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Specifying Sources in General Relativity

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It is demonstrated that the dynamical equations for a massless scalar field, incoherent matter, or incoherent radiation may be put into such forms that they may be solved without a metric. The equations reduce to the usual forms when four conditions on the form of the metric are imposed, which do not restrict its nature. The conservation laws $\tilde{T}_\mu{}^\nu{}_{;\nu}=0$ are thereby satisfied before the integration of the Einstein equations is attempted, ensuring that solutions will exist. The equations of motion of a perfect fluid in isentropic flow are similarly reformulated nonmetrically; but since five conditions must be placed on the metric to reduce the theory to its usual form, there is one real restriction on the metrical geometry implied in this case.

I. INTRODUCTION

When one considers a macroscopic, classical field created by certain sources, two approaches may be taken toward the sources. They may be regarded as given functions of the space-time coordinates; or they may be regarded as themselves built up from other field or particle variables (possibly including the original field, as in the gravitational case with which we shall be concerned), obeying their own equations of motion.

In the first case, the physical system being considered is regarded as open. We do not concern ourselves with how the source fields are created, but just take them as given and investigate how their presence affects the field with which we are concerned.[1]

In the second case we generally regard the physical system as closed, in the sense that we are interested in the interaction between our field and various other fields and particles; so that one is concerned with a coupled set of equations for a group of interacting fields and particles that may be regarded as forming an isolated system.

In the first (open system) approach, the mathematical problem is that of finding the solution to a set of inhomogeneous partial differential equations for the field in question, the inhomogeneity being provided by the source fields. Examples are the inhomogeneous scalar wave equation (which may be looked upon as the equation for a massless spin-0 field); or Maxwell's equations with given source fields. What makes these equations simple to solve (in principle in any background metric, and in practice in Minkowski space) is that the behavior of the external sources is easily prescribed before solution of the field equations. In the case of the scalar wave equation $\Box\varphi=\rho$, the source $\rho(x,y,z,t)$ may be given quite arbitrarily. In the case of Maxwell's equations $\tilde{F}^{\mu\nu}{}_{,\nu}=\tilde{\jmath}^\mu$, the charge-current four-vector density $\tilde{\jmath}^\mu$ must obey the conservation law $\tilde{\jmath}^\mu{}_{,\mu}=0$, which forms an integrability condition for Maxwell's equations. However, this condition is easily satisfied quite independently of the field equations; e.g., by setting $\tilde{\jmath}^\mu=\tilde{K}^{[\mu\nu]}{}_{,\nu}$, where $\tilde{K}^{[\mu\nu]}$ is any antisymmetric tensor density of second rank.[2]

In the case of the Einstein equations with sources,

$$G_\mu{}^\nu=\kappa T_\mu{}^\nu\,, \tag{1.1}$$

where $G_\mu{}^\nu$ is the Einstein tensor and $T_\mu{}^\nu$ is the stress-energy tensor of the sources of the gravitational field, the situation appears to be quite different. The conservation laws $T_\mu{}^\nu{}_{;\nu}=0$ form a set of four integrability conditions for the field equations (1.1). Since the covariant divergence involves the metric tensor $g_{\mu\nu}$, the set of field variables for which we are attempting to solve (1.1), it seems at first sight as if the sources may not be prescribed arbitrarily if they are to obey the covariant divergence law. However, recent work by Pereira[3] has indicated that, locally at any rate, this problem really does not occur. He has shown that if we take the stress-energy tensor field $\tilde{T}^{\mu\nu}$ (in the form of a contravariant tensor density) as arbitrary functions of x^μ (the coordinates of the manifold); and if we take *any* metric $\bar{g}_{\mu\nu}(\bar{x})$, locally there always exists a nonsingular coordinate transformation $x^\nu=x^\nu(\bar{x}^\mu)$ such that when we transform the metric by this coordinate transformation, the covariant divergence of the untransformed stress-energy tensor with respect to the original metric in the new coordinates vanishes. Thus, if we write Eqs. (1.1) in the equivalent form

$$\tilde{G}^{\mu\nu}=\kappa\tilde{T}^{\mu\nu}(x)\,, \tag{1.2}$$

the integrability conditions will automatically be satisfied in the course of solving the equations, a solution to which must always exist locally for any specified $\tilde{T}^{\mu\nu}(x)$.

