Recoil Damping in Heavy-Ion Transfer Reactions*

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An earlier suggestion by the authors that the unexpected features of angular distributions in heavy-ion transfer reactions at energies above the Coulomb barrier may be explained by including recoil and finite-range effects in a direct-reaction theory, is examined in detail. It is shown that the finite mass of the transferred particle may be taken into account approximately by the inclusion of a recoil phase factor in the transfer function of the usual distorted-wave Born amplitude. The implications of modifying the transfer function are worked out with the help of a sharp-cutoff diffraction model for the scattering of the strongly absorbing nuclear cores. Simple, closed expressions for the transfer differential cross sections are obtained. Unlike the earlier work, these expressions are valid for arbitrary angular momentum transfers, and intrinsic spins are included. When the zero-range limit is used or the mass of the transferred particle is neglected, the model predicts extreme diffraction oscillations in the angular distributions. However, if finite-range and recoil terms are retained, then, at sufficiently high energies and large angular momentum transfers, the theory gives strong damping of the diffraction oscillations. The resulting structureless angular distributions fall off with a $1/q^3$ dependence on the linear momentum transfer q, in excellent agreement with experiment. The theory is applied to the recent experimental results of Birnbaum, Overley, and Bromley for the $C^{12}(N^{14}, N^{13})C^{13}$ reaction. Substantial damping of the angular distributions is predicted.

I. INTRODUCTION

In recent experimental studies[1–4] of reactions induced by heavy ions at energies well above the Coulomb barrier, angular distributions for both single nucleon and cluster transfer have shown an almost complete absence of structure. The angular distributions are monotonic decreasing functions of the linear momentum transfer[5] in remarkable contrast to the oscillatory distributions found in elastic heavy-ion scattering and in transfer reactions initiated by protons and deuterons. The smoothness of the angular distributions is also unexpected from previous theoretical considerations of the reaction mechanism. At incident energies well above the Coulomb barrier, the strong nuclear interaction is expected to dominate the Coulomb repulsion between the heavy nuclear cores. Phenomenological diffraction models,[6–9] which take account of the strong absorption in the entrance and exit channels, predict large oscillations in the transfer angular distributions when Coulomb damping becomes negligible. The disagreement with experiment is rather striking in view of the successful application of strong absorption models to both elastic scattering and transfer reactions at lower energies[7–9] as well as to reactions induced by lighter projectiles.[10–13]

Two different explanations for the smoothness of the transfer distributions have been proposed. Both include strong absorption and both rely on the interference of amplitudes corresponding to angular momentum transfers of different parity to wash out diffraction oscillations, but the mechanisms responsible for this interference are quite different in the two explanations. Dar and Kozlowsky point out[14] that if there is strong configuration mixing in the bound states of the transferred particle or if core excitation takes place during transfer, both odd and even angular momentum transfers contribute to the reaction, resulting in a smoothing of the oscillations predicted by the theory of Ref. 9. On the other hand, in a previous publication[15] the present authors proposed that the mixing of odd and even angular momentum transfers is kinematical in origin and is independent of any specific assumptions about the structure of the nuclei involved.

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It was suggested in Ref. 15 that if terms of order μ, where μ is the ratio of the mass of the transferred particle (or cluster) to the mass of one of the heavy cores, were retained in the usual distorted-wave Born amplitude, that damping of the transfer angular distributions results. The consequences of retaining such "recoil" terms were evaluated in a particular sharp-cutoff diffraction model in which harmonic-oscillator wave functions were used for the bound states of the transferred particle to the cores, and intrinsic spins were ignored. The chief limitation of this earlier work, which prevented a detailed comparison of the model with experiment, was that the transferred particle was taken to have zero angular momentum with respect to the donor core in the entrance channel.

The purpose of this paper is to remove these limitations so that a quantitative test of the model with the recent experimental results of Birnbaum, Overley, and Bromley[4] is possible. In particular, the diffraction model is extended to include arbitrary angular momentum transfers and intrinsic spins.

In Sec. II it is shown within the context of the distorted-wave Born approximation that the finite mass of the transferred particle may be taken into account approximately by the inclusion of a "recoil" phase factor in the usual transfer function which contains the nuclear structure information.

With the assumption that the bound states of the transferred particle are adequately described by harmonic-oscillator states, simple expressions for the modified transfer function are obtained in Sec. III. The modified transfer function is split into two parts, one containing a Weyl operator which includes the recoil factor, and the other containing the finite-range potential responsible for the transfer. This decomposition shows that the recoil factor produces a relative displacement of the bound-state wave functions in momentum space as well as the displacement in configuration space which is customary in direct reaction amplitudes. The details of the evaluation of the matrix element of the Weyl operator are given in the Appendix.

In Sec. IV the expression for the modified transfer function found in Sec. III, together with the ring-locus diffraction model[10–12] for the scattering wave functions, is used to obtain analytic expressions for the transfer differential cross section.

The transfer angular distribution is the sum of two terms, one of which falls off smoothly with a $1/q^3$ dependence on the linear momentum transfer q, the other has oscillations characteristic of a diffraction theory. In Sec. V, we define a damping parameter as the ratio of the magnitudes of the oscillatory and monotonic parts of the angular distribution. In general, the degree of damping predicted by the theory, as measured by the damping parameter, increases as the magnitude of the maximum angular momentum transfer which is possible in the reaction increases, and also as the incident energy increases.

In Sec. V A we note that if either recoil is neglected or if the zero-range approximation is made, there is no damping of the angular distributions. In Sec. V B the results for the special case discussed in Ref. 15 are recovered, the damping parameter taking a particularly simple form. The effects of intrinsic spin are considered in Sec. V C.

Finally, in Sec. V D the results of calculations of the damping parameter for the reaction $C^{12}(N^{14}, N^{13})C^{13}$ are compared with the experimental results of Ref. 4.

II. GENERAL FORMULATION

We consider rearrangement process of the type

$$(a+c)+b \rightarrow a+(b+c), \tag{1}$$

where the transferred particle or cluster c, which is initially bound to the core a in the nucleus $(a+c)$ and is bound in the final state to the nucleus b, has mass m_c much smaller than the masses m_a and m_b of a and b. It is assumed that the internal degrees of freedom of the three nuclear systems a, b, and c are undisturbed by the transfer and that the many-body interactions between them may be simulated by effective two-body potentials V_{ab}, V_{ac}, and V_{bc}.

