Separable Approximations to the t Matrix*

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We extend Reiner's comparison of several separable approximations for the off-energy-shell t-matrix elements $t(p, k; s)$ with those for a local, central square-well potential for negative energy s. We treat other separable potentials besides the Noyes approximation considered by Reiner, namely, the Weinberg series terminated at one, two, or three terms, and the unitary-pole approximation. We also consider a larger range of momenta and energy than treated by Reiner. We find that the unitary-pole approximation is in general the best of the four one-term separable approximations used. The off-energy-shell values of $t(p, k; s)$ have an error of less than 5% of the value of $t(0, 0; 0)$. The Weinberg series converges rapidly, two terms giving in general an error of less than 1% of $t(0, 0; 0)$. We further compare phase shifts for positive energy and the effective-range parameters. By definition, the Noyes prescription gives the exact value. The unitary-pole approximation gives good results, and again the two-term Weinberg series is very satisfactory. We discuss qualitatively problems arising in the more realistic case (for the two-nucleon potential) from the use of other shapes for the attractive potential, of a strong short-range repulsion, and of tensor forces.

I. INTRODUCTION

DIFFERENT separable approximations are compared with the exact value for the S-wave off-energy-shell t matrix $t(p, k; s)$ by Reiner[1] for a central, attractive, local square-well potential. He considers three separable approximations: (i) Noyes (specified below as Noyes-Reiner to distinguish between two different interpretations of the Noyes approximation[2] for negative energy s); (ii) a Lovelace approximation[3]; and (iii) the Guennéguès approximation.[4] Reiner finds that the Noyes-Reiner approximation is quite accurate (roughly good to 1%) in the range of momenta $0\leq p\leq 2$, $0\leq k\leq 2$ and energy $-1\leq s\leq 0$. (Here, and throughout this paper, momenta are given in units of $\hbar/b$, and energy in units of $\hbar^2/Mb^2$ for a system of two nucleons of mass M, attracting each other with a square well of range b.) The Lovelace and Guennéguès approximations are in general much less accurate than the Noyes-Reiner approximation, and will not be considered further in this paper. However, we shall treat another Lovelace approximation,[3] which we denote as the unitary-pole approximation.[5]

We shall extend Reiner's work by considering several other separable approximations to the t matrix, and by working with a larger range of momenta and energy.

A separable t matrix can be written as the product of functions of the two different momenta

$$t_S(p, k; s)=g(p, s)g(k, s). \quad (1)$$

If the two-body potential $v(p, k)$ is itself separable,[6] then the solution of the Lippmann-Schwinger equation gives a separable t matrix of the form of Eq. (1), with the additional feature that the function $g(p, s)$ can be written as the product of a function of p and a different function of s.

We shall also consider what Mitra[7] calls a rank-N separable t matrix ($N\geq 2$)

$$t_N(p, k; s)=\sum_{\nu=1}^{N} g_\nu(p, s)g_\nu(k, s). \quad (2)$$

This form is used in phenomenological fits[8] to phase shifts, and also arises in truncating the expansions in Sturmian functions, performed by Rotenberg,[9] by Weinberg,[10] and by Ball and Wong.[11]

It is well known that the use of a separable t matrix greatly simplifies a number of nuclear physics calculations. For instance, the Faddeev equations for the trinucleon[12] reduce from a two-dimensional integral equation to a one-dimensional integral equation for a central spin-independent separable t matrix. Even rank-2 tensor t matrices give only six coupled one-dimensional integral equations.[13] Also, the Bethe-Goldstone equation for nuclear matter is relatively easy to solve[14] for a separable t matrix.

A completely phenomenological potential is determined to fit values of the t matrix on the energy shell (i.e., phase shifts), but does not give a unique prescription for off-energy-shell values of the t matrix unless we are willing to make specific assumptions (e.g., locality, or separability) concerning the phenomenological potential. At present meson theory[15] provides

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only a little help in providing an unambiguous extrapolation from on-shell to off-shell matrix elements.

A historical note seems in order. Some years ago one of us (J. S. L.) argued[16] that nuclear physics was different from atomic physics, since for the former we were not able to establish the locality of the two-nucleon potential by comparison of the phase shifts for states of different angular momenta. This implied that "local potential" meant a potential independent of the angular momentum. In this extreme sense a Majorana exchange potential or a quadratic spin-orbit potential would be judged "nonlocal." Currently the term "local potential" includes a potential such as Reid's[17] which explicitly depends on the orbital, spin, and total angular momentum of the system.

The current controversy[18] involves this question: "Given on-shell t-matrix elements for a state of specified L, S, and J, should we fit these data with a local potential, or with some specified nonlocal form?" Alternatively, "How shall we extrapolate the t matrix off the energy shell?"[19] An epistemological question immediately arises: Is there actually *any* method of determination of the "correct" method of extrapolation of the t matrix?

We might try to solve the above epistemological problem with any of three techniques: Occam's razor, use of other experiments, or pragmatically. (i) The assumption of a local potential does seem the simplest assumption[18]; so if we make this assumption, calculate various nuclear properties, and obtain good agreement with experiment, then Occam's razor justifies our initial assumption. But these "calculations giving good agreement with experiment" are at present incomplete for local potentials involving both repulsive cores and tensor forces, so Occam's justification cannot be applied yet. (ii) We might appeal to *other experiments*, namely, those involving three or more particles, to determine the values of the off-energy-shell two-nucleon t matrix. But any particular method, such as the analysis of elastic electron-deuteron scattering, suffers from the serious drawback of introducing meson-exchange effects, which are only poorly known.[18] In a similar manner, the three-nucleon problem introduces poorly known three-body forces.[20] Careful consideration of various experiments might lead to a method of sorting out which effects were due to the extrapolation procedure for the t matrix, which were due to meson-exchange effects, and which due to three-body forces. We believe that this sorting out has not yet been accomplished. (iii) We might dodge the unsolved epistemological problem in a *pragmatic* (or even opportunistic) manner, by using a separable t matrix as an *approximation* to the "true" t matrix, whatever that mysterious object is. Suppose that a local and a separable t matrix agree "well enough" with each other in a "specified region" of p, k, and s space. It is then *plausible* to assume that all extrapolation methods will agree well enough, since locality and separability represent extreme assumptions. Then the separable t matrix would be close enough to the "true" t matrix.

The reader should be aware, from our chosen title, that we are at present adopting the third pragmatic viewpoint. We immediately face the problem of providing meaning for the vague terms used above: "well enough" and "specified region." The most reliable way to check our approximation is to solve a particular problem (such as the binding energy of the trinucleon) both with and without approximation, and see if they agree "well enough." This test has a discouraging feature: If we are able to solve the problem exactly, why bother with the approximation? Of course, we hope that an approximation which is successful in one problem will then also be successful in a similar problem which is too hard to solve exactly. (Thus we might compare separable and local central potentials in the trinucleon problem, and if they agree, gain confidence in the use of the separable approximation for a tensor potential.) But how do we know that the second problem is indeed similar?

Three such comparisons for the trinucleon use spin-$\frac{1}{2}$ independent central potentials. (i) Osborn[21] compares trinucleon binding energies for a local Yukawa potential and a separable Yamaguchi[6] potential. He finds that the binding energies agree within 10% if the potentials have a strength similar to that of the nucleon-nucleon singlet or triplet states. There is a serious disagreement for much stronger potentials. (ii) Ball and Wong[11] also use a Yukawa potential, and evaluate the trinucleon binding energy by use of a Sturmian series.[9,10] They find that a single term gives less than half the binding energy found for a four-term series [$N=4$ in Eq. (2)]. Even a rank-2 separable potential misses the rank-4 result by 30%. (iii) Finally, Fuda[22] uses a separable approximation to Tabakin's[19] rank-2 separable potential. Fuda finds that the trinucleon binding energies agree to within 0.15 MeV, which is only 2% of Tabakin's value. We note the disagreement among these workers concerning the validity of the separable approximation.

Other workers have compared local and separable potentials for calculations of nucleon-nucleon brems-

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strahlung,[23] for quasi-three-body reactions,[24] and for residual reactions in nuclei.[25] The effects of separability are in general only of order 10%.

While it may well be misleading to know the experimental answer, we cannot help gaining a little confidence in the use of a separable t matrix for the trinucleon problem from the good agreement with experiment found by several workers.[7,26,27] These workers use tensor t matrices that give good values of the low-energy parameters of the nucleon-nucleon system, and that reproduce good agreement with the experimental binding energy for the trinucleon. They also[28] produce reasonable agreement with the experimental trinucleon form factors for elastic electron scattering.

