From (C8) and (C9) we find that the coefficients of the elliptical integrals all vanish at T_c which renders continuity in the internal energy at the critical point. To investigate the nature of the singularity of the specific heat, we note that the asymptotic expression for the energy as $T \to T_c^{\pm}$ is

$$E \to M(y,z)K(k),$$

where $M(y,z)$ is regular in T and vanishes identically at T_c. The specific heat then takes the following asymptotic expression:

$$C \to \frac{dM}{dT}K(k)+M\frac{d}{dT}K(k).$$

The second term on the right-hand side is finite at T_c. Therefore, the singular behavior of C is due entirely to the first term. It can be shown that dM/dT is regular at T_c. It follows that the singularity of the specific heat arises from that of the elliptical integral K and is logarithmic, as is understood for the usual Ising model.

Note added in manuscript. (1). Professor C. Domb has kindly called our attention to an earlier paper [C. Domb and R. B. Potts, Proc. Roy. Soc. **A210**, 125 (1951)], in which some of the results of Sec. II have been discussed from the point of view of the method of transfer matrix. Their approximation for the equivalent neighbor model yields $e^{-2K}=\frac{1}{3}(\sqrt{10}-1)=0.72076$ for the transition temperature as compared to the value 0.68946 of the present paper and the presumably exact Padé value 0.6837.

(2). We also received a preprint by N. W. Dalton and D. W. Wood in which the Padé analysis of Ising model with higher neighbor interactions of Ref. 8 is extended to include the low-temperature expansions as well as models with nonequivalent higher neighbors.

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Exact Solution for a Linear Chain of Isotropically Interacting Classical Spins of Arbitrary Dimensionality*

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The isotropic Hamiltonian $\mathcal{H}^{(\nu)}=-J\sum_{j=1}^{N-1}\mathbf{S}_j\cdot\mathbf{S}_{j+1}$ is considered for an open linear chain of N ν-dimensional vector spins $\mathbf{S}_j$; $\mathcal{H}^{(\nu)}$ reduces to the $S=\frac{1}{2}$ Ising, planar, and Heisenberg models for $\nu=1$, 2, and 3. The thermodynamic properties (including the susceptibility) of $\mathcal{H}^{(\nu)}$ are found for ferromagnetic ($J>0$) and antiferromagnetic ($J<0$) exchange interactions for all temperatures T and all spin dimensionalities ν. The manner in which the various properties depend upon T and ν is studied; in particular we find (a) that although the chain of spins does not display long-range order except at $T=0$ for any value of ν, most of the properties vary monotonically with ν (in such a way that, e.g., the degree of "short-range order" decreases with increasing ν); and (b) that as the spin dimensionality increases without limit, all of the calculated properties approach precisely those predicted by the Berlin-Kac spherical model.

I. INTRODUCTION

THERE exist comparatively few nontrivial statistical mechanical models which have been solved exactly in more than one dimension.[1] One motivation for considering exactly soluble one-dimensional models is that their solutions may possibly aid in judging the validity of approximation techniques which are used in three dimensions.[2] A second motivation is that results discovered for one-dimensional models are sometimes generalizable to higher dimensionalities. Finally, a one-dimensional model may serve as a reasonable approximation to some special physical system. For example, there exist materials in which the magnetic ions may be considered to form "linear chains" so that interactions between spins within the chains are ap-

* A different derivation of the partition function is presented in H. E. Stanley, Proceedings of the 1968 IUPAP Conference on Statistical Mechanics, Kyoto, J. Phys. Soc. Japan (to be published).

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[1] Two notable examples are the two-dimensional Ising model in zero field [L. Onsager, Phys. Rev. **65**, 117 (1944)] and, more recently, the various two-dimensional "ferroelectric" models [see, e.g., E. H. Lieb, 1968 Boulder Lectures in Theoretical Physics (to be published)]. For a comprehensive introduction to exactly soluble models of interacting particles in one-dimension, see E. H. Lieb and D. C. Mattis, *Mathematical Physics in One Dimension* (Academic Press Inc., New York, 1966).