[1] This question is discussed for fields with gauge groups by D. G. Boulware and S. Deser, Nuovo Cimento **30**, 1009 (1963). They analyze the gravitational case on a spacelike hypersurface, assuming the metric given on the hypersurface, and show that while certain components of the stress-energy tensor may be given arbitrarily, the other components depend on the gravitational field variables nonlocally in time. This does not contradict the results of Pereira discussed in the text, which assume the stress-energy tensor to be given as a function of the coordinates throughout space-time. Recently, the question of the construction of a quantum theory of gravitation starting from the concept of sources of gravitons in flat space-time has been discussed by J. Schwinger, Phys. Rev. **173**, 1264 (1968).

[2] A tilde over any tensorial symbol denotes that it is a tensorial density of weight $+1$; a German symbol represents a tensorial density of weight -1. Once a metric has been introduced, this will, of course, be equivalent to multiplication or division, respectively, of the corresponding tensor by $\sqrt{(-g)}$. We use the signature -2 for the metric.

[3] Carlos Pereira, Ph.D. thesis, Case-Western Reserve University, 1967 (unpublished).

Although this result is mathematically satisfying, its physical relevance is more doubtful. First of all, once one has solved for the given metric, it is by no means guaranteed that the metrical properties of the given $\tilde{T}^{\mu\nu}(x)$ will be such as to make it a physically reasonable stress-energy tensor. For example, the time-time component in a local Lorentz frame (physical component) need not be positive for all Lorentz frames.[4] Perhaps more important, the very notion of an open system, in this sense, in general relativity is open to question. In the case of Maxwell's equations, since we can move charges by nonelectromagnetic forces, it is quite reasonable to assume, on the macroscopic level, that arbitrary motions of sources consistent with the conservation of charge may be produced; and then to study the fields with which they are associated. In general relativity, on the other hand, there are no forces which are not associated with an inertial stress-energy tensor; and thus, by the principle of equivalence, with an active gravitational stress-energy tensor, i.e., a right-hand side for Eqs. (1.1) or (1.2).[5] These considerations do not make meaningless the ideas of open systems in general relativity or of a macroscopic stress-energy tensor given as a function of the space-time coordinates, as we have tried to argue elsewhere.[6] But they do indicate the need for extreme caution in interpreting entirely arbitrary stress-energy tensors of this form as having any physical meaning. Clearly, there are interesting problems here which require further investigation.

When the problem is examined from the point of view of what components of the stress-energy tensor may be specified on a spacelike hypersurface, for example, serious difficulties arise.[1]

In the remainder of this paper, however, we shall confine ourselves to the second class of problems: closed systems of coupled fields and/or particles. An example of this is the coupled Maxwell-Dirac fields. Here the current vector for the Maxwell fields is formed from the Dirac fields, while the Dirac fields in turn are affected by the presence of an electromagnetic field in the usual way (minimal coupling). Such problems almost always involve coupled nonlinear partial differential equations, in which the sources may not be freely specified, but are known as functions of the space-time variables only after solving the entire coupled problem.

In the gravitational case, the stress-energy tensor is to be thought of as built up from some set of dynamical variables describing the nongravitational fields and particles, and from the metric tensor as well. (The principle of equivalence usually implies that, for some choice of dynamical variables, only the metric field and not its derivatives need enter.) Its divergence then vanishes as a consequence of the nongravitational dynamical equations obeyed by the dynamical variables. Since these equations usually involve the metric, we are usually faced with the problem of solving these dynamical equations and the gravitational equations (1.1) as a coupled set, again without the freedom to prescribe the behavior of the sources in advance.

However, we shall show that there are certain cases in which, by suitably choosing the set of dynamical variables, the dynamical equations they obey may be so formulated as not to involve the metric (i.e., as tensorial equations in a bare manifold). If only four components of the metric are then needed to relate these dynamical variables in such a way that the theory reduces to the usual form, and the divergence of the usual stress-energy tensor vanishes, then the integrability conditions for the Einstein equations may be satisfied without restricting the nature of the metric. The problem of solving the dynamical equations and the Einstein equations is thus split into two separate parts. Thus, sources may be specified for the Einstein equations before they are solved, with the assurance that the integrability conditions are satisfied beforehand.

