Low-Frequency Magnetoacoustic Effects in Elastic Solids*

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The magnetic field dependence of the stiffness and attenuation are computed in the low-frequency limit for an elastic solid. The direct interaction of the field with the lattice ions is determined by including the Lorentz forces in a derivation of the elastic equations. The results lead to mode coupling, but the influence of the field on the magnitude of the complex elasticity in conducting solids is negligible compared to measured results. The contribution of the conduction electrons to the elasticity is computed using the self-consistent electric field resulting from the ion motion and electron reaction. Substantial agreement is found between the magnetic field dependence and magnitude of the computed values and measured changes in the complex elasticity for a longitudinal wave propagating in polycrystalline tantalum and in niobium. Expressions for the field dependence of the propagation parameters for transverse waves with different polarizations are also obtained. The anisotropy of the change in loss factor for a single crystal of niobium is used to indicate relative average Fermi velocities in various crystal planes.

I. INTRODUCTION

THE magnetic field dependence of the elastic properties of metals reflect the interaction of the lattice ions and charge carriers with the magnetic field and with each other. A Lorentz force acts on the moving lattice ions in addition to an internal electric field resulting from the reaction of the electrons to the ion motion and the magnetic field. The elastic properties are investigated by determining the velocity and attenuation of an acoustic wave propagating in the metal in the presence of a magnetic field. From these parameters the field dependence of the adiabatic stiffness and loss factor can be obtained.

Of particular interest is wave propagation in an isotropic medium at frequencies much lower than the cyclotron frequency and the scattering frequency of the charge carriers. The change in elastic properties due to motion of the lattice ions in the field is determined by computing the reaction to motion in the magnetic field of a charged elastic continuum, having a charge density equal to the ion density. Coupling of longitudinal and transverse displacements occurs in the presence of the field and modifies the pure mode directions in anisotropic solids. The field dependence of the elastic modulus and loss factor resulting from direct interaction of the magnetic field with the lattice ions is computed in Sec. II for an isotropic propagation medium.

The electronic contribution to the field dependence of the complex modulus is determined by using a self-consistent electric field which results from the motion of the lattice during the passage of the sound wave. This self-consistent electric field, as calculated from the development by Cohen, Harrison, and Harrison,[1] acts on the lattice ions as well as the charge carriers. The free-charge-carrier contribution to the total field dependence of the complex modulus is shown in Sec. II to be the major portion of the observed changes. Comparison with observed field dependence of the stiffness and attenuation is presented in Sec. III for niobium and tantalum.

II. LATTICE ELASTICITY

In separating the effects of a magnetic field on the elasticity of a metal, the lattice is represented as a charged elastic continuum. As an elastic wave propagates in the lattice, each of the lattice ions moves periodically with the velocity amplitude of the wave. If a magnetic field is applied, a Lorentz force acts on the lattice ion which is in addition to the normal elastic restoring resulting from the ion displacement. The stress introduced by the presence of the magnetic field is proportional to the propagation direction and polarization of the elastic wave. Of particular interest in this investigation are low-frequency waves propagating perpendicular to the magnetic field in an isotropic medium with particle displacements (θ,ϕ,ξ) in the (x,y,z) directions. If the wave propagates in the z direction with the field (B) in the y direction, the net stress acting on an elemental elastic volume is the sum of the static elastic reaction forces and the Lorentz force. The magnetic field couples the displacements of the simple modes, so that a shearing stress results from the passage of a longitudinal wave, and a compressional stress from a shear displacement in the x direction. Elastic deformations and their associated accelerations do not include body forces, since these forces lead to acceleration of the elastic element as a whole and do not effect the propagation of a wave. The Lorentz force, which effects the elastic reaction of the ions, is included as part of the stress whose spatial derivative is associated with the elastic accelerating forces. The self-consistent stresses for a wave propagating in the z direction, including stresses generated by the Lorentz forces, are given approximately by

$$\begin{aligned} S_z &= -S_{0z}+Q(\partial\theta/\partial t)B, \\ S_x &= -S_{0x}+Q(\partial\xi/\partial t)B, \\ S_y &= -S_{0y}, \end{aligned} \tag{1}$$