The above calculations[26,27] show that the trinucleon binding energy is sensitive both to the poorly known singlet effective range and to the poorly known percentage of D state in the deuteron. Thus a change of only 0.15 F in the singlet effective range changes the trinucleon binding energy by 0.3 MeV. A change from 5.5 to 7% D-state probability in the deuteron also gives a 0.5-MeV change. Further, the binding energy depends on the shape chosen[26] for the separable potential. These comparisons give us a criterion for how well the separable and local t matrices should agree: There is at present an appreciable experimental error in the determination of the t matrix on the energy shell. An extrapolation error, small compared to the on-shell error, can be neglected.

The "relevant region" for which the separable t matrix needs to be a "good enough" approximation clearly depends on the problem under consideration. For the trinucleon ground state (central, spin-independent separable case) the Faddeev equations reduce to a one-dimensional integral equation for the spectator function $\chi(q)$:

$$\chi(q)=2\int \frac{g(|\mathbf{q}+\tfrac{1}{2}\mathbf{k}|,-E_T-\tfrac{3}{4}q^2)\,g(|k+\tfrac{1}{2}\mathbf{q}|,-E_T-\tfrac{3}{4}q^2)\,\chi(k)\,d^3k}{q^2+\mathbf{q}\cdot\mathbf{k}+k^2-E_T}. \tag{3}$$

The energy $s=-E_T-\frac{3}{4}q^2$ is more negative than the trinucleon energy $-E_T$. The relevant range of the momentum variables $|\mathbf{q}+\frac{1}{2}\mathbf{k}|$ and $|\mathbf{k}+\frac{1}{2}\mathbf{q}|$ is limited by the sharp decrease of the spectator function with increasing momentum, namely,[28] a factor of order 100 as momentum increases from 0 to 3 F^{-1}. We *estimate* that the relevant momentum range is from 0 to 3 F^{-1}, and the corresponding energy range $-300<s<-8$ MeV, where we have used experimental knowledge for the upper limit of energy. For a square well of range b about 2 F, this estimate gives momenta in the range $0\leq p\leq 6$ and energy $-30\leq s\leq -1$. (Compare the much smaller ranges given above for Reiner's work.[1])

Another way of estimating the validity of the separable approximation for the trinucleon binding energy is to use perturbation theory[29] expressed in terms of the difference between the "true" and the approximate t matrix. Our estimates above, based on Eq. (3), could be checked in detail by this procedure.

Of course, the "relevant region" depends on the calculation attempted. Thus, if we are concerned with form factors for electron-trinucleon scattering at high momentum transfers, then we are sensitive to the (very small) spectator function and t matrices at corresponding high values of momentum. Or if we treat continuum states of the three-nucleon system,[30] we are concerned also with positive values of the energy s.

Another satisfactory way of testing an approximation is to evaluate higher terms in a (hopefully rapidly converging) series. We use the Weinberg series[9,10] to give us such a test, and find that the two-term Weinberg series [$N=2$ in Eq. (2)] gives quite good accuracy for a square well in the momentum and energy range given above.

In our units, the depth V_0 of the square well is a pure number, with $\frac{1}{4}\pi^2$ corresponding to an infinite scattering length. We cannot expect a separable approximation to work unless V_0 is of order $\frac{1}{4}\pi^2$. For if $V_0\ll\frac{1}{4}\pi^2$, then the Born approximation will give us a good solution $t(p,k;s)=v(p,k)$ for the Lippmann-Schwinger equation, and we know that $v(p,k)$ is *not* separable for a local potential. On the other hand, if $V_0\gg\frac{1}{4}\pi^2$, then the local potential would have more than one bound state, and it is well known[6] that a separable potential can have at most one bound state. We shall emphasize two values of V_0 in the neighborhood of $\frac{1}{4}\pi^2$, namely, $V_0=3.526$ and $V_0=1.960$. These are in the right region to correspond to experimental singlet and triplet nucleon-nucleon potentials.

In Sec. II we quote Reiner's exact solution[31] for the S-wave t matrix for a local central square well. We also present explicit forms for the t matrix, for the Weinberg series,[8,9] for the Noyes approximation,[2] and for the unitary-pole approximation.[3,5] We follow Osborn's discussion[21] of the ambiguity in the Noyes approxima-

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tion for negative energy, denoting Reiner's choice as the Noyes-Reiner approximation, and Osborn's choice as the Noyes-Osborn approximation. The unitary-pole approximation uses a t matrix for a separable potential that has the same eigenfunction and eigenvalue for the bound state as those for a local square well.

In Sec. III we compare numerical values of the phase shift for the exact solution, and for different separable approximations. We also compare effective-range parameters. Here the Noyes "approximation" is exact, since it is designed to give no error for on-energy-shell matrix elements.

In Sec. IV we compare numerical results for our t matrices for negative s, and the momentum range estimated above for the trinucleon binding energy problem. This section is highly numerical, and clumsy, since we find it difficult to compare many functions of three independent variables.

In Sec. V we draw conclusions concerning the accuracy of various separable approximations to the exact t matrix in this particular example. We also make qualitative remarks about the use of the separable approximation for t matrices for central potentials of different shape, and for tensor potentials.

Note added in proof. Sitenko, Kharchenko, and Petrov [Phys. Letters **28B,** 308 (1968)] study the convergence of the Weinberg series for a local central Hulthén potential, and find a rate of convergence similar to that for a Yukawa potential.[11] One of us (A.H.L.) has independently obtained similar results for a Hulthén potential. E. Harms (unpublished) studies a new series called the unitary-pole expansion, which has the UPA as its first term. Harms's series converges more rapidly than the Weinberg expansion.

II. ANALYTIC EXPRESSIONS FOR THE t MATRIX

We first present Reiner's[3,1] exact analytical expression for the off-energy-shell t matrix for S-wave scattering by an attractive central local square well. We then present and discuss briefly four different separable approximations to this t matrix, which we designate as (i) the Weinberg series, terminated at N terms,[9–11] (ii) the Noyes-Reiner approximation,[1,2] (iii) the Noyes-Osborn approximation,[2,21] and (iv) the unitary-pole approximation.[5] We also present effective-range parameters for the positive-energy on-shell t matrices.

As stated earlier, we measure lengths in terms of the range b of the square well, where b is of order 2 F for the nucleon-nucleon potential. Then momenta p and k are measured in units of $\hbar/b$. We estimated above that, for purposes of calculation of the binding energy of the ground state of the trinucleon, we would be concerned with the accuracy of our approximations in the range $0\leq p\leq 6$ and $0\leq k\leq 6$. Energy s is given in units of $\hbar^2/Mb^2$, for nucleon mass M, or roughly 10 MeV. We shall treat the range $-30\leq s\leq 22$. The depth V_0 of the square well equals $\frac{1}{4}\pi^2$ for infinite scattering length.

We now present the basic equations, and the normalization used for the t matrix.

For a central force, the Lippmann-Schwinger equation separates into separate equations for each partial wave. The S-wave equation reads, with our normalization,

$$t(p,k;s)=v(p,k)+\int_0^\infty v(p,q)4\pi q^2dq(s-q^2)^{-1}t(q,k;s). \quad (4)$$