We shall illustrate this procedure for several cases where it works completely: the massless scalar field, incoherent matter, and incoherent radiation. We shall also discuss one case where it does not work completely, but only "three-quarters" of the way. This is the case of isentropic flow of a perfect fluid, where it is of some interest to see that all the dynamical equations may be formulated nonmetrically. However, five conditions on the 10 components of the metric are then needed to reduce our formulation to the usual one.

II. MASSLESS SCALAR FIELD

In the usual formulation of the massless scalar field, Φ obeys the scalar wave equations $\Box\Phi \equiv g^{\mu\nu}\Phi_{;\mu\nu}=0$, as a consequence of which the stress-energy tensor $T_\mu{}^\nu = \Phi_{,\mu}\Phi^{,\nu}-\frac{1}{2}\delta_\mu{}^\nu(\Phi_{,\kappa}\Phi^{,\kappa})$ has a vanishing divergence. We shall reformulate the theory in an equivalent way, using variables such that no metrical concepts enter into the equations replacing the wave equation. Only four components of the metric then will be seen to enter into the relationships making the new formulation equivalent to the usual one.

We shall take as our dynamical variables a contravariant vector density $\tilde{U}^\alpha$, and a scalar field Φ. We define the field v_α by

$$v_\alpha=\Phi_{,\alpha}, \tag{2.1}$$

and postulate the field equation (involving no metric)

$$\tilde{U}^\alpha{}_{,\alpha}=0. \tag{2.2}$$

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[4] This requirement has been stressed by J. L. Synge, *Relativity: the General Theory* (North-Holland Publishing Co., Amsterdam, 1960), pp. 184–187.

[5] We have discussed the advantages of defining inertial, passive gravitational and active gravitational stress-energy tensors, and formulating the principle of equivalence in terms of these concepts, in *Boston Studies in the Philosophy of Science*, edited by R. S. Cohen and M. Wartofsky (D. Reidel, Dordrecht, Holland, in press), Vol. 6.

[6] J. Stachel, in *Boston Studies in the Philosophy of Science*, edited by R. S. Cohen and M. Wartofsky (D. Reidel, Dordrecht, Holland, 1969), Vol. 5, p. 96.

Alternatively, we might have chosen v_α as a fundamental field, subject to the nonmetrical field equations

$$v_{\alpha,\beta}-v_{\beta,\alpha}=0, \tag{2.3}$$

which locally imply (2.1). This procedure would be more analogous to that which we use in the two hydrodynamical cases to follow. We may solve (2.2) by letting $\tilde{U}^\alpha=\tilde{V}^{[\alpha\beta]}{}_{,\beta}$ where $\tilde{V}^{[\alpha\beta]}$ is any antisymmetric tensor density field. So far, we have introduced no metrical concepts. But if we introduce *any* metric (of signature -2) such that

$$v_\alpha=\mathfrak{g}_{\alpha\beta}\tilde{U}^\beta, \tag{2.4}$$

where $\mathfrak{g}_{\alpha\beta}=g_{\alpha\beta}/\sqrt{(-g)}$ is the metric tensor density of weight -1, then (2.1) and (2.2) imply that the scalar wave equation for Φ holds in this metric. We now define the tensor $\tilde{T}_\alpha{}^\beta$ by

$$\tilde{T}_\alpha{}^\beta=v_\alpha\tilde{U}^\beta-\tfrac{1}{2}\delta_\alpha{}^\beta(v_\kappa\tilde{U}^\kappa). \tag{2.5}$$

For any metric obeying (2.4), this is the usual stress-energy tensor for the massless scalar field; it is easily checked that its divergence then vanishes as a consequence of (2.1) and (2.2).