[2] For example, many approximation schemes (such as extrapolation from high-temperature expansions) have been tested on the Ising model for one-dimensional and two-dimensional lattices.

preciably stronger than interactions between spins belonging to different chains.[3]

In this paper we consider a system of isotropically interacting classical spins of arbitrary dimensionality situated on a one-dimensional (linear chain) lattice. Our Hamiltonian,[4] then, is

$$\mathcal{H}^{(\nu)} \equiv -J \sum_{j=1}^{N-1} \mathbf{S}_j \cdot \mathbf{S}_{j+1}, \qquad (1a)$$

where $\mathbf{S}_j$ is a ν-dimensional classical spin (or "vector") of magnitude $\lambda^{1/2}$ and localized on site j. Thus if $[\sigma_1(j), \sigma_2(j), \cdots, \sigma_\nu(j)]$ are the Cartesian coordinates of $\mathbf{S}_j$, we require that

$$\sum_{n=1}^{\nu} \sigma_n{}^2(j) = \lambda; \quad (j=1,2,\cdots,N). \qquad (1b)$$

Here $-J\lambda$ is the energy of a pair of nearest-neighbor spins[5]; thus at $T=0$ all nearest-neighbor pairs of spins are parallel if J is positive, and antiparallel if J is negative. The Hamiltonian $\mathcal{H}^{(\nu)}$ reduces to the $S=\frac{1}{2}$ Ising model,[6] classical planar model,[7] and classical Heisenberg model,[8] respectively for $\nu=1$, 2, and 3.[9]

Here we obtain exact expressions, valid for arbitrary ν, for the free energy, entropy, specific heat, internal energy, and susceptibility for the system described by (1). Although we shall see that the linear chain of ν-dimensional spins does not display infinite-range (or even long-range) order at any temperature except $T=0$, we shall comment upon two fairly striking features of our solution: (a) At high (and also at low) temperatures, most of the various thermodynamic functions calculated vary monotonically with spin dimensionality ν. (b) As ν increases without limit, the free energy and susceptibility approach those obtained for the Berlin-Kac spherical model,[10]

$$\mathcal{H}^{SM} \equiv -J \sum_{j=1}^{N-1} \mu_j \mu_{j+1},$$

where the spins μ_j are continuous variables subject only to the single constraint

$$\sum_{j=1}^{N} \mu_j{}^2 = N.$$

In Sec. II we derive the partition function $Q_N{}^{(\nu)}$, from which follow the free energy, entropy, specific heat, and internal energy. These thermodynamic quantities are each plotted as functions of T and $1/T$ for representative values of ν. In Sec. III we obtain the susceptibility from the two-spin correlation function, and display the antiferromagnetic case graphically.

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[3] Hence the intrachain interactions may be treated exactly, and the interchain interactions approximated, say, by a molecular field. Such a treatment was first carried out for Ising ($\nu=1$) interactions by H. Sato [J. Phys. Chem. Solids **19**, 54 (1961)], A. H. Cooke, D. T. Edmonds, C. B. P. Finn, B. R. Heap, and W. P. Wolf [in *Proceedings of the Seventh International Conference on Low Temperature Physics, 1960*, edited by G. M. Graham and A. C. Hollis (University of Toronto Press, Toronto, 1960), p. 107], and J. W. Stout and R. C. Chisholm [J. Chem. Phys. **36**, 979 (1962)], and for Heisenberg ($\nu=3$) interactions by H. E. Stanley and T. A. Kaplan [J. Appl. Phys. **38**, 975 (1967)]. We have generalized this idea to all ν [H. E. Stanley (to be published)].

[4] This Hamiltonian was first considered for general lattices by H. E. Stanley [Phys. Rev. Letters **20**, 589 (1968)] in an attempt to ascertain by high-temperature approximations the dependence of critical properties upon dimensionality of spins.

[5] Note that in Ref. 4 the energy of a pair of nearest-neighbor spins was $-2J\nu$; thus the J of this paper is called $2J$ in Ref. 4, and the parameter λ of this paper was chosen for convenience to be ν in Ref. 4. We shall leave the parameter λ unspecified for all the expressions which we shall derive, but when we actually plot the various functions we shall take $\lambda=\nu$ in order to facilitate study of the dependence upon spin dimensionality. We can equally well choose $\lambda=1$ (spins of unit length), in which case we must "renormalize" the exchange integral $J\rightarrow J_0/\nu$ in order to take the $\nu\rightarrow\infty$ limit. Then the Hamiltonian

$$\mathcal{H}^{(\nu)} = -J \sum_{j=1}^{N-1} \mathbf{S}_j \cdot \mathbf{S}_{j+1}$$