The "S-wave potential in momentum space" uses integrals with the spherical Bessel functions $j_0(pr)$ and (j_0kr):

$$v(p,k)=(2\pi^2)^{-1}\int_0^\infty j_0(pr)v(r)j_0(kr)r^2dr. \quad (5)$$

In our normalization, there is a proportionality factor of $2\pi^2$ to get from the on-energy-shell t matrix to phase shifts or scattering length a; e.g.,

$$2\pi^2t(0,0;0)=a. \quad (6)$$

We translate Reiner's[1] Eq. (4.3) into our normalization and notation, writing the energy s as $q^2+i\epsilon$ for positive s (later taking the limit as ϵ goes to zero from above), and as $s=-q^2$, for negative s:

$$t(p,k;s)=-(V_0/2\pi^2)\{\phi(p,k)+V_0[\cos\beta-iqj_0(\beta)]^{-1}\chi(p,k;\beta)+iq\psi(p,k;\beta)\}, \quad (7)$$

where

$$\phi(p,k)=[j_0(p-k)-j_0(p+k)]/2pk, \quad (8)$$

$$\chi(p,k;\beta)=[\phi(p,k)\cos\beta-\phi(\beta,k)\cos p]/(p^2-\beta^2), \quad (9)$$

$$\psi(p,k;\beta)=[\phi(\beta,k)j_0(p)-j_0(\beta)\phi(p,k)]/(p^2-\beta^2), \quad (10)$$

$$\beta=(V_0+q^2)^{1/2}. \quad (11)$$

The Weinberg series[9–11] expands the central potential V in energy-dependent orthogonal sets $|\psi_\nu(s)\rangle$, that obey the modified Schrödinger equation [for potential $V/\eta_\nu(s)$]:

$$[H_0+V/\eta_\nu(s)]\,|\psi_\nu(s)\rangle=s\,|\psi_\nu(s)\rangle. \quad (12)$$

The potential is expanded in terms of the set $|\psi_\nu\rangle$ (frequently called Sturmian functions), giving

$$V=\sum_{\nu=1}^{\infty}\frac{V\,|\psi_\nu(s)\rangle\langle\psi_\nu(s^*)\,|\,V}{\langle\psi_\nu(s^*)\,|\,V\,|\,\psi_\nu(s)\rangle}. \quad (13)$$

Using (13), the Lippmann-Schwinger equation (4) is solved, giving the t matrix t_W expanded in terms of the set $|\psi_\nu(s)\rangle$:

$$t_W(p,k;s)=\sum_{\nu=1}^{\infty}\frac{\langle p\,|\,V\,|\,\psi_\nu(s)\rangle\langle\psi_\nu(s^*)\,|\,V\,|\,k\rangle}{[1-\eta_\nu(s)]\langle\psi_\nu(s^*)\,|\,V\,|\,\psi_\nu(s)\rangle}. \quad (14)$$

We note that Weinberg wrote this series explicitly for the case of the t matrix on the energy shell for positive s, where the relevant quantity is the phase shift $\delta(q)$. For negative s, or off the energy shell, Weinberg

sums the series (2.11) only for values of ν such that the Born approximation fails ($|\eta_\nu|$ larger than unity). Rotenberg[9] and Wong[11] write the series (14) explicitly. [Note that Ball's $\phi_n(p, s)=\langle p \mid V \mid \psi_n(s)\rangle$.] Straightforward evaluation of the matrix elements in (14) gives for our attractive square well of unit range and depth V_0

$$t_W(p, k: s) = -\frac{V_0}{\pi^2}\sum_{\nu=1}^{\infty}\frac{(\beta_\nu^2-s)\sin^2\beta_\nu g_\nu(p)g_\nu(k)}{(\beta_\nu^2-s-V_0)[1+(-s)^{1/2}j_0^2(\beta_\nu)]}, \quad (15)$$

where

$$g_\nu(p)=[\cos p+(-s)^{1/2}j_0(p)]/(p^2-\beta_\nu^2) \quad (16)$$

and $\beta_\nu(s)$ is a solution of the transcendental equation

$$\tan\beta_\nu(s)=-(-s)^{-1/2}\beta_\nu(s). \quad (17)$$

Also, the quantities $\beta_\nu(s)$ and $\eta_\nu(s)$ are related as follows:

$$\beta_\nu^2=s+V_0/\eta_\nu(s). \quad (18)$$

The phase shifts $\delta(q)$ are written following Weinberg's Eqs. (97) and (98):

$$\delta(q)=\sum_{\nu=1}^{\infty}\delta_\nu(q), \quad (19)$$

where the "elemental phase shifts" $\delta_\nu(q)$ are

$$\delta_\nu(q)=-\arg[1-\eta_\nu(s)]. \quad (20)$$

Here $\eta_\nu(s)$ is found from Eqs. (17) and (18).

We can treat the Weinberg expansion for an infinite repulsive core by taking the limit as V_0 approaches negative infinity in the above equations. This procedure gives the convergent series obtained independently by Fuda[32] and by Laroze (private communication). On the other hand, if we put $s=0$ in (17), we obtain $\beta_\nu=(\nu+\frac{1}{2})\pi$. We can then sum the Weinberg series (15) explicitly, obtaining agreement with Reiner's (7).

The scattering length a_N and effective range ρ_N for the N-term Weinberg series can be found from Eqs. (15)–(20) by straightforward but lengthy manipulation[33] of the expression $k\cot\delta$ near $k=0$. The scattering lengths add in the same manner as do Weinberg's elemental phase shifts:

$$a_N=\sum_{\nu=1}^{N}A_\nu, \quad (21)$$

$$A_\nu=-2V_0(\nu-\tfrac{1}{2})^{-4}\pi^{-4}[1-V_0/(\nu-\tfrac{1}{2})^2\pi^2]^{-1}.$$

The effective range ρ_N is given by a recursion relation

$$\rho_N=\frac{1}{a_{N-1}+A_N}\left[\left(2-\frac{\rho_{N-1}+R_N}{a_{N-1}+A_N}\right)a_{N-1}A_N + \rho_{N-1}a_{N-1}+R_NA_N\right], \quad (22)$$

where

$$R_N=\frac{(N-\frac{1}{2})^2\pi^2}{V_0}\left[\frac{4}{3}\left(\frac{V_0}{(N-\frac{1}{2})^2\pi^2}-\frac{7}{4}\right) + \frac{2}{(N-\frac{1}{2})^2\pi^2}\left(\frac{7}{2}-\frac{V_0}{(N-\frac{1}{2})^2\pi^2}\right)\right].$$

The Noyes approximation uses a separable form adjusted to give the exact result for the t matrix on the energy shell, i.e., for real $p=k=q$ for positive $s=q^2+i\epsilon$,

$$t_N(p, k; s)=g_N(p, s)g_N(k, s), \quad (23)$$

where

$$g_N(p, s)=t(p, q; s)/[t(q, q; s)]^{1/2}. \quad (24)$$

By definition, the Noyes approximation gives exact phase shifts. As Osborn[21] points out, there is an ambiguity in the use of Eq. (24) for negative s, since one cannot be on the energy shell with real momenta for negative energies. The two possibilities discussed by Osborn are (i) fit at imaginary $q=s^{1/2}$ or (ii) fit at real $iq=(-s)^{1/2}$. Reiner[1] uses positive imaginary q; we designate this the Noyes-Reiner (NR) approximation

$$g_{NR}(p, s)=t(p, q; s)/[t(q, q; s)]^{1/2},$$

or

$$t_{NR}(p, k; s)=-(V_0^2/2\pi^2)\phi(p, \beta)\phi(k, \beta)\exp(-iq) \times[\cos qj_0(\beta)-j_0(q)\cos\beta]^{-1}[\cos\beta-iqj_0(\beta)]^{-1}. \quad (25)$$

The last expression on the right of course uses Eq. (7).

The second alternative, of fitting the t matrix at real momentum by taking $(|s|)^{1/2}$, we designate as the Noyes-Osborn (NO) approximation, since Osborn uses it in his thesis. We merely replace q by iq in (24)

$$g_{NO}(p, s)=t(p, iq; s)/[t(iq, iq; s)]^{1/2} \quad (26)$$

and use this function in (23) to obtain $t_{NO}(p, k; s)$.

Finally, we consider the unitary-pole approximation (UPA) discussed by Lovelace[3] [his Eqs. (2.53) and (2.54)] and generalized to tensor forces by Fuda.[5,22] For a central potential strong enough to give a bound state at energy $-B$, with eigenfunction $|B\rangle$, the problem reduces to that solved by Yamaguchi[6] in his introduction of separable potentials. Namely, find a separable potential $V=-\lambda f_u(p)f_u(k)$ with strength λ and form factor $f_u(p)$ such that the Schrödinger equation has eigenvalue $-B$ and eigenfunction $|B\rangle$. Then use this separable potential in the Lippmann-Schwinger equation to find the corresponding separable t matrix, which we call t_u:

$$t_u(p, k; s)=-f_u(p)f_u(k)/D(s), \quad (27)$$

where

$$f_u(p)=\langle p \mid V \mid B\rangle=-(B+p^2)\langle p \mid B\rangle \quad (28)$$

and

$$D(s)=\lambda^{-1}+\int_0^\infty\frac{4\pi f_u^2(x)x^2dx}{s-x^2}. \quad (29)$$

We note the similarity to the first term in the Weinberg

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expansion (14), when $s\approx -B$ and $|\psi(s)\rangle\approx|B\rangle$. λ is adjusted so that $D(-B)=0$, i.e., so that the pole in the t matrix has the right position.