* This work is supported by the National Science Foundation.
[1] M. H. Cohen, M. J. Harrison, and W. A. Harrison, Phys. Rev. 117, 937 (1960).

where Q is the charge per unit area with each ion assumed to have one electronic charge and S_{0z} is the field-independent stress proportional to the strain. The resulting elastic equations, derived by equating the derivative of the stress to the accelerating forces which are not associated with body forces, are

$$\begin{aligned} v_L^2\frac{\partial^2\xi}{\partial z^2}+\frac{QB}{\rho}\frac{\partial^2\theta}{\partial t\partial z}&=\frac{\partial^2\xi}{\partial t^2},\\ v_\mu^2\frac{\partial^2\theta}{\partial x^2}+\frac{QB}{\rho}\frac{\partial^2\xi}{\partial x\partial t}&=\frac{\partial^2\theta}{\partial t^2},\\ v_\mu^2\frac{\partial^2\phi}{\partial y^2}&=\frac{\partial^2\phi}{\partial t^2}, \end{aligned} \tag{2}$$

where ρ is the mass density.

For zero magnetic field these are the wave equations for an isotropic medium describing the propagation of one longitudinal wave (ξ) and two shear waves (θ,ϕ) in the z direction with a velocity of v_L for the longitudinal wave or v_μ for the shear waves. The magnetic field couples the longitudinal wave with shear displacement perpendicular to the field, so that these simple pure modes do not propagate independently. Only the shear wave with the ion displacements parallel to the magnetic field is unaffected when the electron currents are ignored. The dispersion relation with a magnetic field present is obtained by substituting the plane-wave displacements in the z and x directions (ξ,θ) in the first two simultaneous equations and solving the resulting determinant of the coefficients. The resulting second-order equation in v^2 can be simplified for $QB<1$ (which is the pertinent experimental limit) to give the fractional change in elastic modulus as a function of field as

$$\Delta\mathfrak{L}/\mathfrak{L}=Q^2B^2/4v_L^2\rho \tag{3}$$

and

$$\Delta\mu/\mu=Q^2B^2/4v_\mu^2\rho, \tag{4}$$

where $\mathfrak{L}$ is the longitudinal elastic modulus and μ is the shear modulus.

If B is given in gauss, then $Q=(e/c)(\rho/M)^{2/3}$ with c the velocity of light, e the electronic charge, and M the mass of the ion. The factor Q is of the order of 10^{-3} to 10^{-5}. The observed fractional change in modulus for aluminum in a 10-kG field is 10^{-5}, while the value calculated from Eq. (3) is at the most 10^{-10}. The ratio (10^5) between the measured and computed changes is large enough to indicate that the direct action of the magnetic field on the lattice ions is not the dominant influence on the elastic properties of metals in a magnetic field. However, it might be in semiconductors or insulators if techniques could be devised to observe such small changes in the elastic properties.

The coupling of the differential equations leads to two modes which are not purely transverse or longitudinal. The ratio of the transverse to longitudinal displacements for these modes is determined by substituting one of the roots into the differential equation. The mode of interest in this investigation travels at a rate close to the longitudinal velocity, giving a ratio of transverse to longitudinal displacement amplitudes as

$$\theta/\xi=QB/\rho v_L<10^{-6} \quad \text{at} \quad 10\,\text{kG}. \tag{5}$$

This particular mode has a transverse component much smaller than the longitudinal displacement, even in the maximum magnetic fields considered, and is assumed in the following to be purely longitudinal in displacement.

The change in the lattice component of the attenuation is computed by considering the moduli of Eqs. (3) and (4) to be complex and by decomposing the imaginary parts into the losses associated with the complex Lame's constants.[2] The change in attenuation as a function of field results from an increase in the shear component, but is of negligible magnitude compared to the observed values even at the maximum fields. The direct interaction of the lattice ions with the field is thus a negligible contribution to the field dependence of the complex modulus, and the observed changes in elasticity appear to result from the interaction of the field with the conduction electrons.