The amplitude for rearrangement scattering which includes distorting potentials in the initial and final channels is

$$T_{fi}=\langle\Phi_f^{(-)}\,|\,V_i+V_f^{\dagger}(E-H+i\epsilon)^{-1}V_i\,|\,\Phi_i^{(+)}\rangle. \tag{2}$$

The initial state $\Phi_i^{(+)}$ describes the scattering of the systems $(a+c)$ and b interacting by a potential W_i,

$$|\,\Phi_i^{(+)}\rangle=[1+(E+i\epsilon-H_i-W_i)^{-1}W_i]\,|\,\Phi_i\rangle. \tag{3}$$

The operator H_i is expressed in terms of the complete Hamiltonian of the system H by $H_i=H-V_{ab}-V_{bc}$, and Φ_i is the energy eigenstate of H_i in which $(a+c)$ has definite momentum $\mathbf{k}_i$ relative to b. Similarly, the final state is distorted by a potential W_f,

$$|\,\Phi_f^{(-)}\rangle=[1+(E-i\epsilon-H_f-W_f)^{-1}W_f]\,|\,\Phi_f\rangle. \tag{4}$$

The residual interactions V_i and V_f in the incident and final channels, respectively, are defined by

$$V_i=V_{ab}+V_{bc}-W_i$$

and

$$V_f=V_{ab}+V_{ac}-W_f. \tag{5}$$

It is shown in Ref. 16, for example, that in order for Eq. (2) to be an exact expression for the amplitude, the potential W_i must be chosen such that the state

[16] K. R. Greider and L. R. Dodd, Phys. Rev. **146**, 671 (1966).

$|\Phi_i^{(+)}\rangle$ has no component in the final channel. This condition is satisfied in a natural way, which is well suited to the heavy-ion transfer problem, by taking W_i and W_f in the coordinate representation as functions of the vectors $\mathbf{r}_i$ and $\mathbf{r}_f$, respectively, which join the centers of mass of the two systems in the incident and final channels, as shown in Fig. 1. The final and initial states are then each simply products of a wave function of relative motion and a wave function describing the internal structure, i.e.,

$$\langle \mathbf{r}_i, \mathbf{r}_{ac} | \Phi_i^{(+)}\rangle = \psi_i(\mathbf{r}_{ac})\chi_i^{(+)}(\mathbf{k}_i, \mathbf{r}_i)$$

and (6)

$$\langle \mathbf{r}_f, \mathbf{r}_{bc} | \Phi_f^{(-)}\rangle = \psi_f(\mathbf{r}_{bc})\chi_f^{(-)}(\mathbf{k}_f, \mathbf{r}_f).$$

Here $\psi_i(\mathbf{r}_{ac})$ is the single-particle wave function of c with respect to the core a, and $\psi_f(\mathbf{r}_{bc})$ is the final bound state of c and b. In the limiting case where the masses m_a and m_b of the cores a and b are infinite, the vectors $-\mathbf{r}_i$ and $\mathbf{r}_f$ coincide with the vector $\mathbf{r}\equiv\mathbf{r}_{ab}$ and the potentials W_i and W_f may be used to remove completely the core-core interaction V_{ab} from the residual interactions without violating the restriction imposed on the interaction W_i in the last paragraph. With the further assumption that the term in Eq. (2) involving the full Green's function is small, the usual distorted-wave Born amplitude is obtained:

$$T_{fi}^0 = \int \chi_f^{(-)*}(\mathbf{k}_f, \mathbf{r})\chi_i^{(+)}(\mathbf{k}_i, \mathbf{r})G_{fi}^0(\mathbf{r})d\mathbf{r}, \quad (7)$$

with the transfer function $G_{fi}^0(\mathbf{r})$, which contains the nuclear-structure information, taking the form

$$G_{fi}^0(\mathbf{r}) = \int \psi_f^*(\mathbf{r}'-\mathbf{r})V_{ac}(\mathbf{r}')\psi_i(\mathbf{r}')d\mathbf{r}'. \quad (8)$$

If the masses m_a and m_b are finite but much larger than the mass of the transferred particle m_c, we may still write the amplitude in the simple form (7), but with some important modifications to the transfer function, which lead to significant changes in the predicted cross sections.

The distorted wave $\chi_i^{(+)}(\mathbf{k}_i, \mathbf{r}_i)$ may be written formally as an amplitude function $B_i(\mathbf{k}_i, \mathbf{r}_i)$ modulating a plane wave $e^{i\mathbf{k}_i\cdot\mathbf{r}_i}$. This representation is always possible but is most useful when the phase of the modulating factor B_i is more slowly varying than the phase of the plane wave. For our purpose, we require that the phase of the distorted wave at the nuclear surface be given locally by the phase of the corresponding plane wave. That is, $B_i^{(+)}(\mathbf{k}_i, \mathbf{r}_i)$ should be a smooth function of $\mathbf{r}_i$. Then, since the coordinate space integrations in (2) are limited by the ranges of the bound states, the amplitude functions $B_i^{(+)}(\mathbf{k}_i, \mathbf{r}_i)$ and $B_f^{(-)}(\mathbf{k}_f, \mathbf{r}_f)$ may be approximated by $B_i^{(+)}(\mathbf{k}_i, -\mathbf{r})$ and $B_f^{(-)}(\mathbf{k}_f, \mathbf{r})$, and the wave functions $\chi_i^{(+)}$ and

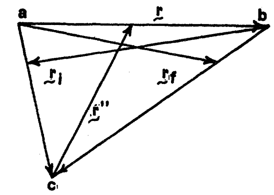

FIG. 1. Coordinate system.

$\chi_f^{(-)}$ replaced in the matrix element (2) by

$$\chi_i^{(+)}(\mathbf{k}_i, \mathbf{r}_i) \approx B_i^{(+)}(\mathbf{k}_i, -\mathbf{r}) \times \exp\{i\mathbf{k}_i\cdot[\mathbf{r}(\mu_{ba}\mu_{ca}-1)-\mu_{ca}\mathbf{r}'']\}$$

and

$$\chi_f^{(-)}(\mathbf{k}_f, \mathbf{r}_f) \approx B_f^{(-)}(\mathbf{k}_f, \mathbf{r}) \times \exp\{i\mathbf{k}_f\cdot[\mathbf{r}(1-\mu_{cb}\mu_{ab})-\mu_{cb}\mathbf{r}'']\}, \quad (9)$$

where $\mu_{ca} = m_c/(m_c+m_a)$, etc.

It is seen from Eqs. (6) and (9) that the approximation Eq. (9) is valid if in the neighborhood of the nuclear surface the amplitude functions satisfy the conditions

$$a\,|\nabla B_i^{(+)}|/|B_i^{(+)}| \ll \mu_{ca}^{-1}$$

and (10)

$$b\,|\nabla B_f^{(-)}|/|B_f^{(-)}| \ll \mu_{cb}^{-1},$$

with a and b the ranges of the initial and final bound states. The conditions (10) are satisfied by typical optical-model wave functions describing elastic scattering at medium energies accompanied by strong absorption[17] and may be expected to hold for heavy-ion scattering. For Coulomb waves, the condition (10) becomes

$$k_i a/\eta + 1 \gg k_i a \mu_{ca}, \quad (11)$$

where η is the usual Sommerfeld parameter. Thus, the above approximation is valid for $\eta > 1$ as well as small η provided that the incident energy is sufficiently great. For the reactions considered in Sec. V D, the condition (11) is satisfied, and we shall ignore Coulomb distortion in the scattering states entirely.

When m_a and m_b are finite, there is incomplete cancellation of the core-core potential V_{ab} in the residual potentials V_i and V_f in the matrix element (2) in addition to the mixed coordinate dependence of the scattering states. But on expansion in powers of μ_{ca} and μ_{cb} of all functions appearing in the amplitude (2), except the rapidly varying phases of the distorted waves, we have, to zero order in μ_{ca} and μ_{cb} from Eqs.