The UPA has indeed been used extensively in the trinucleon problem, since the Yamaguchi form for $f(p)$ is just the UPA for the ground-state eigenfunction of a local Hulthén potential.[6] For our attractive square well, of unit range and depth V_0 (larger than $\frac{1}{4}\pi^2$), Eq. (28) gives

$$f_u(p)=[\cos p+B^{1/2}j_0(p)]/(p^2+B-V_0) \quad (30)$$

and (27) gives

$$D(s)=\tfrac{1}{2}(B+s)(V_0+B^{1/2}) \times(V_0-B)^{-1}(V_0-B-s)^{-1}(V_0)^{-1} +\exp(2iq)(V_0-B-s)^{-2}[i(s-B)/2q+B^{1/2}] +\tfrac{1}{2}i(B+s)(V_0-B-s)^{-2}q^{-1}. \quad (31)$$

For positive energy, $s=q^2+i\epsilon$, the UPA phase shift $\delta_u(q)$ is found from the value of the t matrix on the energy shell [see Eqs. (27), (30), and (31)]:

$$q\cot\delta_u(q)=iq-[D(q^2+i\epsilon)][2\pi^2 f_u^2(q)]^{-1}. \quad (32)$$

The UPA effective-range parameters are found from the expansion of this expression near $q=0$. The scattering length a_u is given by

$$a_u=(1+B^{1/2})^2/[(V_0+B^{1/2})B/2V_0+B+B^{1/2}]. \quad (33)$$

The effective range is

$$\rho_u=2(1+B^{1/2})^{-2}[-(1+B^{1/2})(1+\tfrac{1}{3}B^{1/2})/a_u -\tfrac{1}{2}(V_0+B^{1/2})(V_0-2B)V_0^{-1}(V_0-B)^{-1} +1+\tfrac{2}{3}B+2B^{1/2}]. \quad (34)$$

If $V_0<\frac{1}{4}\pi^2$, we have an antibound state, and we must be careful in the sign of $B^{1/2}$ in Eqs. (30) and (31), namely, we change all $B^{1/2}$ to read $-B^{1/2}$, since the pole is on the other Riemann sheet. [For any well depth, B is found from simultaneous solutions of two equations (compare (17) and (18)): (i) $\beta^2=V_0-B$ and (ii) $\tan\beta=\pm\beta B^{-1/2}$. In the second equation we use the negative sign for a bound state and positive sign for an antibound state.]

III. NUMERICAL COMPARISONS FOR POSITIVE ENERGIES

In this section we shall present results for $t(p, k; s)$ for positive energy $s=q^2+i\epsilon$, with a negligible positive imaginary part ϵ, for the special case where we stay on the energy shell: $p=k=q$. (The formulas in Sec. II could, of course, be evaluated also for use in the off-energy-shell positive-energy region explored, for instance, by nucleon-nucleon bremsstrahlung.) By definition, the Noyes approximation agrees with the exact t matrix on the energy shell, and we do not have the ambiguity that occurs for negative energy and leads to the NR and NO approximations, respectively. We need, therefore, only to consider the N-term Weinberg approximations and the UPA. We shall treat phase shifts for local square-well potentials of two different depths: $V_0=3.526$, giving a bound state at -0.214, and $V_0=1.96$, giving an antibound state at -0.076. (Note that energies are given in units of $\hbar^2/Mb^2$ or roughly 10 MeV in the c.m. system, if we choose the nucleon-nucleon case with range b of roughly 2 F.)

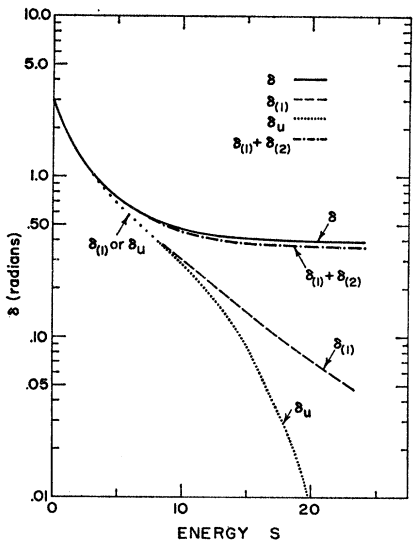

FIG. 1. Phase shifts δ versus c.m. energy s, in units of $\hbar^2/Mb^2$. The solid line shows the exact phase shift δ for an attractive square well; the dashed line shows the one-term Weinberg elemental phase shift $\delta_{(1)}$; the dot-dashed line shows the sum of the first two Weinberg phase shifts $\delta_{(1)}+\delta_{(2)}$; the dotted line shows the UPA δ_u. See Table I for numerical values, and references to the text.

Table I and Fig. 1 present phase shifts for the deeper square well. The Weinberg "elemental phase shifts" are found from Eqs. (17)–(20), while the UPA phase shifts are found from Eq. (32). The figure illustrates two main results. First, the single-term Weinberg elemental shift $\delta_{(1)}$ and the UPA shift δ_u agree well with each other up to s of 7 (or some 150-MeV laboratory energy), but start disagreeing noticeably from the exact phase shifts at s of about 2 (or only 40-MeV laboratory energy). The UPA is slightly closer to the exact results for $s\leq 5$. That is, a single separable potential can fit both the bound-state energy and wave function exactly, and give a good fit to the moderate-energy phase shifts (up to 40-MeV laboratory), but fails at higher energies. However, the two-term Weinberg series gives excellent agreement with the exact phase shifts (i.e., within 0.01 rad) up to $s=8$ (or about 160-MeV laboratory energy). The small disagreements at still higher energies, but below $s=22$, are accounted for by the third elemental phase shift: The plot of the first three elemental phase shifts lies on top of the plot of the exact shifts. We conclude that either a single-term Weinberg, or UPA, t matrix gives a fair fit to the phase shifts, but that the two-term Weinberg gives an excellent fit, in this particular case.

TABLE I. Comparison of phase shifts, for well with bound state. For the nucleon-nucleon case, the laboratory energy is roughly $20s$ MeV. The Weinberg elemental phase shifts (in rad) are calculated from Eqs. (17)–(20), and the UPA phase shifts from Eq. (32); both use depth $V_0=3.526\hbar^2/Mb^2$.

		Sums of Weinberg "elemental shifts"			
Energy s	UPA	$\delta_{(1)}$	$\delta_{(1)}+\delta_{(2)}$	$\delta_{(1)}+\delta_{(2)}+\delta_{(3)}$	Exact
1	1.489	1.473	1.492	1.494	1.495
2	1.167	1.147	1.177	1.180	1.181
3	0.970	0.941	0.980	0.984	0.986
4	0.807	0.787	0.838	0.842	0.844
5	0.678	0.665	0.727	0.732	0.735
6	0.572	0.564	0.640	0.646	0.648
7	0.481	0.480	0.570	0.577	0.580
8	0.402	0.410	0.515	0.523	0.526
9	0.334	0.350	0.472	0.481	0.484
10	0.275	0.299	0.439	0.448	0.452
12	0.179	0.219	0.397	0.408	0.412
14	0.109	0.162	0.378	0.391	0.395
16	0.059	0.120	0.370	0.386	0.390
18	0.027	0.091	0.368	0.385	0.390
20	0.009	0.070	0.364	0.384	0.390
22	0.001	0.054	0.357	0.380	0.386

Table II shows similar results for a shallower potential, $V_0=1.960$. The region of validity of the different approximations is very nearly the same as for a deeper well.

We note that Weinberg[10] also finds a good, though not as rapid, convergence for his numerical example with a Hulthén local potential: See his Fig. 4, Eq. (121), and numerical values below his Eq. (98). He finds the first four elemental phase shifts as 127°, 13°, 3°, and 1°, respectively, so that here four terms are needed to obtain the phase shift accurate within 0.01 rad. Weinberg's numerical example is for a Hulthén potential of magnitude twice that sufficient to give a bound state at zero energy, while we are here concerned with the more favorable case of square wells of depth within 40% of the critical $\frac{1}{4}\pi^2$ value.