III. ELECTRON INTERACTION

Passage of the sound wave through the lattice perturbs the local charge-carrier distribution, with its reaction to the lattice displacement contributing to the stiffness of the metal. The presence of a magnetic field alters the charge-carrier reaction to the lattice motion and leads to what is the major contribution to the field dependence of the elastic properties. The net stiffness of the metal is represented to a first approximation by the sum of the stiffness due to the interactions of the lattice ions (L) and the stiffness due to the reaction of the charge carrier to the displacements of the ions (E). The complex elastic constant for a metal is treated as the sum of complex moduli due to the lattice and the charge carriers. The elastic constant of the charge carriers is computed by determining the velocity of sound in an isolated gas acted on by the same self-consistent field generated by the passage of the acoustic wave. The reaction of the charge carriers to the wave is coupled to the lattice by this same electric field. The fractional change in elastic modulus, consistent with the negligible direct effect of the field on the lattice elasticity, is given by

$$\Delta\mathfrak{L}/\mathfrak{L}=\mathrm{Re}\Delta E'/\rho v_0^2, \tag{6}$$

where v_0 is the zero-field wave velocity, ρ the density, and $\Delta E'$ the change in the electron modulus as a function of applied magnetic field. The change in the loss

[2] E. J. Skudrzyk, *Simple and Complex Vibratory Systems* (Pennsylvania State University Press, University Park, Pa., 1968), pp. 103, 105.

factor due to the presence of the field is given by

$$\Delta\eta/\eta_0=\mathrm{Im}\Delta E'/E_0, \tag{7}$$

where the contribution of the charge-carrier damping of the sound wave is assumed to be larger than the attenuation of the lattice even at zero field, and is given by $\mathrm{Im}E_0/\mathrm{Re}E_0$.

The self-consistent field which acts on the charge carriers due to the passage of the sound wave was calculated by Cohen, Harrison, and Harrison[1] using the Boltzmann equation in conjunction with Maxwell's equations. Equations relating the components of the electric field to the ion displacements are given in the Appendix for a wave propagating in the z direction with the magnetic field in the y direction. These equations use the notation of Rodriguez,[3] and are derived on the assumption that the electron relaxation time τ is much less than 1, as is the product of the cyclotron resonance frequency $\omega_c=eB/mc$ and the electron relaxation time.

The internal field generated for a longitudinal wave propagating along the z direction in a magnetic field in the y direction is given approximately by

$$\mathcal{E}_z=-\left\{\frac{\omega^2 v_F^2 m\tau}{2ev_s^2}\left(1-\frac{1}{\sigma_0\tau}\right)+\frac{e\omega^2\tau c^2B^2}{16\pi^2 m\sigma_0^2 v_s^4}-i\left[\frac{mv_F^2\omega}{2ev_s^2}\left(\frac{\omega^2\tau}{4\pi\sigma_0}-1\right)+\frac{\omega\tau eB^2}{4\pi v_s^2\sigma_0 m}\right]\right\}u, \tag{8}$$

where m is the mass of the electron, v_F is the fermi velocity, σ_0 is the zero-field conductivity, v_s is the sound velocity, and u is the complex velocity amplitude of the ion in the z direction. As shown previously, the polarization of the lattice displacement for a low-frequency longitudinal wave does not change appreciably in a magnetic field and the displacement of the ion in the x and y directions induced by the presence of the field is ignored.

The stiffness and losses due to the charge carriers are computed by determining[4] the complex propagation constant γ for a wave propagating in a medium acted on by the self-consistent electric field given in Eq. (8). The complex propagation constant is given by the ratio of the power per unit volume of the wave to the power per unit area. The power per unit area is the product of the stress times the strain[5] and is given by ρvuu^*. The power density is $\mathbf{j}\cdot\mathcal{E}$, where $\mathbf{j}$ is the current density and is given by neu^*, with n the number of particles per unit volume. The real part of the propagation constant is proportional to the loss factor (η) of the material[6]

$$\mathrm{Re}\gamma=\frac{\omega\eta}{2v_s}=\frac{\mathrm{Re}\mathbf{j}\cdot\mathcal{E}}{\frac{1}{2}\rho v_s uu^*}. \tag{9}$$

The loss factor is then given by

$$\eta=\frac{\omega\sigma_0 m^2 v_F^2}{2\rho e^2 v_s^2}\left(1-\frac{1}{\sigma_0\tau}\right)+\frac{\omega^2 c^2 B^2}{16\pi^2\rho\sigma_0 v_s^2}. \tag{10}$$

The first term is field-independent and equivalent to the low-frequency attenuation calculated by Pippard[7] for a spherical fermi surface. The additional terms exhibit a squared field dependence, with no lower-order terms appearing in the calculation.