[17] K. A. Amos, Nucl. Phys. **77**, 225 (1966); I. E. McCarthy and D. L. Pursey, Phys. Rev. **122**, 578 (1961).

(2), (6), and (9),

$$T_{fi}=\int B_f^{(-)*}(\mathbf{k}_f,\mathbf{r})B_i^{(+)}(\mathbf{k}_i,-\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}G_{fi}(\mathbf{r})d\mathbf{r},\quad(12)$$

where the new transfer function, which is to be compared with G_{fi}^0 of Eq. (8),

$$G_{fi}(\mathbf{p},\mathbf{r})=\exp(-i\mathbf{p}\cdot\mathbf{r}\mu_{ba})\int e^{i\mathbf{p}\cdot\mathbf{r}'}\psi_f^*(\mathbf{r}'-\mathbf{r})V_{ac}(\mathbf{r}')\psi_i(\mathbf{r}')d\mathbf{r}',\quad(13)$$

includes a "recoil" factor $e^{i\mathbf{p}\cdot\mathbf{r}'}$ with the recoil momentum,

$$\mathbf{p}=\mu_{ca}\mathbf{k}_i-\mu_{cb}\mathbf{k}_f.\quad(14)$$

The momentum $\mathbf{q}$ is very nearly the momentum transfer,

$$\mathbf{q}=\mathbf{k}_i(\mu_{ba}\mu_{ca}-1)+\mathbf{k}_f(\mu_{ab}\mu_{cb}-1).\quad(15)$$

Clearly, when $1/p$ is of the order of the range of bound states ψ_i and ψ_f, there may be either constructive or destructive interference between the phase factor and the wave functions ψ_i and ψ_f. For example, for wave functions corresponding to small values of angular momentum and, hence, having few nodes, the magnitude of the transfer function is reduced. It will become clear from the following that, in general, the transfer functions G_{fi} differs considerably from the unmodified transfer function G_{fi}^0 in which all terms of order μ_{ca} and μ_{cb} are neglected.

It is important to note that a potential V_{ac} of finite range is essential if recoil effects are to be taken into account. If the zero-range approximation for V_{ac} is made in evaluating the transfer function (13), the recoil factor is unity and the damping effects to be discussed are lost.

For completeness, we also remark that an alternative definition from the transition amplitude may be used in place of Eq. (2):

$$T_{fi}^{(+)}=\langle\Phi_f^{(-)}\mid V_f^\dagger+V_f^\dagger(E-H+i\epsilon)^{-1}V_i\mid\Phi_i^{(+)}\rangle.\quad(2')$$

The argument of this section carries through for (2′), leading to an expression for the transition amplitude identical with Eq. (12), except that G_{fi} of Eq. (13) is replaced by

$$G_{fi}^{(+)}=\exp(i\mathbf{p}\cdot\mathbf{r}\mu_{ab})\int e^{i\mathbf{p}\cdot\mathbf{r}'}\psi_f^*(\mathbf{r}')V_{bc}(\mathbf{r}')\psi_i(\mathbf{r}+\mathbf{r}')d\mathbf{r}'.\quad(13')$$

III. EVALUATION OF THE TRANSFER FUNCTION

In general, the transfer function of Eq. (13) which includes recoil may be computed numerically, but with the aim of understanding the effects of the modified transfer function on the cross section, it is valuable to adopt some simplifying assumptions to obtain analytic expressions for the transfer function. We assume that the initial state of particle c with respect to a in the nucleus $(a+c)$ may be described by a single-particle state $|nlm\rangle$ of a three-dimensional isotropic harmonic oscillator of strength ν. After transfer, particle c is assumed to be bound to the nucleus b in a state $|n'l'm'\rangle$ of an oscillator of strength ν'. Furthermore, the interaction V_{ac} is taken to be local and central. For the present, we neglect spin, which, however, is included in Sec. V C.

The transfer function (13) may be written with the aid of displacement operators in momentum and coordinate space[18,19] as

$$G_{n'l'm',nlm}(\mathbf{p},\mathbf{r})=\exp(-i\mathbf{p}\cdot\mathbf{r}\mu_{ba})\times\langle n'l'm'\mid e^{i\mathbf{r}\cdot\mathbf{P}}e^{i\mathbf{p}\cdot\mathbf{R}}V_{ac}\mid nlm\rangle,\quad(16)$$

$\mathbf{P}$ and $\mathbf{R}$ denoting momentum and position operators, whereas $\mathbf{p}$ and $\mathbf{r}$ are c numbers. A convenient form for the transfer function which separates the potential and recoil operators is obtained by introducing a complete set of final oscillator states in the matrix element (16), so that

$$G_{n'l'm',nlm}(\mathbf{p},\mathbf{r})=e^{i\mathbf{p}\cdot\mathbf{r}(\frac{1}{2}-\mu_{ba})}\sum_{n''}U_{n'l'm',n''lm}(\mathbf{p},\mathbf{r})V_{n''n,l},\quad(17)$$

with

$$U_{n'l'm',nlm}(\mathbf{p},\mathbf{r})=\langle n'l'm'\mid\exp(i\mathbf{r}\cdot\mathbf{P}+i\mathbf{p}\cdot\mathbf{R})\mid nlm\rangle\quad(18)$$

and

$$V_{n'n,l}=\langle n'l'm'\mid V_{ac}\mid nlm\rangle.\quad(19)$$

The Baker-Hausdorff identity[19] has been used to combine the position and momentum displacement operators into the Weyl operator of Eq. (18). We see from Eq. (18) that the transfer function which includes recoil depends on the overlap of oscillator states which are displaced in both momentum space and configuration space. A similar expression holds for the prior form of the transfer function Eq. (13′),

$$G_{n'l'm',nlm}^{(+)}(\mathbf{p},\mathbf{r})=\exp[i\mathbf{p}\cdot\mathbf{r}(\tfrac{1}{2}-\mu_{ba})]\times\sum_{n''}V_{n'n'',l}U_{n''l'm',nlm}(\mathbf{p},\mathbf{r}).\quad(20)$$

The post form (17) and the prior form (20) of the

[18] K. Gottfried, *Quantum Mechanics* (W. A. Benjamin, Inc., New York, 1966), p. 260.