We now consider the accuracy of the Weinberg and UPA approximations for the scattering length a and the effective range ρ, as a function of the depth V_0 of our square well. The N-term Weinberg values of the effective-range parameters are found from Eqs. (21) and (22), while the UPA values are given in Eqs. (33) and (34). Table III gives the numerical results, which we plot in Figs. 2 and 3, as percent errors from the exact values. We see that both the one-term Weinberg expansion and the UPA give scattering lengths correct within 1% in the range of depths $0.8\leq V_0\leq 3.8$. In this same region of well depth the one-term Weinberg expansion gives an effective range in error by 2 of 3%. The UPA does extremely well near the center of this interval, $2\leq V_0\leq 3$. (This comparison in accuracy of effective-range parameters corresponds to the better accuracy of UPA in the low-energy phase shifts, presented in Tables I and II and in Fig. 1.) The two-term Weinberg expansion gives both scattering length and effective range to an accuracy of better than 1% for the very large range of well depths, $0.3\leq V_0\leq 8$.

IV. NUMERICAL COMPARISONS FOR NEGATIVE ENERGY

For negative energy s, we are concerned in general with off-diagonal elements of the t matrix $t(p, k; s)$ i.e., with a function of three independent variables. The amount of numerical results becomes still larger, since we wish to compare several different approximations to the t matrix with the exact result of Reiner's, namely, the Weinberg series, using one, two, or three terms, the NR approximation, the NO approximation, and the UPA. Thus we wish to compare seven different functions of three independent variables. We shall of necessity select a small fraction of our numerical results; further numbers for the Weinberg series are available.[33]

If we first consider one-term separable approximations, we can reduce the number of independent variables from three to two, since we are concerned only with the "form factor" $g(p, s)$ of Eq. (1). This comparison of form factors will tell us how well the four different separable approximations agree with each other. We can compare them with the exact t matrix

TABLE II. Comparison of phase shifts, for well with antibound state. The quantities tabulated are explained in the caption of Table I; these results are for a well of depth $V_0=1.960\hbar^2/Mb^2$.

Energy s	UPA	Sums of Weinberg "elemental shifts" $\delta_{(1)}$	$\delta_{(1)}+\delta_{(2)}$	$\delta_{(1)}+\delta_{(2)}+\delta_{(3)}$	Exact
1	0.821	0.813	0.823	0.824	0.825
2	0.709	0.699	0.714	0.716	0.717
3	0.603	0.596	0.616	0.618	0.619
4	0.512	0.508	0.534	0.536	0.537
5	0.432	0.433	0.465	0.468	0.469
6	0.363	0.369	0.407	0.410	0.412
7	0.303	0.314	0.360	0.363	0.365
8	0.250	0.267	0.320	0.324	0.326
9	0.204	0.227	0.289	0.293	0.295
10	0.164	0.193	0.264	0.269	0.270
12	0.101	0.140	0.230	0.236	0.238
14	0.056	0.102	0.212	0.218	0.221
16	0.027	0.075	0.203	0.212	0.214
18	0.010	0.056	0.200	0.210	0.213
20	0.002	0.043	0.199	0.210	0.213
22	0.000	0.033	0.196	0.209	0.212

for the special case $p=k$ by defining

$$g_T(p, s)=[t(p, p; s)]^{1/2}. \tag{35}$$

We therefore consider five functions of momentum and energy: (i) $g_T(p, s)$ from Eq. (35); (ii) $g_W(p, s)$, using the term $\nu=1$ in Eqs. (15) and (16) for the Weinberg series; (iii) the NR approximation $g_{NR}(p, s)$ from Eq. (25); (iv) the NO approximation $g_{NO}(p, s)$ from Eq. (26); and, finally, (v) the UPA, where we include the s dependence of Eqs. (27)–(29), using

$$g_u(p, s)=f_u(p)[D(s)]^{-1/2}. \tag{36}$$

As discussed above, for application to the trinucleon problem we are concerned with the behavior of these five functions in the approximate range $0\leq p\leq 7$ and $-30\leq s\leq -1$. We shall also treat the case $s=-0.25$, near the pole at -0.214 (for a well of depth $V_0=3.526$), to show how good the pole approximation is in this case. Of course, all four separable approximations agree with each other, and with g_T if we are very close to the bound-state pole.

Table IV gives these five "form factors" for $p=0$ (1) 6, and for $s=-\frac{1}{4}$, $s=-1$, $s=-4$, and $s=-16$. Two more or less representative cases of $s=-\frac{1}{4}$ and $s=-4$ are

TABLE III. Effective-range parameters. $V_0=\frac{1}{4}\pi^2$ gives a bound state at zero energy. a_u and ρ_u are the scattering length and effective range for the UPA, Eqs. (33) and (34); a_1 and ρ_1 refer to one-term Weinberg and a_2 and ρ_2 to two-term Weinberg, Eqs. (21) and (22). The exact scattering lengths are a and ρ.

Well depth V_0	a_u	a_1	a_2	a	ρ_u	ρ_1	ρ_2	ρ
0.5	−0.205	−0.206	−0.208	−0.208	3.072	3.008	2.935	2.923
1.25	−0.837	−0.832	−0.838	−0.839	1.491	1.516	1.485	1.479
2.0	−3.477	−3.468	−3.477	−3.478	1.117	1.144	1.120	1 116
3.0	4.552	4.565	4.551	4.549	0.911	0.937	0.914	0.911
3.5	2.730	2.747	2.730	2.727	0.853	0.878	0.854	0.850
4.0	2.095	2.115	2.095	2.092	0.809	0.833	0.808	0.804
6.0	1.348	1.376	1.343	1.338	0.705	0.730	0.694	0.689
8.0	1.137	1.172	1.121	1.114	0.652	0.678	0.626	0.619
10.0	1.036	1.076	1.002	0.993	0.620	0.647	0.570	0.562
12.0	0.975	1.020	0.914	0.904	0.598	0.626	0.510	0.499

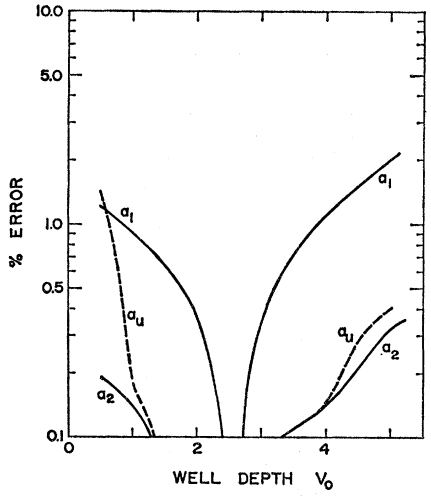

FIG. 2. Percent error in the scattering length versus well depth V_0 in units of $\hbar^2/Mb^2$. a_1 and a_2 show percent errors for one-term and two-term Weinberg series, respectively, while a_u shows percent error for the UPA. See Table III for numerical values, and references to the text.

presented graphically in Fig. 4. All four one-term separable form factors agree well with each other, and with the true form factor $g_T(p, s)$ for $s=-0.25$, up to $p=4$. At higher momentum, all four separable form factors dip to 40% of the value of the true g_T. At the more relevant energy value of -4.0, in the range $0<p<4$ the Weinberg one-term, the NO, and the unitary-pole approximations agree fairly well with each other and with g_T; the NR approximation falls too rapidly. All four approximations becomes quite bad at higher momentum. Note that from the definition, Eq. (25), of the NO approximation, it is exact at $p=[\,|\,s\,|\,]^{1/2}$, or at $p=2.0$ for this value of energy.

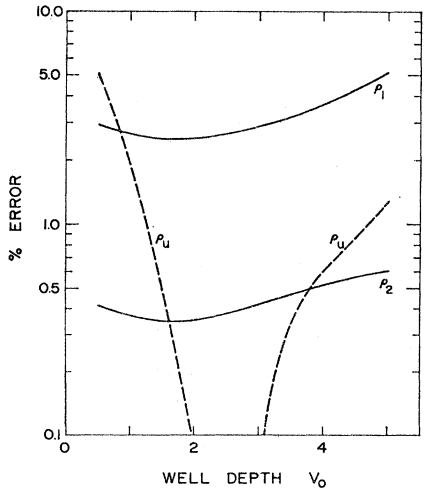

FIG. 3. Percent error in the effective range versus well depth V_0 in units of $\hbar^2/Mb^2$. ρ_1 and ρ_2 show percent errors for one-term and two-term Weinberg series, respectively, while ρ_u shows percent error for the UPA. See Table III for numerical values, and references to the text.