(where the spins are of length $\nu^{1/2}$) becomes identically

$$\mathcal{H}^{(\nu)} = -J_0 \nu^{-1} \sum_{j=1}^{N-1} \mathbf{S}_j \cdot \mathbf{S}_{j+1} = -J_0 \sum_{j=1}^{N-1} \mathbf{s}_j \cdot \mathbf{s}_{j+1},$$

where the spins $\mathbf{s}_j \equiv \nu^{-1/2}\mathbf{S}_j$ are of unit length and the free energy is given by Eq. (14) with $\lambda=1$ and $x\rightarrow x_0 \equiv J_0/kT \equiv J\nu/kT$. Thus x_0 is of order ν and we again use the same asymptotic expansion for Bessel functions with large argument as well as large order.

[6] There exists a small class of magnetic materials whose spin interactions are well described by the Ising model; such materials (e.g., DAG, dysprosium aluminum garnet) have been studied extensively by W. P. Wolf and co-workers [see D. C. Mattis and W. P. Wolf, Phys. Rev. Letters **16**, 899 (1966) and references contained therein; also W. P. Wolf (to be published)]. The $S=\frac{1}{2}$ Ising model has also come to serve as a practical model for a binary alloy and a classical "lattice gas."

[7] The classical planar model has recently received attention as a fairly crude lattice model for the transition in a Bose fluid. See, e.g., V. G. Vaks and A. I. Larkin, Zh. Eksperim i Teor. Fiz. **49**, 975 (1965) [English transl.: Soviet Phys.—JETP **22**, 678 (1966)]; R. J. Bowers and G. S. Joyce, Phys. Rev. Letters **19**, 630 (1967); H. E. Stanley, *ibid.* **20**, 150 (1968). The planar spin model ($\nu=2$) is not to be confused with the (quantitatively similar but qualitatively different) XY model (a "Heisenberg model"

$$\mathcal{H} = -\sum_{\alpha=1}^{3} \sum_{ij} J_\alpha(ij) S_\alpha(i) S_\alpha(j)$$

with $J_1=J_2$ and $J_3=0$), which has been invoked in connection with certain magnetic insulators [such as $Gd_2(SO_4)_3 \cdot 8H_2O$]—see, e.g., D. D. Betts and M. H. Lee [Phys. Rev. Letters **20**, 1507 (1968)]; R. F. Wielinga, J. Lubbers, and W. J. Huiskamp [Physica **37**, 375 (1967)] and references contained therein.

[8] The classical Heisenberg model was first considered in detail by G. Heller and H. A. Kramers [Proc. Roy. Acad. Sci. Amsterdam **37**, 378 (1934)]. The fact that for many materials with $S>\frac{1}{2}$, the classical ($S\rightarrow\infty$) limit is a good approximation for $T\gtrsim T_c$ was realized independently by H. E. Stanley and T. A. Kaplan [Phys. Rev. Letters **16**, 981 (1966)]; G. S. Joyce and R. G. Bowers [Proc. Phys. Soc. (London) **88**, 1053 (1966)]; and P. J. Wood and G. S. Rushbrooke [Phys. Rev. Letters **17**, 307 (1966)].

[9] In the case of the one-dimensional (linear-chain) lattice, $\mathcal{H}^{(\nu)}$ has been solved exactly for $\nu=1$ by E. Ising [Z. Physik. **31**, 253 (1925)] and for $\nu=3$ by M. E. Fisher [Am. J. Phys. **32**, 343 (1964)], and by T. Nakamura [J. Phys. Soc. Japan **7**, 264 (1952)]. We wish to thank Professor S. Katsura and Professor T. Oguchi for calling this (generally overlooked) work of Nakamura to our attention. The case of a *closed* chain (or ring) with $\nu=2$ was treated by G. S. Joyce [Phys. Rev. **155**, 478 (1967)].

[10] T. H. Berlin and M. Kac, Phys. Rev. **86**, 821 (1952).