[19] J. Klauder and E. Sudarshan, *Fundamentals of Quantum Optics* (W. A. Benjamin, Inc., New York, 1968), Chap. 7.

transfer function are equivalent only when either the initial and final single-particle states are identical or when the range of the potential V is much greater than the range of the bound state.[20]

The form of Eqs. (17) and (20) is useful since the recoil effects are isolated in the matrix element of the Weyl operator. In the Appendix the following explicit expression for the matrix elements (18) of the Weyl operator between oscillator states in the $\{nlm\}$ representation is derived:

$$U_{n'l'm',nlm}(\mathbf{p},\mathbf{r}) \propto e^{-p^2/4}e^{-q^2/4} \times \sum_{l_1l_2l_3m_1m_2m_3} i^{l_1+l_2-l_3}[\hat{l}_1\hat{l}_2\hat{l}_3{}^2\hat{l}\hat{l}']^{1/2} Y_{l_1}{}^{m_1*}(\hat{s})Y_{l_2}{}^{m_2}(\hat{t}) \times\begin{pmatrix} l' & l_2 & l_3 \\ m' & m_2 & m_3\end{pmatrix}\begin{pmatrix} l & l_1 & l_3 \\ m & m_1 & m_3\end{pmatrix}\begin{pmatrix} l & l_1 & l_3 \\ 0 & 0 & 0\end{pmatrix} \times\begin{pmatrix} l' & l_2 & l_3 \\ 0 & 0 & 0\end{pmatrix} D_{n'n,l_1l_2l_3}(s,t), \quad (21)$$

where the complex vectors $\mathbf{s}$ and $\mathbf{t}$ are defined by

$$\mathbf{s}=\mathbf{r}+i\mathbf{p},\ \mathbf{t}=\mathbf{r}-i\mathbf{p}, \quad (22)$$

and the notation for the angular momentum algebra, which is explained further in the Appendix, is that of Edmonds.[21] The functions $D_{n'n,l_1l_2l_3}$ are defined in terms of spherical Bessel functions by

$$D_{n'n,l_1l_2l_3}(s,t)=\left[\frac{\partial^n}{\partial u^n}\frac{\partial^{n'}}{\partial u'^{n'}}j_{l_1}(us)j_{l_2}(u't)j_{l_3}(2iuu')\right], \quad u=u'=0. \quad (23)$$

The parameter differentiations are carried out in the Appendix, resulting in simple expressions for the D functions. For small values of n or n', the sums over l_1, l_2, and l_3 are limited to a few terms, making the expression (21) for U quite tractable for practical calculations.

The matrix elements of V, Eq. (19), are easily calculated. They are given for a Gaussian potential in the Appendix.

It is useful to define a transfer function $G_{LM,n'l'nl}$ for transfer of definite angular momentum L and component M:

$$G_{n'l'm',nlm}=\sum_{LM}\hat{L}(-1)^{m'}\begin{pmatrix} l' & l & L \\ -m' & m & M\end{pmatrix}G_{LM,n'l'nl}. \quad (24)$$

The differential cross section for the transfer from a state n, l to a state n', l' is

$$\frac{d\sigma_{n'l',nl}}{d\Omega}=\mu_i\mu_f\frac{k_f}{k_i}\left(\frac{1}{2\pi\hbar^2}\right)^2\hat{l}^{-1}\sum_{mm'}|T_{n'l'm',nlm}|^2, \quad (25)$$

where μ_i and μ_f are reduced masses in the initial and final channels. Defining $T_{LM,n'l'nl}$ in terms of $T_{n'lm',nlm}$ in the manner of Eq. (24), we have from Eq. (12),

$$T_{LM,n'l'nl}=\int B_i^{(+)}(\mathbf{k}_i,-\mathbf{r})B_f^{(-)*}\times(\mathbf{k}_f,\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}G_{LM,n'l'nl}(\mathbf{p},\mathbf{r})d\mathbf{r}. \quad (26)$$

Summation over the magnetic substates in Eq. (25) yields the differential cross section in the form

$$\frac{d\sigma_{n'l',nl}}{d\Omega}=\mu_i\mu_f\frac{k_f}{k_i}\left(\frac{1}{2\pi\hbar^2}\right)^2\hat{l}^{-1}\sum_{LM}\hat{L}\,|T_{LM,n'l'nl}(\mathbf{k}_i,\mathbf{k}_f)|^2. \quad (27)$$

IV. DIFFRACTION MODEL

We turn to the evaluation of the amplitude (26). The transfer function (17) could be used in present distorted-wave Born computations, where the scattering states $\chi_i^{(+)}$ and $\chi_f^{(-)}$ of Eq. (9) are derived from optical potentials. However, here we shall use a diffraction model for the distorted waves which avoids partial-wave expansions, leading to simple results in closed form for the differential cross section (27) which clearly exhibit the effects of the recoil factor.

Because of the strong nuclear absorption for scattering of complex nuclei well above the Coulomb barrier, the distorted wave $\chi_i^{(+)}$ is represented in configuration space by a plane wave which vanishes inside a sphere of radius r_0, given by the sum of the radii of the colliding nuclei, and also in the shadow region.[10–12] Thus the factor $B_i^{(+)}(\mathbf{k}_i,\mathbf{r})$ in Eq. (9) is zero when the vector $\mathbf{r}$ lies within a semi-infinite cylinder of radius r_0 which is aligned in the direction of $\mathbf{k}_i$, capped by a half-sphere, and is unity elsewhere. A similar representation holds for $\chi_f^{(-)}$. If we confine our attention to scattering at forward angles, the product $B_i^{(+)}B_f^{(-)*}$ in the integrand of Eq. (26) vanishes inside a cylinder of infinite length, aligned along the beam direction. Also, $G_{LM,n'l'nl}$ is rapidly diminishing in magnitude with increasing r, so that the principal contribution to the amplitude comes from annular region which lies in the plane

[20] This ambiguity is, of course, not peculiar to the present model but is inherent in all distorted-wave Born theories. It is particularly difficult to decide on the most appropriate form in this case since, unlike stripping theories, there is symmetry between the final and initial channels and a choice cannot be made on the grounds of physical or mathematical simplicity. However, the angular distributions for the post and prior forms have the same qualitative features, and for definiteness we use the post form as the basis of our discussion.

[21] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N.J., 1957).

perpendicular to the momentum vector $\mathbf{k}_i$. The preceding discussion indicates that the amplitude (26) is approximately

$$T_{LM,n'l'nl}=\int_{r_0}^{\infty} r^2dr\int_0^{2\pi} d\phi[e^{i\mathbf{q}\cdot\mathbf{r}}G_{LM,n'l'nl}(\mathbf{r},\mathbf{p})]\theta=\tfrac{1}{2}\pi, \quad (28)$$

where the z axis is taken in the beam direction $\hat{k}_i$, and we use spherical polar coordinates.

