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Exact Solution of the Dirac Equation with an Equivalent Oscillator Potential*

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It is shown that a Hamiltonian obtained by adding an energy term $\lambda^2\rho_2\boldsymbol{\sigma}\cdot\mathbf{r}K|\boldsymbol{\sigma}\cdot\mathbf{L}+1|^{-1}$ to the free-particle Dirac Hamiltonian possesses exact solutions and a discrete spectrum of high degeneracy. In the nonrelativistic limit this leads to an isotropic harmonic oscillator with a spin-orbit coupling term of the Thomas-Frenkel form. The symmetry is likely to be at least as high as SU_3.

INTRODUCTION

THE "symmetric Hamiltonian" introduced by Biedenharn[1] for the Coulomb field differs from the exact Dirac Hamiltonian by the "fine-structure interaction"

$$H_{\mathrm{fs}}\equiv\rho_2\frac{\boldsymbol{\sigma}\cdot\mathbf{r}}{r^2}K\left\{\left[1+\left(\frac{\alpha Z}{K}\right)^2\right]^{1/2}-1\right\},\qquad(1)$$

where K is the Dirac operator $\rho_3(\boldsymbol{\sigma}\cdot\mathbf{L}+1)$, and α the Sommerfeld fine-structure constant. It has been shown that removal of this fine structure restores the symmetry of the nonrelativistic hydrogen atom and the group of this symmetric Hamiltonian has been established as (for bound states) $O(4,2)\otimes SU(2)$.[2]

It is the purpose of this paper to point out that if, on the other hand, an interaction

$$V(\mathbf{r})\equiv\lambda^2\rho_2\boldsymbol{\sigma}\cdot\mathbf{r}K[|(\boldsymbol{\sigma}\cdot\mathbf{L}+1)|]^{-1}\qquad(2)$$

is added to the free-particle Hamiltonian

$$H_p=\rho_1\boldsymbol{\sigma}\cdot\mathbf{p}+\rho_3m_0,$$

the operator $\mathbf{H}=H_p+V$ possesses exact eigenvectors and a discrete eigenvalue spectrum. In units of $\hbar=c=1$, λ is the isotropic harmonic-oscillator parameter

$$\lambda^2=m_0\omega,\qquad(3)$$

where ω is the classical frequency of the oscillator. Furthermore, as will be shown later, the degeneracy is four times the nonrelativistic case and this appears to mean that the group of this Hamiltonian should be at least as high as $SU(3)$.

EIGENFUNCTIONS AND EIGENVALUES

The existence of exact solutions of $\mathbf{H}$ stems from the interesting fact that ladder operators exist which connect the contiguous radial solutions of the Schrödinger equation with the oscillator potential. Closely paralleling the case of the Coulomb field we have

$$\begin{aligned}\left[\left(\frac{d}{dr}+\frac{1}{r}\right)+\frac{l}{r}+\lambda^2r\right]F_{vl}&=2\lambda(v+l+\tfrac{1}{2})^{1/2}F_{v\,l-1},\\ \left[\left(\frac{d}{dr}+\frac{1}{r}\right)-\frac{l+1}{r}-\lambda^2r\right]F_{vl}&=-2\lambda(v+l+\tfrac{3}{2})^{1/2}F_{v\,l+1}.\end{aligned}\qquad(4)$$

These radial functions pertain to the nonrelativistic eigenvalue $\omega(2v+l+\frac{3}{2})$ and are normalized such that

$$\int_0^\infty F_{vl}^2r^2dr=1.\qquad(5)$$

With the help of the basis functions of the spin-angle group $O(3)\otimes SU(2)$, we introduce the spinors[1]

$$\begin{aligned}|v\kappa\mu\rangle&\to\chi_\kappa{}^\mu F_{vl},\\ |v-\kappa\mu\rangle&\to i\chi_{-\kappa}{}^\mu F_{v\bar{l}}.\end{aligned}\qquad(6)$$

Remembering that the pseudoscalar operator $\boldsymbol{\sigma}\cdot\hat{r}=\boldsymbol{\sigma}\cdot\mathbf{r}/r$ essentially takes κ into $-\kappa$

$$\boldsymbol{\sigma}\cdot\hat{r}\chi_\kappa{}^\mu=-\chi_{-\kappa}{}^\mu,\qquad(7)$$

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[1] L. C. Biedenharn, Bull. Am. Phys. Soc. 7, 364 (1962); L. C. Biedenharn and N. V. V. J. Swamy, Phys. Rev. 133, B1353 (1964).