Reiner[1] compares, in his Fig. 3, the NR and the exact values of $t(1, k; s)$ in the range $0\leq k\leq 2.0$ and energy values $s=-1$, 0, 2, and 4. We agree with Reiner's conclusion that the NR approximation is very accurate for $s=-1$ for his range of momentum; compare our Table IV for $s=-1$ and $p=1$. Note, however, that the NR approximation becomes much less successful as s becomes more negative.

We obtain similar results for the well depth $V_0=1.960$, and will not present them here.

We now face the problem of comparing seven different functions of three independent variables for a depth

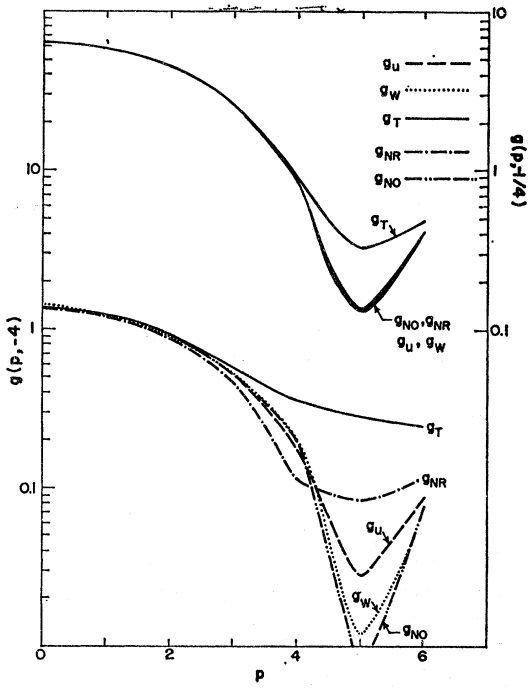

FIG. 4. Form factors $g(p, s)$ versus momentum p in units of $\hbar/b$. The energy s is $-\frac{1}{4}\hbar^2/Mb^2$ drawn above (see ordinate scale on right) and s is $-4\hbar^2/Mb^2$ (ordinate scale on left) below. See Table IV for explanation of form factors, and their numerical values.

$V_0=3.526$. The functions are the various t matrices: (i) the exact t matrix $t(p, k; s)$ from Eq. (7); (ii) the one-term Weinberg series $t_1(p, k; s)$, truncating Eq. (15) at one term; (iii) the two-term Weinberg series t_2, truncating at two terms; (iv) the three-term Weinberg series t_3, truncating at 3; (v) the NR approximation t_{NR}, Eq. (25); (vi) the NO approximation t_{NO}, Eq. (26); and, finally, (vii) the UPA t_u from Eq. (27).

In Tables V, VI, and VII we select three representative points in the momentum (p, k) plane, namely, (1, 1), (1, 4), and (4, 4), respectively. For each (p, k) value we present the values of the seven t matrices in terms of the energy s. Our NR results, for $-1\leq s\leq 0$, agree with Reiner's[1] Fig. 1.

We plot our results (Tables V, VI, and VII) in Figs. 5, 6, and 7, respectively. Instead of plotting the

TABLE IV. Comparison of different form factors. The five form factors tabulated are given in Eqs. (35) and (36) and the intervening paragraph, for a square well of range b and depth $V_0=3.526$ with a bound state at -0.214. These energies, and s, are given in units of $\hbar^2/Mb^2$. The momentum p is given in units of $\hbar/b$.

Energy s	Momentum p	g_T (p, s)	g_W (p, s)	g_{NR} (p, s)	g_{NO} (p, s)	g_{UPA} (p, s)
−0.25	0.0	6.528	6.528	6.528	6.528	6.512
−0.25	1.0	5.944	5.942	5.943	5.943	5.929
−0.25	2.0	4.419	4.419	4.416	4.417	4.412
−0.25	3.0	2.528	2.520	2.512	2.513	2.510
−0.25	4.0	0.920	0.869	0.860	0.861	0.861
−0.25	5.0	0.321	0.127	0.136	0.135	0.133
−0.25	6.0	0.481	0.421	0.426	0.426	0.423
−1.0	0.0	1.915	1.899	1.915	1.915	1.914
−1.0	1.0	1.742	1.733	1.740	1.742	1.743
−1.0	2.0	1.299	1.299	1.284	1.292	1.297
−1.0	3.0	0.776	0.754	0.717	0.732	0.738
−1.0	4.0	0.401	0.277	0.230	0.247	0.253
−1.0	5.0	0.294	0.017	0.058	0.045	0.039
−1.0	6.0	0.258	0.112	0.135	0.128	0.124
−4.0	0.0	1.361	1.448	1.350	1.340	1.356
−4.0	1.0	1.235	1.209	1.218	1.224	1.235
−4.0	2.0	0.920	0.918	0.878	0.920	0.919
−4.0	3.0	0.568	0.549	0.460	0.538	0.523
−4.0	4.0	0.352	0.221	0.110	0.202	0.179
−4.0	5.0	0.290	0.012	0.083	0.007	0.028
−4.0	6.0	0.242	0.065	0.116	0.076	0.088
−16.0	0.0	1.172	1.225	1.084	0.400	1.163
−16.0	1.0	1.061	1.003	0.964	0.404	1.059
−16.0	2.0	0.787	0.775	0.655	0.408	0.788
−16.0	3.0	0.492	0.483	0.287	0.388	0.448
−16.0	4.0	0.329	0.216	0.003	0.329	0.154
−16.0	5.0	0.284	0.038	0.137	0.233	0.024
−16.0	6.0	0.235	0.038	0.124	0.122	0.076

relative error as done by Reiner, we plot instead the absolute error divided by $t(0, 0; 0)$,

$$\gamma_{NR}(p, k; s)=[t_{NR}(p, k; s)-t(p, k; s)]/t(0, 0; 0). \quad (37)$$

The absolute error is used instead of the relative error, since a perturbation calculation[29] gives the energy shift of the trinucleon as proportional to the small change in the t matrix. Also, in future work with strong short-range repulsion, we shall investigate cases where the true t matrix[34] goes through zero, and the relative error is therefore meaningless at that point. We divide by the on-shell t matrix at zero energy to remove ambiguities between different normalizations, e.g., the factor of $2\pi^2$ between Reiner's choice and our choice.

[34] R. Laughlin and B. L. Scott, Phys. Rev. **171**, 1196 (1968).

We see from Fig. 5 that the one-term Weinberg has an error $\gamma(1, 1; s)$ that varies from 1 to about 5%. The error does not get much above 1% for a two-term Weinberg series, and stays below $\frac{1}{2}$% for a three-term series. The error in the NR approximation is similar to that in the one-term Weinberg. While the NO approximation is exact for $s=-1$ [since $p=(|s|)^{1/2}$ in this event], it fails seriously at more negative energies. By contrast, the UPA is by far the best of the one-term approximations, and in fact is better than the *three-term* Weinberg series.

The remarks above about Weinberg series and the NR approximation apply fairly well to Fig. 6 for $\gamma(1, 4; s)$; the over-all errors tend to be somewhat larger in this case. The NO approximation is exact

TABLE V. t matrices $t(1, 1; s)$. Values of the t matrix $t(p, k; s)$ for $p=k=1$ in units of $\hbar/b$, for a square well of range b and depth $V_0=3.526$. Both V_0 and energy s are in units of $\hbar^2/Mb^2$. The exact t matrix is given in Eq. (7), the N-term Weinberg series results t_1, t_2, and t_3 in Eq. (15), the NR approximation t_{NR} in Eqs. (23) and (25), the NO approximation t_{NO} in (23) and (26), and the UPA in (27).

s	Exact	t_1	t_2	t_3	t_{NR}	t_{NO}	t_u
−2	−2.013	−1.968	−2.005	−2.011	−1.995	−2.010	−2.014
−4	−1.526	−1.462	−1.514	−1.522	−1.485	−1.497	−1.525
−6	−1.357	−1.279	−1.341	−1.351	−1.288	−1.263	−1.354
−8	−1.268	−1.179	−1.249	−1.261	−1.171	−1.057	−1.265
−10	−1.213	−1.115	−1.191	−1.205	−1.089	−0.829	−1.209
−12	−1.175	−1.069	−1.150	−1.166	−1.025	−0.580	−1.171
−14	−1.148	−1.034	−1.120	−1.138	−0.973	−0.344	−1.143
−16	−1.127	−1.007	−1.097	−1.116	−0.929	−0.163	−1.121
−18	−1.110	−0.985	−1.078	−1.098	−0.891	−0.054	−1.104
−20	−1.096	−0.966	−1.062	−1.084	−0.858	−0.007	−1.091
−22	−1.085	−0.950	−1.049	−1.072	−0.828	−0.002	−1.079
−24	−1.076	−0.936	−1.037	−1.061	−0.801	−0.020	−1.070
−26	−1.068	−0.924	−1.028	−1.052	−0.777	−0.048	−1.061
−28	−1.061	−0.914	−1.019	−1.045	−0.755	−0.078	−1.054
−30	−1.055	−0.904	−1.011	−1.038	−0.735	−0.105	−1.048

both for $s=-1$ and $s=-16$, and in other regions of s has an accuracy about that of the NR approximation. The UPA is again the most accurate one-term separable approximation, but it is an order of magnitude worse than in Fig. 5, and here is not as good as the two-term Weinberg series.