Now, how much does (2.4) restrict the metric? If we complete $\tilde{U}^\alpha$ by any three linearly independent contravariant vector density fields $\tilde{e}_i{}^\alpha$ $(i=1, 2, 3)$ we can see that, once $\tilde{U}^\alpha$ and v_α are given, only $\mathfrak{g}_{\alpha\beta}\tilde{U}^\alpha\tilde{U}^\beta$ and $\mathfrak{g}_{\alpha\beta}\tilde{U}^\alpha\tilde{e}_i{}^\beta$ are determined by (2.4). Or, as we may loosely say (since no metric exists until we have defined it fully), (2.4) determines only the "parallel-parallel" and "parallel-transverse" parts of $\mathfrak{g}_{\alpha\beta}$ with respect to $\tilde{U}^\alpha$. If we were to adapt a coordinate system to $\tilde{U}^\alpha$ (which can always be done in an infinite number of ways), so that $\tilde{U}^\alpha=\delta_0{}^\alpha$ in such a coordinate system, then this would mean that the components $\mathfrak{g}_{0\mu}$ would be determined in this coordinate system. But the remaining six "transverse-transverse" components of $\mathfrak{g}_{\mu\nu}$ with respect to $\tilde{U}^\alpha$ ($\mathfrak{g}_{ij}$ in the adapted coordinate system) are left completely free by (2.4). This therefore constitutes no restriction on the *nature* of the metric, i.e., these conditions are compatible with any Riemannian structure, but only on its *form* as a function of the x^μ. These six components may be determined from the Einstein equations (1.1), which must have solutions, because we have already satisfied their integrability conditions, $\tilde{T}_\mu{}^\nu{}_{;\nu}=0$.

Note that by the use of the reciprocal tensor densities $\tilde{\mathfrak{g}}^{\mu\nu}=(\sqrt{-g})g^{\mu\nu}$ and $\mathfrak{g}_{\mu\nu}=g_{\mu\nu}/\sqrt{(-g)}$, which obey

$$\mathfrak{g}_{\mu\nu}\tilde{\mathfrak{g}}^{\nu\kappa}=\delta_\mu{}^\kappa,$$

we can write the Einstein equations in the form

$$\tilde{G}_\mu{}^\nu=(\sqrt{-g})G_\mu{}^\nu=\kappa(\sqrt{-g})T_\mu{}^\nu=\kappa\tilde{T}_\mu{}^\nu, \tag{2.6}$$

where $\tilde{G}_\mu{}^\nu=\tilde{G}_\mu{}^\nu(\tilde{\mathfrak{g}}^{\mu\nu},\mathfrak{g}_{\mu\nu})$. The exact form of $\tilde{G}_\mu{}^\nu$ as a function of these variables will not be necessary for our purposes, but may be found in a number of places, e.g.' in Ref. 7.

To summarize the procedure in this case: First we solve (2.2) (for example, by letting $\tilde{U}^\alpha=\tilde{V}^{[\alpha\beta]}{}_{,\beta}$). Then Φ is picked arbitrarily [general solution to (2.3)]. Equation (2.4) is then used to determine the parallel-parallel and parallel-transverse portions of $\mathfrak{g}_{\alpha\beta}$. We have now satisfied the scalar wave equation for Φ, and the divergence condition on the stress-energy tensor, no matter how the rest of the metric is determined. Then we use the Einstein equations in the form (2.6) to determine the rest of the metric.

If we wanted to solve the scalar wave equation in any given background space, such as Minkowski space, we could use the same procedure, up to the application of the Einstein field equations. Since we have only imposed four conditions on the form of the metric, it is still compatible with any Riemannian structure, and we need only choose the six remaining components accordingly. For example, in Minkowski space, we would pick them to satisfy $R_{\mu\nu\kappa\lambda}=0$.