The imaginary part of the propagation constant is the wave vector and is given by

$$\mathrm{Im}\gamma=\omega/v_s. \tag{11}$$

The square of the velocity of the longitudinal wave in the electron gas is then

$$v_s^2=\frac{mv_F^2}{2M}\left(\frac{\omega^2\tau}{4\pi\sigma_0}-1\right)+\frac{B^2}{4\pi\rho}, \tag{12}$$

where M is the lattice-ion mass. The first factor in the velocity ($mv_F^2/2M$) is the velocity computed for the "jellium" model of a metal.[8]

The fractional change in elasticity of a wave traveling in the complete solid is dependent on the charge-carrier contribution to the total stiffness and is determined from Eq. (12). The field dependence is obtained by substituting the electron stiffness into the Eq. (6) and is

$$\Delta\mathcal{L}/\mathcal{L}=B^2/4\pi\rho v_0^2, \tag{13}$$

a result which exhibits a squared field dependence but is independent of both the frequency and conductivity in the low-frequency approximation. The next higher-order approximation (10^{-8} smaller than the above terms) is also dependent on the square of the field and independent of the conductivity but does have a fourth-order frequency dependence. Similar results[9] were obtained fortuitously by adding the Lorentz force to the elastic equation as a body force.

The field dependence of the fractional change in loss factor is determined from Eq. (7), assuming that the major contribution to the attenuation at zero field is due to the charge carriers, and is

$$\frac{\Delta\eta}{\eta}=\frac{e^2c^2B^2}{8\pi^2\sigma_0^2 m^2 v_F^2 v_s^2}, \tag{14}$$

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[3] S. Rodriguez, Phys. Rev. **130**, 1778 (1963).

[4] The complex propagation constant is given by $\gamma=\alpha+iq$, where α is the attenuation coefficient and $q=\omega/v$ is the wave vector.

[5] G. B. Thurston and S. Wu, J. Acoust. Soc. Am. **34**, 653 (1962).

[6] The loss factor is proportional to the ratio of the energy loss per cycle to the energy stored per cycle, and is $\eta=2\alpha v/\omega$, with the attenuation coefficient (α) in nepers.

[7] A. B. Pippard, Phil. Mag. **46**, 1104 (1955).

[8] C. Kittel, *Quantum Theory of Solids* (John Wiley & Sons, Inc., New York, 1963), pp. 142, 144.

[9] S. Rodriguez, Phys. Letters **2**, 271 (1962).

which is proportional to the square of the field and inversely proportional to the square of the conductivity.

Similar procedures are used to determine the velocity and loss factor for the two transverse modes traveling in the z direction. For the wave with a particle displacement (θ) perpendicular to the field direction the charge-carrier gas velocity is

$$v_s^2=\frac{\omega c^2}{4\pi\sigma_0}\left(1-\frac{B^2}{4\pi\rho c^2}\right), \tag{15}$$

which exhibits the squared field dependence similar to the longitudinal mode. The loss factor is

$$\eta=\sigma_0 B^2/\omega\rho v_s^2\left(\frac{16\pi^2\sigma_0^2}{\omega^2}+\frac{c^4}{v_s^4}\right), \tag{16}$$

again with a squared field dependence.

For ion displacement (ϕ) in the direction of the field, the charge-carrier gas velocity is

$$v_s^2=\frac{\omega c^2}{4\pi\sigma_0}\left(1-\frac{neB}{8\pi\rho c\sigma_0}\right), \tag{17}$$

exhibiting only a linear dependence on field, but with the same zero-field value as for the transverse wave with polarization perpendicular to the field. The loss factor is

$$\eta=2necB/\omega\rho v_s^2\left(\frac{16\pi^2\sigma_0^2}{\omega^2}+\frac{c^4}{v_s^4}\right), \tag{18}$$

with a linear field dependence similar to that of the velocity.