II. PARTITION FUNCTION

The normalized partition function corresponding to $\mathcal{H}^{(\nu)}$ of Eq. (1) is

$$Q_N^{(\nu)}(x)=Z_N^{(\nu)}(x)/Z_N^{(\nu)}(0), \tag{2}$$

with[11]

$$Z_N^{(\nu)}(x)=\int\cdots\int d\mathbf{S}_1\cdots d\mathbf{S}_N \prod_{j=1}^{N}[\delta(\lambda-\mathbf{S}_j^2)] \times\exp\left[x\sum_{j=1}^{N-1}\mathbf{S}_j\cdot\mathbf{S}_{j+1}\right], \tag{3}$$

where $x\equiv J/kT$, and where the N δ functions represent the constraints that each spin has magnitude $\lambda^{1/2}$.

Consider first the integration over spin $\mathbf{S}_N$,

$$\mathfrak{z}^{(\nu)}(x)\equiv\int d\mathbf{S}_N\delta[\lambda-\mathbf{S}_N^2]\exp[x\mathbf{S}_{N-1}\cdot\mathbf{S}_N]. \tag{4}$$

In Appendix A it is shown that

$$\mathfrak{z}^{(\nu)}(x)=\tfrac{1}{2}x(2\pi/x)^{\nu/2}I_{\nu/2-1}(\lambda x), \tag{5}$$

independent of the orientation of $\mathbf{S}_{N-1}$; here $I_n(z)$ is a modified Bessel function of the first kind. Thus we obtain

$$Z_N^{(\nu)}(x)=\mathfrak{z}^{(\nu)}(0)[\mathfrak{z}^{(\nu)}(x)]^{N-1}, \tag{6}$$

where, from (5),

$$\mathfrak{z}^{(\nu)}(0)=(\lambda\pi)^{\nu/2}/\lambda\Gamma(\tfrac{1}{2}\nu). \tag{7}$$

Hence, from Eq. (2)

$$Q_N^{(\nu)}(x)=[\mathfrak{z}^{(\nu)}(x)/\mathfrak{z}^{(\nu)}(0)]^{N-1}, \tag{8}$$

or, on using (5) and (7),

$$Q_N^{(\nu)}(x)=[(\tfrac{1}{2}\lambda x)^{1-\nu/2}\Gamma(\tfrac{1}{2}\nu)I_{\nu/2-1}(\lambda x)]^{N-1}. \tag{9}$$

Equation (9) is an exact expression for the normalized partition function of a linear chain containing an arbitrary number N of ν-dimensional spins of magnitude $\lambda^{1/2}$.

In the following we shall calculate exact expressions for the various normalized thermodynamic functions per spin, where by normalized we mean that we divide each function by either J or k in order that it be dimensionless, and we divide by λ in order to facilitate study of the depencence upon spin dimensionality.[5]

The internal energy per spin $E^{(\nu)}\equiv\lim_{N\to\infty}N^{-1}\langle\mathcal{H}^{(\nu)}\rangle$ is directly related to the partition function via $\langle\mathcal{H}^{(\nu)}\rangle=-(\partial/\partial\beta)\ln Q_N^{(\nu)}(x)$. Thus we obtain

$$\bar{E}^{(\nu)}\equiv E^{(\nu)}/\lambda J=-y_\nu, \tag{10}$$

where

$$y_\nu\equiv\frac{1}{\lambda}\frac{\partial}{\partial x}\ln[x^{1-\nu/2}I_{\nu/2-1}(\lambda x)] \tag{11}$$

is the (normalized) nearest-neighbor spin correlation function $\lambda^{-1}\langle\mathbf{S}_j\cdot\mathbf{S}_{j+1}\rangle$. Thus, e.g., $y_1=\tanh\lambda x$, $y_2=I_1(\lambda x)/I_0(\lambda x)$, $y_3=\mathcal{L}(\lambda x)=\coth\lambda x-1/\lambda x$, and

$$y_\infty=2x/\{1+[1+(2x)^2]^{1/2}\}. \tag{12}$$

For ν finite, one may relate the Bessel-function derivative in Eq. (11) to a Bessel function of a higher order, and thereby write

$$y_\nu=I_{\nu/2}(\lambda x)/I_{\nu/2-1}(\lambda x). \tag{13}$$

In Fig. 1 we indicate the temperature dependence of $\bar{E}^{(\nu)}$ for various values of ν. We note that the correlation between nearest-neighbor spins decreases monotonically with increasing spin dimensionality, as one might have expected intuitively.