III. INCOHERENT MATTER AND INCOHERENT RADIATION

By incoherent matter, we mean matter whose stress-energy tensor is of the form $T_\mu{}^\nu=\rho u_\mu u^\nu$, where the matter density ρ must be positive in regions where it does not vanish; and u^μ represents a timelike unit vector field. With our choice of signature, $u_\mu u^\mu=1$. It is well known that the conservation equations for the stress-energy tensor are equivalent to the two equations

$$(\rho u^\alpha)_{;\alpha}=0, \tag{3.1}$$

which expresses the conservation of matter, and

$$u^\alpha u_{\beta;\alpha}=0, \tag{3.2}$$

which expresses the fact that the streamlines are timelike geodesics.[8] A set of equations equivalent to (3.2) is

$$(u_{\alpha,\beta}-u_{\beta,\alpha})u^\beta=0, \tag{3.3}$$

since $u^\alpha u_{\alpha;\beta}=0$ as a consequence of its being a unit vector.

We now choose a set of dynamical variables that allows us to rewrite (3.1) and (3.3) without metric. If we assume that $\tilde{U}^\alpha=\tilde{\rho}u^\alpha$, then we may replace (3.1) by

$$\tilde{U}^\alpha{}_{,\alpha}=0. \tag{3.4}$$

In terms of u_α and $\tilde{U}^\alpha$, (3.3) then takes the form (since $\tilde{\rho}$ cannot vanish wherever the matter is present)

$$(u_{\alpha,\beta}-u_{\beta,\alpha})\tilde{U}^\beta=0. \tag{3.5}$$

[7] A. Papapetrou, Ann. Physik **20**, 399 (1957). The result needed above may be obtained from Eq. (2.2) of this paper.

[8] See, e.g., A. Lichnerowicz, *Théories relativistes de la gravitation et de l'électromagnétisme* (Masson, Paris, 1955); A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics* (W. A. Benjamin, Inc., New York, 1967). Lichnerowicz calls this the case of pure matter.

In this form, neither equation involves a metric. We can now start with the fields u_α and $\tilde{U}^\alpha$ as our fundamental dynamical variables, for which we postulate Eqs. (3.4) and (3.5), and *define* $\tilde{\rho}$ by

$$u_\alpha \tilde{U}^\alpha = \tilde{\rho} > 0. \tag{3.6}$$

If we introduce any metric such that

$$\tilde{\rho} u_\alpha = g_{\alpha\beta} \tilde{U}^\beta, \tag{3.7}$$

then (3.4) and (3.5) will be fully equivalent to (3.1) and (3.3). We can satisfy (3.4) by choosing $\tilde{U}^\alpha = \tilde{V}^{[\alpha\beta]}{}_{,\beta}$, where $\tilde{V}^{[\alpha\beta]}$ is any antisymmetric tensor density of rank 2. Equation (3.5) can always be satisfied by the introduction of three potentials ψ, ξ, and η, where ψ is an arbitrary scalar field, and ξ and η obey

$$\eta_{,\alpha} \tilde{U}^\alpha = \xi_{,\alpha} \tilde{U}^\alpha = 0. \tag{3.8}$$

Then $u_\alpha = \psi_{,\alpha} + \eta \xi_{,\alpha}$ will be the most general solution of (3.5).[9] Since we need to have $\tilde{\rho}$ positive for our physical interpretation, we may pick η and ξ subject to (3.8) and ψ arbitrary except that $u_\alpha \tilde{U}^\alpha = \psi_{,\alpha} \tilde{U}^\alpha > 0$. As in Sec. II, (3.7) implies a restriction on only the four components of $g_{\alpha\beta}$ parallel-parallel and parallel-transverse with respect to $\tilde{U}^\alpha$. In summary, by appropriate choices of $\tilde{U}^\alpha$ and the three scalar potentials ψ, ξ, and η, we have satisfied (3.4)–(3.6) without any metrical considerations. By then choosing four components of $g_{\alpha\beta}$ to satisfy (3.7), we are assured that (3.4) and (3.5) are equivalent to (3.1) and (3.3). It is then easily shown that if we define $\tilde{T}_\alpha{}^\beta$ by

$$\tilde{T}_\alpha{}^\beta = u_\alpha \tilde{U}^\beta, \tag{3.9}$$

this becomes the usual stress-energy tensor for incoherent matter, and its divergence vanishes as a consequence of (3.4). Thus we have satisfied the integrability conditions for the Einstein equations, which may now be solved to find the remaining six components of the metric.