IV. EXPERIMENTAL RESULTS

The magnetic field dependence of the complex elasticity is observed through measurement of the field dependence of the resonance frequency (~40 kHz) and decay time of a longitudinally driven rod (~2 in.×¼ in. diam). The end of the rod is capacitively driven and the vibration amplitude detected by an FM receiver. The resolution of the fractional change in resonance frequency is proportional to the loss factor and is typically better than one-half part per million. Measurements were made at 77°K in fields to 13 kG.

A squared magnetic field dependence of the fractional change in modulus is observed in polycrystalline niobium and tantalum (Fig. 1), in agreement with previous observations[10] and with the field dependence predicted by Eq. (13). The calculated and observed coefficient of B^2 is shown in Table I to be in substantial agreement for the field dependence of the modulus of niobium and in very close agreement for tantalum. Determination of the field dependence of the stiffness of the material does not provide any more information concerning the microscopic state of the charge carriers than is obtained at zero field but does substantiate the procedure used in the calculation. However, the attenuation of the wave calculated by this same procedure does reflect the microscopic state of the charge carriers [Eq. (14)] through the conductivity, charge-carrier mass, and Fermi velocity.

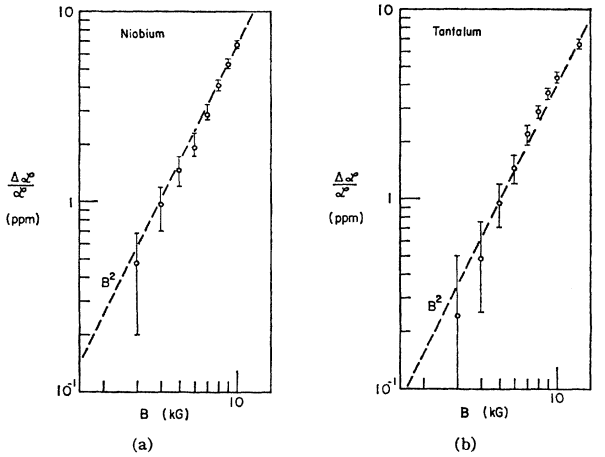

FIG. 1. The magnetic field dependence of the fractional change in stiffness in polycrystalline niobium (a) and tantalum (b) at 77°K. The dashed line depends on the square of the field with a coefficient adjusted to fit the measured values and appearing in Table I.

The observed field dependence of the fractional change in the loss factor of polycrystalline niobium and tantalum (Fig. 2) exhibits a squared dependence similar to the stiffness and in agreement with Eq. (14). The magnitude of the fractional change is determined by using measured values of the zero-field conductivity[11] and the sound velocity, and assuming the charge-carrier mass to be the electron mass. The remaining parameter needed to calculate the loss factor is the Fermi velocity spatially averaged over all the crystal, since the results shown in Fig. 2 are for polycrystalline materials. The values of the Fermi velocity to fit the measured field dependence are shown in Table II. For comparison the Fermi velocity for a free-charge-carrier gas is also presented. This calculation assumes that one hole per atom is contributed on the average,[12] and is in reasonable agreement with the measured values determined from Fig. 2.

TABLE I. Comparison of measured and calculated coefficient of B^2.

	Calculated coefficient (ppm/kG²)	Observed coefficient (ppm/kG²)
Niobium	0.84	0.68
Tantalum	0.42	0.44

[10] G. A. Alers and P. A. Fleury, Phys. Rev. **129**, 2425 (1963).

[11] G. K. White and S. B. Woods, Phil. Trans. Roy. Soc. (London) Ser. **A251**, 273 (1959).

[12] E. Fawcett, W. A. Reed, and R. R. Soden, Phys. Rev. **159**, 533 (1967).