Figure 7 shows similar features for $\gamma(4, 4; s)$. No one-term separable approximation stays below 1% error; but this time the NO approximation is particularly accurate, since it is exact at $s=-16$. The two-term Weinberg series has an accuracy of better than $\frac{1}{4}\%$ for all s treated, and the three-term Weinberg series is more accurate by a factor of at least 3.

These three figures give the over-all impression that while it is hard to make a firm choice among the four one-term separable t matrices, the UPA tends to be

TABLE VI. t matrices $t(1, 4; s)$. See caption to Table V for explanation of notation.

s	Exact	t_1	t_2	t_3	t_{NR}	t_{NO}	t_u
−2	−0.276	−0.335	−0.281	−0.277	−0.233	−0.292	−0.293
−4	−0.200	−0.267	−0.207	−0.202	−0.135	−0.248	−0.222
−6	−0.172	−0.245	−0.181	−0.175	−0.088	−0.251	−0.197
−8	−0.157	−0.234	−0.167	−0.160	−0.059	−0.259	−0.184
−10	−0.148	−0.227	−0.159	−0.151	−0.037	−0.258	−0.176
−12	−0.142	−0.223	−0.153	−0.145	−0.021	−0.235	−0.170
−14	−0.137	−0.219	−0.149	−0.141	−0.008	−0.191	−0.166
−16	−0.133	−0.217	−0.146	−0.137	−0.003	−0.133	−0.163
−18	−0.130	−0.215	−0.144	−0.135	−0.011	−0.075	−0.160
−20	−0.128	−0.214	−0.142	−0.133	−0.018	−0.026	−0.158
−22	−0.126	−0.213	−0.141	−0.131	−0.024	−0.012	−0.157
−24	−0.124	−0.212	−0.140	−0.129	−0.029	−0.039	−0.155
−26	−0.123	−0.211	−0.139	−0.128	−0.033	−0.056	−0.154
−28	−0.122	−0.210	−0.138	−0.127	−0.037	−0.067	−0.153
−30	−0.120	−0.210	−0.138	−0.127	−0.040	−0.072	−0.152

TABLE VII. t matrices $t(4, 4; s)$. See caption to Table V for explanation of notation.

s	Exact	t_1	t_2	t_3	t_{NR}	t_{NO}	t_u
−2	−0.137	−0.057	−0.134	−0.137	−0.027	−0.042	−0.043
−4	−0.124	−0.049	−0.120	−0.123	−0.012	−0.041	−0.032
−6	−0.118	−0.047	−0.114	−0.117	−0.006	−0.050	−0.029
−8	−0.115	−0.046	−0.110	−0.114	−0.003	−0.064	−0.027
−10	−0.113	−0.046	−0.108	−0.111	−0.001	−0.080	−0.026
−12	−0.111	−0.046	−0.106	−0.109	−0.001	−0.096	−0.025
−14	−0.110	−0.047	−0.104	−0.108	0	−0.106	−0.024
−16	−0.109	−0.047	−0.103	−0.107	0	−0.109	−0.024
−18	−0.108	−0.047	−0.102	−0.106	0	−0.105	−0.023
−20	−0.107	−0.047	−0.101	−0.105	0	−0.096	−0.023
−22	−0.106	−0.048	−0.100	−0.104	−0.001	−0.086	−0.023
−24	−0.106	−0.048	−0.099	−0.104	−0.001	−0.076	−0.023
−26	−0.105	−0.048	−0.099	−0.103	−0.001	−0.067	−0.023
−28	−0.105	−0.048	−0.098	−0.102	−0.001	−0.058	−0.022
−30	−0.104	−0.049	−0.097	−0.102	−0.002	−0.050	−0.022

either the best, or one of the better, approximations. On the other hand, the two-term Weinberg series has an accuracy of better than 1% for all cases plotted, and the three-term Weinberg series is still more accurate. As discussed in the Introduction, the validity of a specified approximation must be judged for the resulting accuracy in a particular calculation. We gain the qualitative impression that the UPA may be accurate enough for calculation of the binding energy of the trinucleon with a local square well, and that the two-term Weinberg series should certainly be accurate enough.

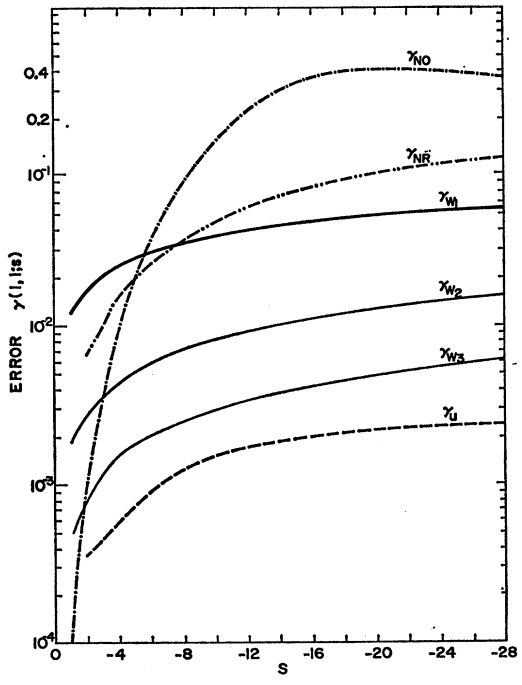

FIG. 5. Relative error $\gamma(1, 1; s)$ for six different separable approximations to the t matrix, plotted versus energy s in units of $\hbar^2/Mb^2$. The momenta $p=k=\hbar/b$. See Eq. (37) for definition of γ; see Table V for numerical values and explanations of the different approximate t matrices used.

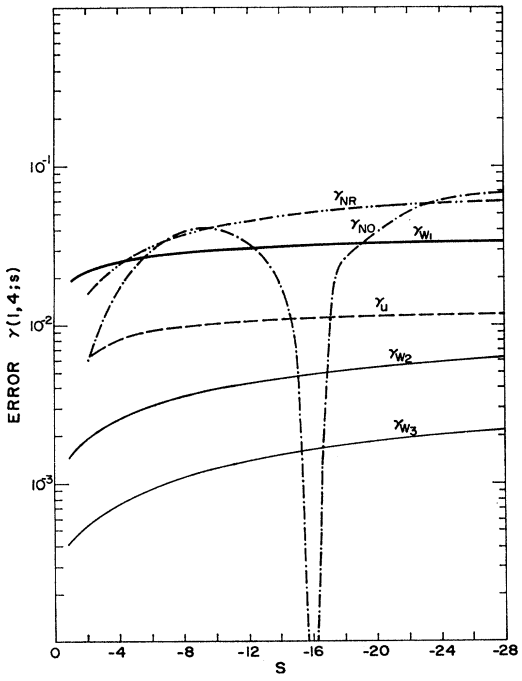

FIG. 6. Relative error $\gamma(1, 4; s)$ versus energy s. The momenta are $p=\hbar/b$ and $k=4\hbar/b$. See captions to Fig. 5 and Table V for explanation of notation; see Table VI for numerical values.

