$$

Thus we finally have for ω_s

$$\omega_s = \frac{\omega_s'}{\gamma_\perp} + \Omega_{\rm Th} = \tfrac{1}{2}g\omega_0 + \frac{\omega_0}{\gamma_\perp}(1 - \gamma_\perp)\,, \tag{A3}$$

which leads directly to Eq. (2).

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[9] Equation (1) of Ref. 1; also quoted in Ref. 2.

[10] We are indebted to Professor G. W. Ford for helpful communications on this point.

[11] Professor A. Rich has reanalyzed the Michigan $g-2$ data (see Refs. 1 and 2) using Eq. (3′) rather than Eq. (6a), finding the new experimental value

$$\tfrac{1}{2}g - 1 = (1\ 159\ 549 \pm 30) \times 10^{-9},$$

to be compared to the theoretical value

$$\tfrac{1}{2}g - 1 = (1\ 159\ 641 \pm 3) \times 10^{-9},$$

where $\alpha^{-1} = 137.0359 \pm 0.0003$ has been used (Ref. 12).

[12] G. W. Ford and A. Rich (private communication).