A similar approach may be used to handle the case of incoherent radiation.[10] Here the stress-energy tensor is of the form $T_\mu{}^\nu = \rho k_\mu k^\nu$, and ρ is normalized (as may always be done) so that ρk^ν obeys the conservation law $(\rho k^\nu)_{;\nu} = 0$. k^ν is the tangent to a family of null geodesics, which then obeys $k^\nu k_\nu = 0$, and $k_{\mu;\nu} k^\nu = 0$.

An equivalent set of equations is again given by

$$(\rho k^\nu)_{;\nu} = 0 \tag{3.10}$$

and

$$(k_{\mu,\nu} - k_{\nu,\mu}) k^\nu = 0. \tag{3.11}$$

These may again be rewritten without metric by using the dynamical variables k_μ and $\tilde{V}^\nu = \tilde{\rho} k^\nu$, which are to obey

$$\tilde{V}^\nu{}_{,\nu} = 0 \tag{3.12}$$

and

$$(k_{\mu,\nu} - k_{\nu,\mu}) \tilde{V}^\nu = 0, \tag{3.13}$$

as before. However, now we want $k_\mu \tilde{V}^\mu = 0$, so that when we solve (3.13) we must pick ψ so that

$$\psi_{,\mu} \tilde{V}^\mu = 0, \tag{3.14}$$

with ξ and η picked as before. Now if we specify ρ, we need a metric such that

$$\rho k_\mu = \mathfrak{g}_{\mu\nu} \tilde{V}^\nu \tag{3.15}$$

in order for our equations to reduce to the usual ones for incoherent radiation. The divergence of the stress-energy tensor $\tilde{T}_\mu{}^\nu = k_\mu \tilde{V}^\nu$ will then vanish, as may easily be checked. Because (3.15) represents only four conditions on the form of the metric, no restrictions on the Riemannian space are implied, and we may proceed as before to solve the Einstein equations with this stress-energy tensor.

IV. ISENTROPIC FLOW OF PERFECT FLUID

It has been shown that the four conservation laws following from the vanishing of the divergence of the stress-energy tensor for a perfect fluid in isentropic flow may be written in a form quite similar to (3.1) and (3.3).[8] If

$$v^\alpha = \left(1 + \int \frac{dp}{\rho}\right) u^\alpha,$$

then the equations of motion which are equivalent to the vanishing of the divergence of the stress-energy tensor

$$T_\mu{}^\nu = \rho\left(1 + \int \frac{dp}{\rho}\right) u_\mu u^\nu - p\delta_\mu{}^\nu$$

take the form

$$(\rho u^\alpha)_{;\alpha} = 0 \tag{4.1}$$

and

$$(v_{\alpha,\beta} - v_{\beta,\alpha}) u^\beta = 0. \tag{4.2}$$

Here ρ is the conserved density of matter, since (4.1) is the equation of continuity of matter; p is the isotropic pressure, related to ρ by some given equation of state $p = p(\rho)$; and $\int dp/\rho$ represents the specific enthalpy of the fluid.

We begin by analogy with the work of Sec. III. We take as our fundamental variables a contravariant vector density $\tilde{U}^\alpha$, and a covariant vector field v_α, which are to be subject to

$$\tilde{U}^\alpha{}_{,\alpha} = 0 \tag{4.3}$$

and

$$(v_{\alpha,\beta} - v_{\beta,\alpha}) \tilde{U}^\beta = 0. \tag{4.4}$$

We may again solve (4.3) by introduction of $\tilde{V}^{[\alpha\beta]}$, and

[9] This question will be discussed more fully in a paper on relativistic hydrodynamics being prepared with J. Plebanski. A brief proof of this result is given in the Appendix.

[10] Incoherent radiation is discussed, e.g., in R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford University Press, Oxford, 1934).