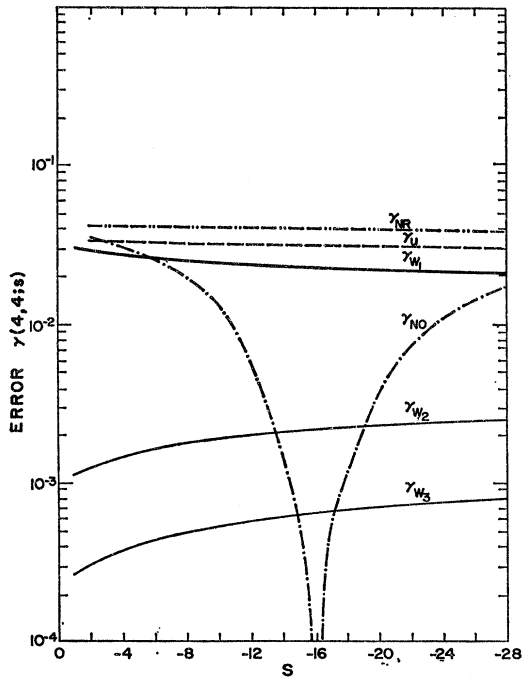

FIG. 7. Relative error $\gamma(4, 4; s)$ versus energy s. The momenta $p=k=4\hbar/b$. See captions to Fig. 5 and Table V for explanation of notation; see Table VII for numerical values.

We have available, on request, machine calculations of our six different separable approximations for a much greater variety of values of momenta p and k and of negative energies s. We have results both for a depth $V_0=3.526$ treated above, and for $V_0=1.960$, for an antibound state.[33] The general impressions on accuracy given above are not changed by this plethora of numerical results.

V. DISCUSSION

We have presented above a small fraction of our tabulated results on the accuracy of various separable approximations to the t matrix for a local, central attractive square well of range b. We have concentrated on the special case of a square well deep enough to have a single weakly bound state. For this case, we find that the UPA, which is exact at the bound state, gives a good approximation both for phase shifts at modest positive energies (up to c.m. energy s of $2\hbar^2/Mb^2$) and for the off-shell t matrix in the range of momenta and negative energies relevant for the calculation of the trinucleon binding energy. Other one-term separable approximations (one-term Weinberg, NR, or NO) are of similar, or not quite as good, accuracy at negative s; of course, the two Noyes approximations give exact phase shifts. The two-term Weinberg approximation in general gives a t matrix accurate to 1% of the value of $t(0, 0; 0)$: This should be sufficient accuracy for any current applications of the two-nucleon t matrix.

We hope that the accuracy of one- or two-term separable approximations is a general feature of short-range potentials, of strength sufficient to give a bound state (or antibound state) near zero energy. If this hope is justified, then we could use one- or two-term separable potentials with confidence in a variety of problems of nuclear physics. We shall briefly mention three difficulties that *may* arise when more realistic nuclear potentials are considered: (i) the shape of the attractive portion of the central potential, (ii) the presence of a strong short-range repulsion, and (iii) the tensor character of the potential.

First, we consider our use of a square well, rather then, for instance, a Yukawa or Hulthén shape. We note that the square well gives rapid convergence in the terms in the Weinberg series.[15] Part of this convergence is due to the term $[1-\eta_\nu(s)]^{-1}$ in (14) and is characteristic for having $\eta_1(s)$ near unity for the values of s under consideration. (This factor makes the first term large compared to subsequent terms, but helps only a little on the ratio of the second to the third or subsequent terms.) The remainder of the convergence is due to the rapid decrease of $g_\nu(p)$ with increasing ν, for the range of p being considered. The analytical result, (16), can easily be understood pictorially by looking at the matrix element $\langle p \mid V \mid \psi_\nu(s)\rangle$ in (14)

$$\langle p \mid V \mid \psi_\nu(s)\rangle = \int \langle p \mid r\rangle V(r) \langle r \mid \psi_\nu\rangle d^3r.$$

For $0\leq p\leq 2\pi$, the $\langle p \mid r\rangle$ has no nodes for $0\leq r\leq 1$, and the potential V has the same sign in this interval. But by definition $\langle r \mid \psi_\nu(s)\rangle$ has ν nodes, so we can expect more and more destructive interference as ν increases. This feature of destructive interference for high ν is quite effective for an attractive square well. We obtain the asymptotic form for large ν from (15), putting $\beta_\nu\approx\nu\pi$. Provided that $|s|$ is small, $p\ll\beta_\nu$, $k\ll\beta_\nu$, and that $V_0\ll\beta_\nu^2$, we find that the individual terms in (15) are proportional to ν^{-4}. This destructive interference can be expected to persist (perhaps not quite as vigorously) for other shapes of potentials. We note that the very small ratio (of order 1%) of the second to the first term in the Weinberg expansion (15) is the result of the product of three small factors, each of order of magnitude $\frac{1}{5}$.

Published results on the convergence of the Weinberg expansion for other shapes $V(r)$ or on the accuracy of other separable approximations to the t matrix seem to be rather rare. Weinberg[10] gives an explicit formula for the "elemental phase shifts" for a Hulthén potential, and obtains the numbers quoted in Sec. II. Weinberg's Eq. (121) gives the asymptotic behavior of the elemental phase shift for large ν as ν^{-3}, which represents reasonably rapid convergence. The rapidity of convergence for off-shell t-matrix elements for the local Hulthén potential and the agreement with the UPA (here the Yamaguchi potential) clearly deserve further study. As remarked in the Introduction, the

recent work of Ball and Wong[11] indicates moderately slow convergence for the Weinberg expansion as applied to a local Yukawa potential, for calculation of the binding energy of the trinucleon.

Of course, the separable approximation may be much worse for P, or higher angular momentum states, than it is for S states. Recently Fuda[32] at the State University of New York, Buffalo, and Laroze at Rensselaer have developed the Weinberg series for a hard-core potential. This amounts to putting the well depth V_0 as $-\infty$ in Eq. (15). Fuda considers the special case $p=k=s=0$, and finds that successive terms in the Weinberg expansion go as the inverse squares of successive odd integers: This convergence is less rapid than the inverse fourth power found above for an attractive square well or the inverse cube for the elemental phase shifts for the Hulthén potential. The more realistic case of a hard core combined with an attractive short-range potential is now being studied by Fuda.

Laughlin and Scott[34] have recently shown that a local potential with a core and an attractive region gives off-energy-shell values for the t matrix that cannot be fitted with a one-term separable potential. It has been argued for some time that the change of sign of the phase shift for such a local potential cannot be fitted by a one-term separable approximation. Tabakin, therefore, used two terms for his central[19] and tensor[8] potential fits to phase parameters. Very recently, Tabakin[35] has shown that he can fit a phase shift with changes sign, using a single separable potential, with the form factor containing a node. We believe that there are serious physical objections to this fit, namely, that Tabakin has a node in his ground-state wave function. We shall shortly present these objections in detail. It seems clear to us that a *really good* separable fit to a local potential with a core and attraction must use at least two terms in the separable expansion. However, as stated in the Introduction, Fuda's work[22] shows that in the case[19] treated, a one-term separable approximation works *well enough* for purposes of calculation of the ground-state energy of the trinucleon.

[35] F. Tabakin, Phys. Rev. **177**, 1443 (1969).

Finally, any realistic nuclear potential V includes a strong tensor force. Of course, the Weinberg expansion (13) still applies. Now the ket $|\psi_\nu(s)\rangle$ would contain both S and D waves. The rapidity of convergence of the Weinberg series in this case or the validity of the UPA[5] clearly deserves further study. We note that with a tensor force there is a special difficulty with the use of a one-term separable potential, as noted by Yamaguchi.[36] Namely, a one-term separable potential in the spin triplet, including a tensor force, will give a zero phase shift for the $^3D_1$ state. If we want to match the observed $^3S_1$ phase shift, the mixing parameter ϵ, and also the $^3D_1$ phase shift, we must include two or more terms in the spin triplet potential.[8] This argument indicates the need to include at least two terms in the Weinberg series for the triplet system.

All our remarks above apply to off-shell matrix elements at negative energy and to on-shell at positive energy. The area of off shell at positive energy clearly deserves further investigation.

In conclusion, we have extended Reiner's work on the accuracy of different one-term separable t matrices as an approximation to the exact t matrix for a local central attractive square-well potential. The UPA is particularly convenient, and among the most accurate; it is good to roughly 1% for a range of momenta and energies expected to be of significance for a calculation of the ground state of the trinucleon. The Weinberg series converges rapidly for this case, the two-term Weinberg series having an accuracy of better than 1% in the relevant region.

ACKNOWLEDGMENTS

We wish to thank M. G. Fuda for calling the Weinberg series to our attention, and for informing us about his work (Ref. 32) with it. We wish to thank F. Tabakin for sending us his recent work (Ref. 36) and for his detailed and frank discussions on this subject. We also thank T. Brady, L. Laroze, and E. Harms for their help on portions of this calculation.

[36] Y. Yamaguchi and Y. Yamaguchi, Phys. Rev. **95**, 1635 (1954).