The generalized adiabatic conditions that are assumed here,

$$k_i\simeq -k_f, \qquad \mu_{ca}\simeq\mu_{cb}, \quad (29)$$

imply with Eqs. (14) and (15) that the recoil momentum $\mathbf{p}$ is in the direction of the z axis and that the momentum transfer $\mathbf{q}$ lies in the annulus. Consequently, in evaluating the transfer function on the annulus in Eq. (28), the vectors $\mathbf{t}$ and $\mathbf{s}$ of Eq. (22) have spherical polar coordinates

$$\mathbf{s}=(s,\cos^{-1}(ip/s),\phi)$$

and

$$\mathbf{t}=(t,\cos^{-1}(-ip/t),\phi),$$

with

$$s=t=(r^2-p^2)^{1/2}. \quad (30)$$

We note that s and t are real for $p\leq r$ and equal on the annulus, and that the angular dependence of the transfer function contained in Eq. (21) takes the simple form

$$[Y_{l_1}^{m_1*}(\hat{s})Y_{l_2}^{m_2}(\hat{t})]_{\text{on annulus}}\propto P_{l_1}^{m_1}(ix)P_{l_2}^{m_2}(ix)\times\exp[i(m_2-m_1)\phi],$$

with the arguments of the associated Legendre polynomials being pure imaginary, $x=p(r^2-p^2)^{-1/2}$. Thus the ϕ dependence of the integrand in Eq. (28) is simply $\exp[i(qr\cos\phi-M\phi)]$, and the integration over the azimuthal angle ϕ is readily performed, yielding

$$T_{LM,n'l'nl}=2\pi i^M e^{-p^2/4}\int_{r_0}^{\infty} e^{-r^2/4}J_M(qr)g_{LM,n'l'nl}\times(r,p)r^2dr, \quad (31)$$

with

$$g_{LM,n'l'nl}(p,r)=[e^{iM\phi}e^{p^2/4}e^{r^2/4}G_{LM,n'l'nl}(\mathbf{p},\mathbf{r})]_{\text{on annulus}}, \quad (32)$$

which from Eqs. (20), (21), and (24) is a polynomial in r and p.

If the product of the momentum transfer q and diffraction radius r_0 is greater than the order of the cylindrical Bessel function,

$$qr_0\gg M, \quad (33)$$

we may use the asymptotic form

$$J_M(qr)\sim\cos(qr-\tfrac{1}{2}M\pi-\tfrac{1}{4}\pi)(qr)^{-1/2}$$

in Eq. (31). For heavy-ion reactions at energies above the Coulomb barrier, this approximation is valid for a range of scattering angles in the forward direction, and thus does not conflict with our earlier assumption of forward scattering. Since the polynomial $g_{LM,n'l'nl}$ is a slowly varying function of r compared with the other factors in the integrand, it may be taken outside the integration. Finally, by integrating by parts, we obtain an asymptotic expansion in powers of the inverse momentum transfer q^{-1}, and the amplitude (31) becomes

$$T_{LM,n'l'nl}=2\pi i^M e^{-p^2/4}e^{-r^2/4}(qr_0)^{-3/2}r_0^3 g_{LM,n'l'nl}(p,r_0)\times\sin(qr_0-\tfrac{1}{2}M\pi-\tfrac{1}{4}\pi). \quad (34)$$

Substitution of Eq. (34) into Eq. (27) gives our final result for the differential cross section

$$\frac{d\sigma_{n'l',nl}}{d\Omega}=\mu_i\mu_f\left(\frac{k_f}{k_i}\right)\left(\frac{r_0^2}{\nu}\right)^3\hbar^{-4}e^{-\frac{1}{2}p^2}(qr_0)^{-3}\times\hat{l}^{-1}\sum_{LM}\hat{L}\,|\,g_{LM,n'l'nl}(p)\,|^2\,[1-(-1)^M\sin(2qr_0)]. \quad (35)$$

The explicit form of the transfer polynomial g is found from Eqs. (32), (17), (21), and (24):

$$g_{LM,n'l'nl}(p)=\sum_{n''}u_{LM,n'l'n''l}(p)V_{n''n,l}, \quad (36)$$

where

$$u_{LM,n'l'nl}(p)=4\pi^{5/2}C_{n'l',nl}\times\sum_{l_1l_2l_3m_1m_2} i^{l_1+l_2-l_3}(-1)^{l'+M}[\hat{l}_1\hat{l}_2\hat{l}_3][\hat{l}\hat{l}']^{1/2}\times\begin{pmatrix} l_1 & l_2 & L\\ -m_1 & m_2 & M\end{pmatrix}\begin{Bmatrix} l' & l & L\\ l_1 & l_2 & l_3\end{Bmatrix}\begin{pmatrix} l & l_1 & l_3\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l' & l_2 & l_3\\ 0 & 0 & 0\end{pmatrix}\times\left(\frac{(l_1-m_1)!}{(l_1+m_1)!}\frac{(l_2-m_2)!}{(l_2+m_2)!}\right)^{1/2}P_{l_1}^{m_1}\left(\frac{ip}{s_0}\right)P_{l_2}^{m_2}\times\left(\frac{ip}{s_0}\right)D_{n'n,l_1l_2l_3}(s_0,s_0). \quad (37)$$

The polynomial D is defined by Eq. (23), and

$$s_0=(r_0^2-p^2)^{1/2}.$$

The scattering angle θ is related to p and q by Eqs. (14), (15), and (29),

$$p\approx 2\mu_{ca}k_i\cos(\tfrac{1}{2}\theta), \qquad q\approx 2k_i\sin(\tfrac{1}{2}\theta). \quad (38)$$

Whereas the evaluation of the transfer function in Sec. III was quite general under the conditions listed there, this relatively simple expression for differential cross section has been obtained with the diffraction model by making the additional restriction to forward angles and the assumptions of general adiabatic conditions Eq. (29) and large momentum transfer Eq. (33). We stress that the formulation of Sec. II is of greater generality than the particular diffraction model of this section.

V. DIFFERENTIAL CROSS SECTIONS

The differential transfer cross section (35) falls off sharply with increasing momentum transfer q or scattering angle with a q^{-3} dependence characteristic of a diffraction theory. Apart from the over-all q^{-3} dependence, the angular distribution is the sum of two components, one of which is monotonic in the scattering angle and the other of which has characteristic diffraction oscillations of period $(2qr_0)$.

It is useful to define a damping parameter λ, which is the ratio of the magnitude of the oscillating part to the smoothly varying part of the angular distribution, as

$$\lambda_{n'l',nl}=(-1)^{l'-l} \times\Big[\sum_{LM}(-1)^M\hat{L}\,|\,g_{LM,n'l'nl}\,|^2\Big/\sum_{LM}\hat{L}\,|\,g_{LM,n'l'nl}\,|^2\Big]. \quad (39)$$

This parameter provides a measure of the washing out of the angular distribution of the diffraction model when recoil is included.

A. Angular Distribution without Recoil

The cross section with the modified transfer function (8) is simply obtained by putting the recoil momentum equal to zero in the final result (37). In this case the factors $P_{l_1}^{m_1}(0)$ and $P_{l_2}^{m_2}(0)$ imply that l_1+m_1 and l_2+m_2, and hence $l_1+m_1-l_2-m_2=l_1-l_2+M$, are even for a nonvanishing contribution to the sum in Eq. (37), but then the 3-j symbols in Eq. (37) require that l_1+l+l_3 and $l'+l_2+l_3$ be even. Hence, there is a contribution to the cross section for a given M component of the angular momentum transfer only if $M-l+l'$ is even, resulting in the selection rule[9] that if

$$L_{\min}=|\,l-l'\,|\text{ is even, then } M \text{ is even},$$

and if

$$L_{\min}=|\,l-l'\,|\text{ is odd, then } M \text{ is odd}. \quad (40)$$

Thus, only odd or even M, but not both, contribute to the cross section:

$$\frac{d\sigma_{n'l',nl}}{d\Omega}\propto\sum_{LM}\hat{L}^2\,|\,g_{LM,n'l'nl}\,|^2\,[1\pm\sin(2qr_0)], \quad (41)$$

the positive sign being taken when $L_{\min}$ is odd, the negative sign when $L_{\min}$ is even. We note that in both cases $\lambda_{n'l',nl}=1$ and the angular distribution exhibits extreme diffraction oscillations, the minima going to zero. Furthermore, there is a phase rule[10]: Diffraction maxima for even $L_{\min}$, or parity preserving transitions, correspond to diffraction minima for odd $L_{\min}$, or parity changing transitions.