[2] I. A. Malkin and V. I. Manko, P. N. Lebedev Physical Institute, Moscow, USSR, Report No. N 9, 1968 (unpublished).

[3] L. C. Biedenharn, Phys. Rev. 126, 845 (1962).

that the χ_κ^μ are eigenvectors of $(\boldsymbol{\sigma}\cdot\mathbf{L}+1)$,

$$(\boldsymbol{\sigma}\cdot\mathbf{L}+1)\chi_\kappa^\mu=-\kappa\chi_\kappa^\mu, \tag{8}$$

and also that the operator $\boldsymbol{\sigma}\cdot\mathbf{p}$ has a representation

$$\boldsymbol{\sigma}\cdot\mathbf{p}=-i\frac{\boldsymbol{\sigma}\cdot\mathbf{r}}{r}\left[\left(\frac{d}{dr}+\frac{1}{r}\right)-\frac{\boldsymbol{\sigma}\cdot\mathbf{L}+2}{r}\right], \tag{9}$$

it is now a straightforward calculation to prove that the four-component spinor

$$\begin{bmatrix} |v\kappa\mu\rangle \\ \dfrac{s_\kappa(E-m_0)}{2\lambda(v+|\kappa|+\frac{1}{2})^{1/2}}|v-\kappa\mu\rangle \end{bmatrix}=\Phi_{v\kappa\mu} \tag{10}$$

satisfies the eigenvalue equation

$$\mathbf{H}\Phi_{v\kappa\mu}=[m_0^2+4\lambda^2(v+|\kappa|+\tfrac{1}{2})]^{1/2}\Phi_{v\kappa\mu}. \tag{11}$$

The s_κ multiplying the small component of $\Phi_{v\kappa\mu}$ is a phase factor $\kappa/|\kappa|$. In deriving the above solution use has, of course, been made of the algebra of the Dirac ρ operators and the Pauli σ operators, as also the fact that ρ and σ commute. We employ the standard representation with ρ_3 and σ_3 diagonal. Since $(\boldsymbol{\sigma}\cdot\mathbf{L}+1)^2$ commutes with $\mathbf{H}$, it happens that both κ and $-\kappa$ appear in the solution.

DISCUSSION

$V(\mathbf{r})$ can be regarded as a "potential" if the requirement of reflection invariance is relaxed. The discrete energy levels are then given by

$$E=[m_0^2+4\lambda^2(v+|\kappa|+\tfrac{1}{2})]^{1/2}, \tag{12}$$

with the ground-state energy $(m_0^2+b\lambda^2)^{1/2}$. The energies are independent of the quantum number μ since J_z commutes with $\mathbf{H}$. They are also independent of the sign of κ and further they depend on the radial quantum number v and the Dirac quantum number κ only through the combination $v+|\kappa|$. It is easy to see that the degenerate states belonging to the first six energy levels are, respectively, 4, 12, 24, 40, 60, and 84 in number, whereas in the Schrödinger case, the corresponding states are 1, 3, 6, 10, 15, and 21 in number.

A Foldy-Wouthuysen transformation of $\mathbf{H}$ yields a nonrelativistic Hamiltonian correct to order $(1/m_0)^2$

$$H''=(2m_0)^{-1}\mathbf{p}^2+(2m_0)^{-1}\lambda^4r^2 \\ +\omega s_{-\kappa}[(\boldsymbol{\sigma}\cdot\mathbf{L}+2)+\tfrac{1}{2}]. \tag{13}$$

This is precisely the oscillator Hamiltonian with spin-orbit coupling when we remove the constant additional term by an appropriate scaling of the energy levels. It is noteworthy that, relative to the nonrelativistic potential, the spin-orbit energy can be written

$$\frac{2}{\lambda^2}\frac{1}{r}\frac{\partial V}{\partial r}\mathbf{S}\cdot\mathbf{L},$$

which is of the familiar Thomas-Frenkel form for this nonelectric force. To a first approximation, Eq. (12) gives a nonrelativistic (NR) discrete spectrum

$$E_{\mathrm{NR}}=\omega(2v+2|\kappa|+1). \tag{14}$$

While the levels are equally spaced, the "zero-point energy" becomes 3ω, twice the well-known value for a three-dimensional oscillator.

The high degeneracy of the discrete levels leads us to suspect that the symmetry group of this Hamiltonian must be at least as high as $SU(3)$. This question as well as the determination of the generators of this group is currently under study.