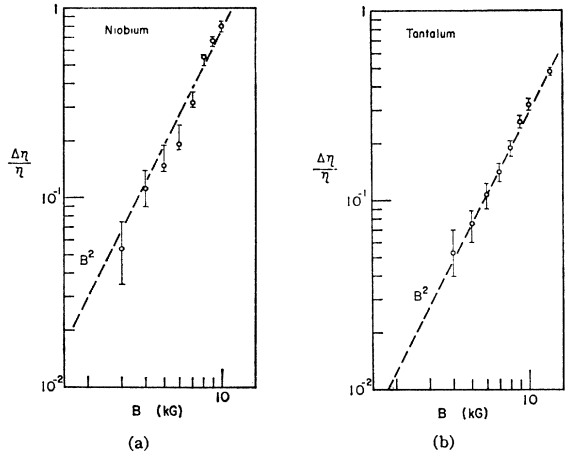

FIG. 2. The magnetic field dependence of the fractional change in loss factor for polycrystalline niobium (a) and tantalum (b) at 77°K. The dashed curve depends on the square of the field and is adjusted to fit the experimental values consistent with the measured conductivity, charge-carrier mass, and sound velocity to give the Fermi velocity in Table II.

The fractional change in modulus for a single crystal of niobium exhibits the same general behavior as the polycrystalline material. A squared field dependence of the change in modulus is observed for different magnetic field directions which are perpendicular to the displacement and propagation direction of the wave (Fig. 3). A slight anisotropy noted in the fractional change in modulus at 13 kG is exhibited in the polar plot of Fig. 4. The deviation from isotropy is at a maximum with the field in the [110] direction, but

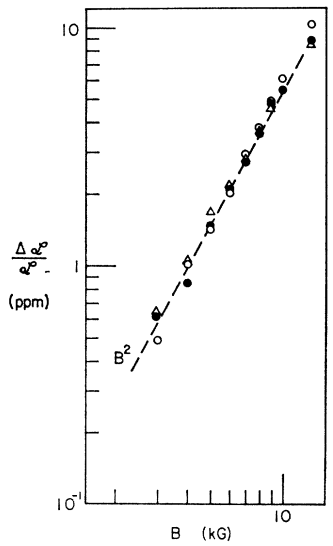

FIG. 3. The fractional change in elastic modulus in a single crystal of niobium as a function of direction and magnitude of the magnetic field. The modulus changes are almost identical for the magnetic field in the [111] (△), [001] (●), and [110] (○) directions and perpendicular to the direction of the ion displacement $\xi-[1\bar{1}0]$. A squared field dependence similar to the polycrystalline niobium is obtained.

TABLE II. Comparison of calculated Fermi velocity from Eq. (14) and for a free-charge-carrier gas.

	Measured Fermi velocity (10^8 cm/sec)	Free-charge-carrier gas Fermi velocity (10^8 cm/sec)
Niobium	1.56	1.86
Tantalum	2.49	1.87

even there it is only 15%. The general isotropy of the fractional change in modulus suggests that the internal electric fields, generated transverse to the ion motion, do not induce an appreciable transverse ion displacement.

The fractional change in loss factor for the single crystal also exhibits a squared magnetic field dependence

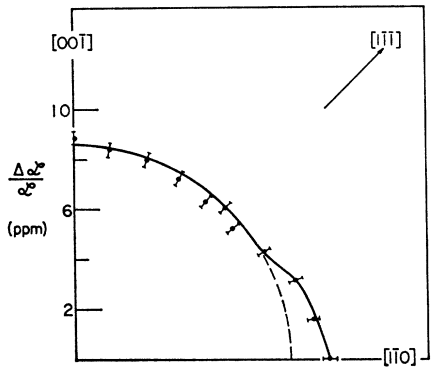

FIG. 4. The fractional change in modulus at a magnetic field of 13 kG as a function of field direction. The changes in modulus are approximately isotropic, indicating a minimal transverse ion displacement due to the presence of the magnetic field.

similar to the polycrystalline niobium. The anisotropy in the magnitude of the change in loss factor is shown in Fig. 5 by the separation of the curves for the [110], [111], and [100] field directions. The anisotropy of the fractional change in loss factor is more obvious in the

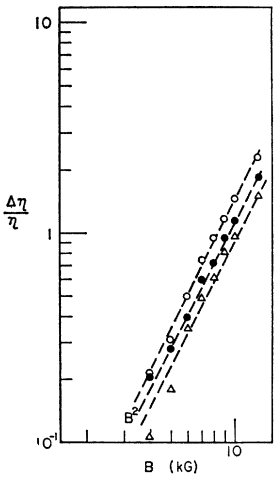

FIG. 5. The fractional change in loss factor in a single crystal of niobium as a function of the magnitude and direction of the field. A quadratic field dependence is observed similar to that of the polycrystalline niobium. The magnitude of the change in loss factor is anisotropic and largest for the field in the [110] (○) direction compared to the [111] (△) and [001] (●) directions.