B. Angular Distributions for $l\to0$ Transitions

For nonvanishing p both odd and even M terms contribute to the sum in Eq. (36), resulting in a partial cancellation of the diffraction oscillations. The dependence of this effect on the angular momentum transfer is most easily seen in the special case where the final single-particle state is the $1s$ state. Then, since $l_3+2w\leq\min\{n',n\}$ and $n'=0$ in the expression for the D function, the sum on w reduces to a single term and l_3 is zero. Also, from the properties of the $6j$ symbol in Eq. (37), $l_2=0$, $l_1=l'$, and $L=l'$ for a nonvanishing contribution to the transfer function. With these simplifications in the transfer function (37), the differential cross section (35) reduces to

$$\frac{d\sigma_{00,nl}}{d\Omega}\propto e^{-p^2/2}(qr_0)^{-3}\,\Big|\sum_{n'}D_{0n',l00}C_{00,n'l}V_{n'n,l}\Big|^2 \times\sum_m\frac{(l-m)!}{(l+m)!}P_l^m\Big(\frac{ip}{s_0}\Big)P_l^{m*}\Big(\frac{ip}{s_0}\Big) \times[1-(-1)^m\sin(2qr_0)].$$

With the help of the addition theorem for the Legendre polynomials,[22] written in the form

$$P_l(z^2-(z^2-1)\cos\psi)=P_l^2(z)+2\sum_{m=1}^{l}\frac{(l-m)!}{(l+m)!} \times[P_l(z)]^2\cos(m\psi),$$

and the relation

$$P_l^{m*}(ix)=P_l^m(-x)=P_l^m(ix)(-1)^{l+m},$$

the summation over m yields

$$\frac{d\sigma_{00,nl}}{d\Omega}\propto\left[P_l\Big(1+\frac{2p^2}{s_0^2}\Big)-(-1)^l\sin(2qr_0)\right],$$

and the damping parameter is simply

$$\lambda_{00,nl}=P_l^{-1}[(r_0^2+p^2)/(r_0^2-p^2)]. \quad (42)$$

The damping increases strongly with increasing angular momentum transfer and recoil momentum. The case of zero angular momentum transfer is exceptional; since $P_0(x)=1$, there is no damping for any value of the recoil momentum.

C. General Transfers with Spin

In the general transfer $nl\to n'l'$, the angular distribution consists of a sum of terms, one for each possible

[22] E. T. Whittaker and G. N. Watson, *Course of Modern Analysis* (Cambridge University Press, New York, 1963), 4th ed., p. 326.

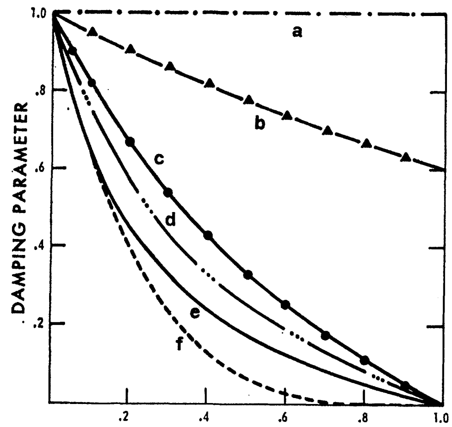

FIG. 2. Dependence of the damping on the recoil momentum for transfer between various single-particle states. Note that the dimensionless quantity $(p^2/r_0^2\nu^2)$ is plotted along the abscissa. The curve a shows the damping parameter for $1s_{1/2}\to 1s_{1/2}$ transitions; b, $1p_{1/2}\to 1p_{1/2}$; c, $1s_{1/2}\to 1p_{1/2,3/2}$; d, $1s_{1/2}\to 1d_{3/2,5/2}$; e, $1p_{1/2}\to 1d_{3/2}$; f, $1p_{1/2}\to 1d_{5/2}$.

angular momentum transfer. Each of these terms includes a monotonic and fluctuating component, which is damped according to the magnitude of the angular momentum transfer and the recoil momentum as in the simple case of $l\to 0$ transitions discussed above. For nonzero recoil momentum, all transfer functions $g_{LM,n'l'nl}$ contribute to the cross section (35), and hence the selection rule (40) is no longer valid. In a given reaction the degree of damping predicted by the model depends on the recoil momentum, which is determined by the incident energy, the range ρ of the interaction, and the oscillator parameters ν and ν', which fix the range of the initial and final single-particle states.

The inclusion of the spin in the model is straightforward. The spins of the cores a and b enter the cross section only through some over-all statistical factors and do not change the shape of the differential cross section. The spin of the transferred particle, which we now take to be a nucleon of spin $\frac{1}{2}$ does, however, modify the angular distribution. The differential cross section for transfer from the single-particle state with total angular momentum j and orbital angular l to the state with quantum numbers n', j', and l' becomes

$$\frac{d\sigma_{n'j'l',njl}}{d\Omega}\propto e^{-p^2/2}(qr_0)^{-3}\sum_{LM}\hat{L}\begin{Bmatrix} j & j' & L\\ l' & l & \frac{1}{2}\end{Bmatrix}^2 |g_{LM,n'l'nl}|^2 \times[1-(-1)^M\sin(2qr_0)].$$

Thus, the damping is now determined by

$$\lambda_{n'j'l',njl}=(-1)^{l'-l}\left(\sum_{LM}(-1)^M\hat{L}\begin{Bmatrix} j & j' & L\\ l' & l & \frac{1}{2}\end{Bmatrix}^2 |g_{LM,n'l'nl}(p)|^2 \Bigg/ \sum_{LM}\hat{L}\begin{Bmatrix} j & j' & L\\ l' & l & \frac{1}{2}\end{Bmatrix}^2 |g_{LM,n'l'nl}(p)|^2\right). \quad (43)$$

The damping parameters for various single-particle transitions are plotted as functions of increasing recoil momentum (which corresponds to increasing incident energy) in Fig. 2. In our calculations we have used a Gaussian potential $V_{ac}=V_0e^{-\rho r^2}$. Since the results are insensitive to the parameter ρ, provided it is not much larger than the oscillator strength ν, the damping parameters for Fig. 2 are shown for $\rho=0$. (The effects of varying the range of the potential may be seen in Fig. 4 for the $1p_{1/2}\to 1p_{1/2}$ and $1p_{1/2}\to 1d_{5/2}$ transitions.) The damping parameters for the $1p_{1/2}\to 1p_{1/2}$ and $1s_{1/2}\to 1s_{1/2}$ transitions are atypical since they include a constant component for transfer of zero angular momentum, which is independent of the recoil momentum. The other transitions shown consist of nonzero angular momentum transfers, leading to complete damping at $p=r_0$. Transitions in which there is no angular momentum transfer should provide valuable experimental tests of the assumptions of the theory, which predicts strong oscillations in the angular distributions for these transitions in contrast with the smooth behavior predicted for other transfers.