(4.4) by introduction of ξ, η, and ψ, where $\xi_{,\mu}\tilde{U}^{\mu}=\eta_{,\mu}\tilde{U}^{\mu}=0$. Let $\psi_{,\mu}\tilde{U}^{\mu}=\tilde{\lambda}$; then we shall demand that $\tilde{\lambda}>0$ for the physical interpretation, but leave ψ otherwise unrestricted. This ensures that $v_{\mu}\tilde{U}^{\mu}=\psi_{,\mu}\tilde{U}^{\mu}=\tilde{\lambda}>0$. Now we must relate v_{μ} and $\tilde{U}^{\nu}$ so that our theory is equivalent to the usual hydrodynamics of a perfect fluid. It is clear that the lowered components of $\tilde{U}^{\nu}$ must be parallel to v_{μ}:

$$v_{\mu}=K\mathfrak{g}_{\mu\nu}\tilde{U}^{\nu}, \quad K>0. \tag{4.5}$$

If K is given, this becomes four conditions on the 10 components of the metric, as before. In addition, we must introduce the stress-energy tensor

$$\tilde{T}_{\mu}{}^{\nu}=v_{\mu}\tilde{U}^{\nu}-\tilde{p}\delta_{\mu}{}^{\nu}. \tag{4.6}$$

We shall show that if $p=p(\lambda)$ and $K=K(\lambda)$ are picked as suitably related functions of the scalar λ, whose scalar density was introduced above, this is equivalent to the choice of an equation of state, and that the theory then does reduce to ordinary hydrodynamics. However, note that we are forced to use scalar functions of a scalar variable here. It would not make sense to use scalar densities, since we require the argument and value of the function giving the equation of state to remain unchanged by a coordinate transformation. Thus, in addition to the four conditions on the metric (4.5), we must also assume the determinant of the metric known so that we can go from scalar densities to scalars. But this imposes a fifth condition on the metric, so that we have one real restriction on the Riemannian space in this case. However, let us proceed under this assumption. Let us define ρ by $\rho=(\lambda/K)^{1/2}$ and $\mu(\lambda)$ by $K=e^{\mu}$ (always possible since K has been assumed positive). Then either pick $p(\lambda)$ arbitrarily, and define

$$\mu(\lambda)=\int\frac{2p'(\lambda)-1}{\lambda}d\lambda; \tag{4.7}$$

or, pick $\mu(\lambda)$ arbitrarily [equivalent to picking $K(\lambda)$], and define

$$p(\lambda)=\frac{1}{2}\left(\lambda+\int\lambda\mu'(\lambda)d\lambda\right), \tag{4.8}$$

where a prime denotes the derivative with respect to λ. In either case, by elimination of λ, it follows that p and ρ are related to each other by an equation of state. It follows from a result of Lichnerowicz[8] that the local speed of sound is given by

$$V^2=\left(1+\int\frac{dp}{\rho}\right)^{-1}\left(\frac{dp}{d\rho}\right).$$

In terms of μ and λ, this means that $V^2=(1+\lambda\mu')/(1-\lambda\mu')$. Thus, if we pick $\mu(\lambda)$ in such a way that $\mu'(\lambda)\leq 0$, we will assure that the resulting equation of state keeps the local speed of sound less than the speed of light.

Using the relationship (4.7) or (4.8) between $K(\lambda)$ and $p(\lambda)$, it may now be verified that the theory does indeed reduce to the usual hydrodynamical relationships in the form given at the beginning of this section, and that the divergence of the stress-energy tensor (4.6) vanishes as a consequence of (4.3)–(4.5).

However, when we turn to the Einstein equations, since we have imposed five conditions on the metric components in this case, we cannot hope to always find a solution if we have picked the various functions arbitrarily. Thus, it seems that this approach leads to the conclusion that one real restriction on the nature of the metric is contained in the four conservation equations, so that we may say we have only been "three-quarters" successful in this case.

V. CONCLUSIONS

The basis of our procedure is to choose dynamical variables such that their form as functions of the points of the manifold may be specified in such a way that the conservation equation for the stress-energy tensor can then be satisfied without imposing any real restrictions on the nature of the metric, but only on the functional form of certain of its components in some coordinate system. Then the field equations for the metric may be solved to determine the metrical significance of our nongravitational fields.