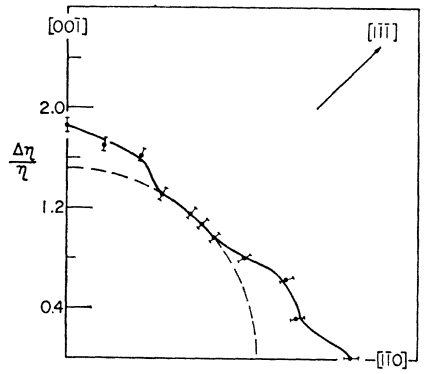

FIG. 6. The fractional change in loss factor at a field of 13 kG as a function of magnetic field orientation. A decided anisotropy is noted, with the minimum change for the field in the [111] direction.

polar plot of Fig. 6 for a field of 13 kG oriented in a variety of directions. If the induced transverse displacement is small, as is indicated by the isotropy of the change in modulus, the expression for the fractional change in loss factor [Eq. (14)] derived for an isotropic medium can be used as a first approximation for the single crystal if the Fermi velocity is assumed to be anisotropic. The fractional change in the loss factor is then a function of the Fermi velocity which is an average over the plane perpendicular to the field direction. The average Fermi velocity determined from Fig. 5 is largest in the (111) plane compared to the (110) and (100) planes. The average Fermi velocities computed from Fig. 5 and Eq. (14) are seen in Table III to be the order of magnitude expected and comparable to the polycrystalline values. The values determined from the fractional change in attenuation would tend to be an upper limit, since the measured zero-field loss factor includes contributions from support losses and lattice attenuation. The relative magnitude of the average fermi velocity for various field directions is not affected by the zero-field values, and can be determined accurately.

TABLE III. The average Fermi velocity in a plane perpendicular to the magnetic field, computed from Fig. 5 and Eq. (14).

Plane	Average Fermi velocity (10^8 cm/sec)
(001)	1.28
(100)	1.12
(111)	1.41

V. CONCLUSIONS

The direct interaction of the magnetic field with the lattice ions is negligible compared to the observed field dependence of the complex elasticity. The charge-carrier contribution to the field dependence of the elasticity is calculated using a low-frequency approximation and is seen to agree with the observed changes in the niobium and tantalum. From the field dependence of the loss factor an average fermi velocity can be computed, which for polycrystalline niobium and tantalum is close to the fermi gas approximation. Application of the isotropic calculation to single-crystal niobium provides a means of determining the fermi velocity averaged over a plane perpendicular to the magnetic field direction.

ACKNOWLEDGMENTS

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APPENDIX

The self-consistent electric field generated by lattice-ion motion and the associated charge-carrier reaction was calculated by Cohen, Harrison, and Harrison. This development requires knowledge of the resistivity tensor which at low frequencies represents the electron as being driven in phase or resistively by the ion motion. The simultaneous equations relating the ion displacements to the electric fields for a wave propagating in the z direction are given in the following, using the notation of Rodriguez:

$$\mathcal{E}_x(1-i\beta)-\mathcal{E}_z\frac{i\omega\omega_c\tau}{4\pi\sigma_0}=\frac{nei\omega\omega_c\tau}{\sigma_0}\xi, \tag{19}$$

$$\mathcal{E}_y(1-i\beta)=(nei\omega/\sigma_0)\xi, \tag{20}$$

$$-\mathcal{E}_x i\beta\omega_c\tau+\mathcal{E}_z\left(1+\frac{i\omega}{4\pi\sigma_0}\right)=\frac{(1-i\omega\tau)mv_F^2\omega^2}{2e(1+\omega^2\tau^2)v_s^2}\xi$$
$$-\frac{nei\omega}{\sigma_0}(\phi-\theta\omega_c\tau), \tag{21}$$

where $\beta=c^2\omega/v_s^2 4\pi\sigma_0$ and the magnetic field is in the y direction.