D. Damping in the Reaction $C^{12}(N^{14}, N^{13})C^{13}$

As a specific test of the damping mechanism proposed here, we discuss in this section the reaction $C^{12}(N^{14}, N^{13})C^{13}$, $E_{\text{c.m.}}=68.5$ MeV, recently examined experimentally by Birnbaum, Overley, and Bromley.[4] The final states of C^{13} most strongly excited were identified as the ground state 0.0 MeV$(J^\pi=\frac{1}{2}^-)$, and the 3.85-MeV$(\frac{5}{2}^+)$, the 7.65-MeV $(\frac{3}{2}^+)$, and the 9.51-MeV$(\frac{7}{2}^-)$ states. These states are believed to have dominant parentage [C^{12}(ground state)$+n$], supporting the assumption of Sec. II of single-particle transfer without core excitation for this reaction. The transition to the ground state is taken here as the transfer of a $1p_{1/2}$ nucleon in N^{14} to a $1p_{1/2}$ orbit in C^{13} and the transition to the 3.85-MeV level as a $1p_{1/2}\to 1d_{5/2}$ transfer. The N^{13} angular distributions for the ground and 3.85-MeV states are plotted in Fig. 3 together with the over-all q^{-3} dependence predicted by the present theory, showing good agreement. The experimental curves show negligible diffraction structure, requiring theoretically a small value of the damping parameter of

Eq. (43). The reciprocal of the recoil momentum at $E_{c.m.}=68.5$ MeV is approximately 1.5 F, which is comparable with the diffraction radius $r_0 \approx 6$ F, so that damping is expected. Calculations with the aid of Eq. (43) and Eq. (35) show that the degree of damping depends quite critically on the choice of the oscillator strength ν. Figure 4 shows the damping parameter plotted against the oscillator strength for transitions to the ground and 3.85-MeV states, at $E_{c.m.}=68.5$ MeV. In shell-model calculations,[22] the oscillator strength is taken as $\nu \approx 0.3$ F^{-2}. For this value, the present theory predicts oscillations in the angular distribution for transitions to the ground state, whose amplitude is within the experimental error. On the other hand, for this value of ν, the damping parameter for transitions to the 3.85-MeV level is about 0.6, whereas no oscillations are discernible in the experimental results. A smaller value of $\nu \sim 0.2$ F^{-2} yields damping parameters for both transitions of about 0.3, which is more consistent with experiment. A smaller value of ν in the scattering calculation is perhaps reasonable when the obvious shortcomings of the oscillator single-particle state are considered. The transfer function is evaluated on the diffraction radius $r=r_0$, so that the principal contribution to the integral of Eq. (13) comes from the overlap of the tails of the bound-state wave functions. The oscillator wave functions fall

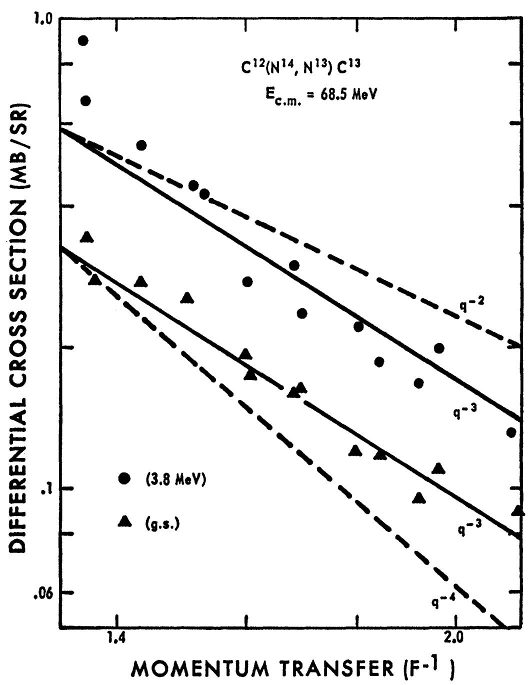

FIG. 3. Comparison of the experimental transfer cross sections of Ref. 4 for the reaction $C^{12}(N^{14}, N^{13})C^{13}$ with the q^{-3} dependence on the linear momentum transfer q given by the present theory. Angular distributions for transitions to the ground state and the 3.8-MeV level of C^{13} are shown. The theoretical curves are arbitrarily normalized.

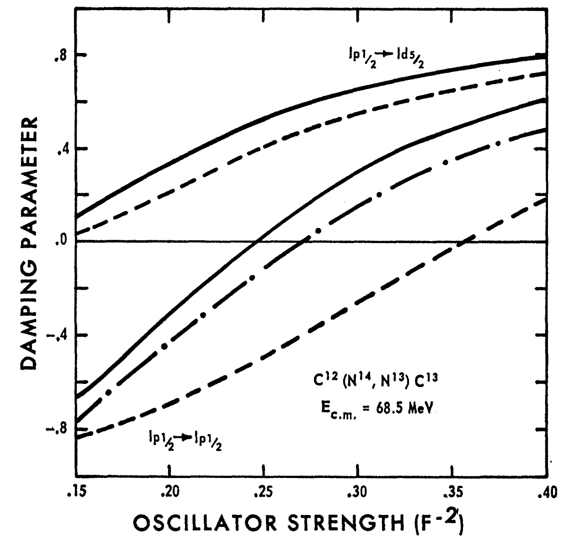

FIG. 4. Predicted damping in angular distributions for neutron transfer to the ground state and 3.8-MeV levels of C^{13} in the reaction $C^{12}(N^{14}, N^{13})C^{13}$. The diffraction radius r_0 is taken to have the value 6 F. The solid curves were calculated with infinite range ($\rho=0$) of the interaction V_{ac}. For the dashed curves the range parameter ρ was taken with twice the value of the oscillator strength ν. For the broken curve, $\rho=\nu$.

off too quickly at the nuclear surface; this may be compensated by decreasing the oscillator strength.

In addition, it must be emphasized that our model for the scattering wave functions is extreme, the sharp nuclear surface leading to complete diffraction oscillations in the absence of recoil. In a more realistic calculation, a diffuse nuclear surface should lead to some damping apart from recoil effects.

In view of the simplifications of the model and the more general limitations of the distorted-wave Born approximation, including the post-prior discrepancy mentioned previously, it cannot be concluded from the limited agreement with experiment demonstrated in this section that the mechanism of recoil damping provides a complete explanation for the absence of structure in the angular distributions of transfer reactions at energies above the Coulomb barrier. This work does show, however, that a simple distorted-wave direct-reaction model can, when recoil is included, give substantial damping of angular distributions. The question of whether the recoil mechanism alone is sufficient to explain the experimental phenomena is open, but the present model suggests that recoil must play a major role in a more realistic theoretical treatment of the transfer mechanism, and that the picture of the transfer reaction as a direct transfer between strongly absorbing cores is not in disagreement with experiment.