It would be interesting to know whether there are other cases in which this procedure may be applied either fully or partially and, in general, whether some criterion for its applicability can be given. We note that in the cases we have considered where it works, as well as the one case where it does not fully work, the dynamical equations for the nongravitational fields follow fully from the conservation law. Hence this criterion, which one might first think of, cannot be necessary and sufficient. In the case of the electromagnetic field, it is easy to introduce dynamical variables in terms of which the Maxwell equations may be written nonmetrically.[11] However, the usual constitutive equations relating the dynamical variables involve so many components of the metric that the method seems of little practical interest.

We have been able to use the ideas of this paper, together with restrictive symmetry assumptions, to actually solve the equations for the fields in a few trivially simple cases, where the solutions were already known by other methods. For example, the $1/r$ static solution to the massless scalar field equation in Minkowski space may be easily be transformed into a nonmetrical solution which can be fitted into any static spherically symmetric Riemann space. It remains to be seen whether the method will prove useful in actually finding interesting solutions of the relevant field equations.

[11] A further discussion, with references, of this point is found in J. Stachel, Acta Phys. Polon. (to be published).

None of the work of this paper (nor that of Pereira discussed in the Introduction) should be interpreted as contradicting the idea that the conservation laws $T_\mu{}^\nu{}_{;\nu}=0$ constitute four conditions on the motion of the sources of the gravitational field which, under certain conditions, may even serve to determine that motion completely.[12] In the case of incoherent matter, for example, those conditions require that the streamlines of the matter velocity field be geodesics of the Riemann space; so that if the matter is confined to timelike world tubes of narrow cross section (filamentary matter), these filaments must be timelike geodesics of the space-time.[8] Now, if we are given a congruence of curves in a bare manifold, there are an infinite number of ways to metrize the manifold in such a way as to make the congruence timelike and geodesic. The work of Sec. III shows if the congruence is defined by Eqs. (3.4)–(3.6), then any metric obeying (3.7) will make the congruence geodesic. Thus, once the Einstein field equations are solved, the matter will indeed be moving along timelike geodesics of the Riemann space.

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[12] For the problem of motion in general relativity see, e.g., L. Infeld and J. Plebanski, *Motion and Relativity* (Pergamon Press, Inc., New York, 1961); or J. Goldberg, in *Gravitation: an Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962), pp. 102–129.

APPENDIX

Schouten[13] proves that a simple covariant multivector is a product of gradients if and only if its rotation vanishes. For a bivector $F_{\mu\nu}$, the vanishing of its rotation is equivalent to the vanishing of the divergence of its dual bivector density $\tilde{F}^{\kappa\lambda}=\frac{1}{2}\tilde{\epsilon}^{\kappa\lambda\mu\nu}F_{\mu\nu}$. A criterion for a bivector to be simple is that its determinant vanish. We can thus restate Schouten's result in the case of a bivector in the form: A covariant bivector whose determinant vanishes and whose dual has vanishing divergence is the antisymmetrized product of two gradients. Another form of this result may be gotten by noting that the rotation of a bivector vanishes if and only if it is the curl of some vector (locally). It follows that a covariant bivector which is the curl of a vector and whose determinant vanishes is the antisymmetrized product of two gradients. In this latter form we may use this result to find the general solution to the set of equations which we have indicated form much of the basis of relativistic hydrodynamics:

$$(v_{\mu,\nu}-v_{\nu,\mu})U^\mu=0, \qquad \text{(A1)}$$

It is clear that $v_{\mu,\nu}-v_{\nu,\mu}$ satisfies the conditions of our theorem, so that we may write

$$v_{\mu,\nu}-v_{\nu,\mu}=\xi_{,\mu}\eta_{,\nu}-\xi_{,\nu}\eta_{,\mu}, \qquad \text{(A2)}$$

where ξ and η are two scalar fields. But then $v_\mu-\xi\eta_{,\mu}$ (or $v_\mu-\eta\xi_{,\mu}$ as well) is a vector whose curl vanishes, so that it must be a gradient (this is actually the "zeroth-order" case of Schouten's result, although much better known otherwise). Thus finally we have

$$v_\mu=\xi\eta_{,\mu}+\psi_{,\mu} \qquad \text{(A3)}$$

as the general solution of (A1).

[13] J. Schouten, *Ricci Calculus* (Springer-Verlag, Berlin, 1954), p. 84.