APPENDIX

In this appendix the derivation of Eq. (21) for the matrix elements of the Weyl operator $U(\mathbf{p}, \mathbf{r})$ in the

$\{n, l, m\}$ representation is outlined. It should be mentioned that an expression for $U(\mathbf{p}, \mathbf{r})$ in terms of ladder operators in the $\{n_x, n_y, n_z\}$ representation is well known[19,23,24] and that explicit forms for $U(\mathbf{p}, \mathbf{r})$ in the $\{nlm\}$ representation may be found from this expression with the help of the transformation brackets $\langle n, l, m \mid n_x, n_y, n_z\rangle$. Alternatively, as shown here, Eq. (21) may be derived from an integral representation of the oscillator wave functions.

In configuration space,

$$U_{n'l'm',nlm}(\mathbf{p},\mathbf{r}) = \langle n'l'm' \mid U(\mathbf{p},\mathbf{r}) \mid nlm\rangle = \exp(-\tfrac{1}{2}i\mathbf{p}\cdot\mathbf{r}) \int \psi_{n'l'm'}^*(\mathbf{r}'-\mathbf{r})e^{i\mathbf{p}\cdot\mathbf{r}'}\psi_{nlm}(\mathbf{r}')d\mathbf{r}'. \quad \text{(A1)}$$

The evaluation of this integral rests on the following integral representation of the oscillator wave functions:

$$\psi_{nlm}(\mathbf{r}) = C_{nl}(-1)^m e^{-r^2/2}\left[\frac{\partial^n}{\partial u^n} e^{u^2}\int e^{2i\mathbf{u}\cdot\mathbf{r}} Y_l^m(\hat{u})d\hat{u}\right]_{u=0}, \quad \text{(A2)}$$

where

$$C_{nl} = [\tfrac{1}{2}(n-\tfrac{1}{2}l)!\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}l+\tfrac{3}{2})]^{1/2}/2^{1/2}i^l\pi^{3/2}n!$$

and $\mathbf{r}$ is dimensionless, since we measure p and r in terms of the oscillator strength ν.

The equivalence of Eq. (A2) with the usual definition of the oscillator wave functions,[25]

$$\psi_{nlm}(\mathbf{r}) = (-1)^m[(\tfrac{1}{2}n-\tfrac{1}{2}l)!/\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}l+\tfrac{3}{2})]^{1/2} r^l e^{-\frac{1}{2}r^2} \times L_{\frac{1}{2}n-\frac{1}{2}l}{}^l(r^2)Y_l^m(\hat{r}),$$

is easily demonstrated by expanding the exponential $e^{2i\mathbf{u}\cdot\mathbf{r}}$ of Eq. (A2) in partial waves,[20]

$$e^{2i\mathbf{u}\cdot\mathbf{r}} = 4\pi\sum_{lm} i^l j_l(2ur)Y_l^{m*}(\hat{u})Y_l^m(\hat{r}), \quad \text{(A3)}$$

performing the parameter integrations, and using the fact that the spherical Bessel function is a generating function for the Laguerre polynominals,[26]

$$(2/\sqrt{\pi})e^{u^2}j_l(2ur) = \sum_{p=0}^{\infty}[u^{2p+l}/\Gamma(p+l+\tfrac{3}{2})]L_p{}^{l+1/2}(r^2)r^l.$$

After substitution of Eq. (A2) in Eq. (A1), the integration with respect to $\mathbf{r}'$ is easily performed, yielding

$$U_{n'l'm',nlm}(\mathbf{p},\mathbf{r}) = C_{nl}C_{n'l'}^*(2\pi)^{3/2}(-1)^{m+m'}e^{-p^2/4}e^{-r^2/4}\left[\frac{\partial^{n'}}{\partial u'^{n'}}\frac{\partial^n}{\partial u^n} \times \iint d\hat{u}\,d\hat{u}'e^{i\mathbf{u}\cdot\mathbf{s}}e^{i\mathbf{u}'\cdot\mathbf{t}}e^{-2\mathbf{u}\cdot\mathbf{u}'}Y_l^m(\hat{u})Y_l^{m*}(\hat{u}')\right]_{u=0,u'=0}, \quad \text{(A4)}$$

with $\mathbf{s}=\mathbf{r}+i\mathbf{p}$ and $\mathbf{t}=\mathbf{r}-i\mathbf{p}$.

On expansion in the partial-wave series (A3) of each of the exponentials involving the parameters $\mathbf{u}$ and $\mathbf{u}'$, the parameter integrations in Eq. (A4) may be carried out, resulting in Eq. (21) given in Sec. III.

An explicit expression for the D function of Eq. (23) is obtained by substituting the series expansions of the spherical Bessel functions and carrying out the parameter differentiations,

$$D_{n'n,l_1l_2l_3}(s,t) = \tfrac{1}{8}\pi^{3/2}\sum_w i^{n+n'-l_1-l_2-l_3}(\tfrac{1}{2})^{n+n'-2l_3-4w} \times\{s^{n-l_3-2w}t^{n'-l_3-2w}n!n'!/[\Gamma(\tfrac{1}{2}l_1+\tfrac{3}{2}+\tfrac{1}{2}n-\tfrac{1}{2}l_3-w)\Gamma(\tfrac{1}{2}l_2+\tfrac{3}{2}+\tfrac{1}{2}n'-\tfrac{1}{2}l_3-w)]\}, \quad \text{(A5)}$$

where the sum is over non-negative integers w which satisfy both $2w\leq n-l_1-l_3$ and $2w\leq n-l_2-l_3$.

The matrix elements of the Gaussian potential used in the calculations of Sec. V are also easily found with the aid of Eq. (A2) for the oscillator wave functions. The result is

$$\langle n'lm \mid e^{-\rho R^2} \mid nlm\rangle = [(\tfrac{1}{2}n-\tfrac{1}{2}l)!(\tfrac{1}{2}n'-\tfrac{1}{2}l)!\Gamma(\tfrac{1}{2}n+\tfrac{1}{2}l+\tfrac{3}{2})\Gamma(\tfrac{1}{2}n'+\tfrac{1}{2}l+\tfrac{3}{2})]^{-1/2} \times(1/1+\epsilon)^{3/2}(\epsilon/1+\epsilon)^{\frac{1}{2}(n+n')}\sum_w \epsilon^{-l-2w}/[\tfrac{1}{2}(n-l)-w]![\tfrac{1}{2}(n'-l)-w]!w!\Gamma(l+\tfrac{3}{2}+w), \quad \text{(A6)}$$

where the sum is over non-negative integers w which satisfy both $2w\leq n-l$ and $2w\leq n'-l$, and ϵ is defined by $\epsilon=\rho/2\nu$.

[23] W. W. True, Phys. Rev. **130**, 1530 (1963).
[24] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[25] P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Co., New York, 1953), Vol. II, p. 1663.
[26] Reference 25, Vol. I, p. 785.