The entropy $S^{(\nu)}$ may be obtained from the free energy per spin $\psi^{(\nu)}=-kT\lim_{N\to\infty}N^{-1}\ln Q_N^{(\nu)}(x)$ by means of $\psi^{(\nu)}\equiv E^{(\nu)}-TS^{(\nu)}$ or, equivalently, $S^{(\nu)}=(-\partial/\partial T)\psi^{(\nu)}$. Since from Eq. (9)[11]

$$\bar{\psi}^{(\nu)}\equiv\psi^{(\nu)}/\lambda J=-(\lambda x)^{-1}\ln[(\tfrac{1}{2}\lambda x)^{1-\nu/2} \times\Gamma(\tfrac{1}{2}\nu)I_{\nu/2-1}(\lambda x)], \tag{14}$$

we have

$$\bar{S}^{(\nu)}\equiv S^{(\nu)}/\lambda k=\lambda^{-1}\ln[\exp(-\lambda x y_\nu)(\tfrac{1}{2}\lambda x)^{1-\nu/2} \times\Gamma(\tfrac{1}{2}\nu)I_{\nu/2-1}(\lambda x)]. \tag{15}$$

The free energy and the entropy are plotted in Figs. 2 and 3. For $\nu>1$ the entropy diverges logarithmically as $T\to 0$ and this is reflected in the fact that the specific heat does not approach zero as $T\to 0$.

Finally we calculate the specific heat $C^{(\nu)}=(\partial/\partial T)E^{(\nu)}=T(\partial/\partial T)S^{(\nu)}$. From Eqs. (10) and (11), or, alternatively, from Eq. (15), we obtain

$$\bar{C}^{(\nu)}\equiv C^{(\nu)}/\lambda k=x^2(\partial y_\nu/\partial x), \tag{16}$$

or, for ν finite, simply

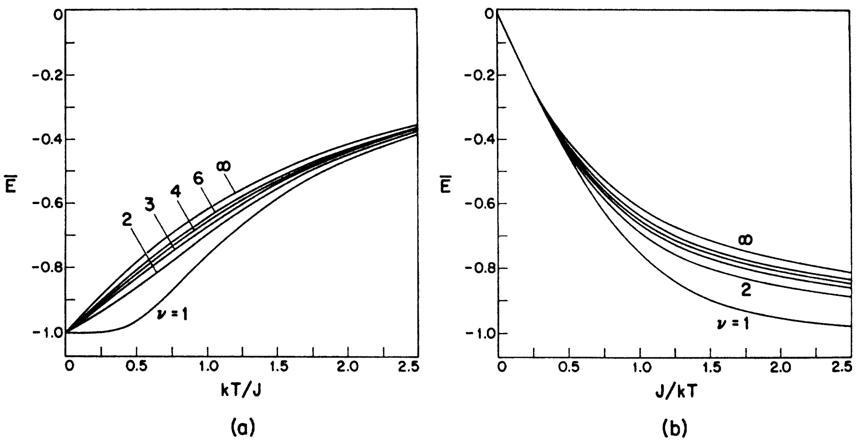
$$\bar{C}^{(\nu)}=\lambda x^2\{1-y_\nu[(\nu-1)/\lambda x+y_\nu]\}, \tag{17}$$

with y_ν given by Eq. (13). The specific heat is plotted against $t(\equiv kT/J)$ and $x(\equiv 1/t)$ for various spin dimensionalities in Fig. 4. (We note that the maximum in the specific heat occurs at the highest temperature for $\nu=1$, becomes a broad flat region at $T\cong 0$ for $\nu=3$, and disappears altogether for $\nu>3$). The presence of a maximum in $C^{(\nu)}$ for $\nu=1$ and 2 at a nonzero temperature is reflected in the initial upward curvature of the corresponding energy curves in Fig. 1(a).

Low-temperature and high-temperature expansions are developed in Appendices B and C, respectively, for $\bar{E}^{(\nu)}$, $\bar{\psi}^{(\nu)}$, $\bar{S}^{(\nu)}$ and $\bar{C}^{(\nu)}$. Also, simplified expressions for $\nu=1$, 2, 3, and ∞ of $\bar{\psi}$, $\bar{E}$, and $\bar{C}$ are given in Table I.

[11] For $\nu=1$, the Ising model, we associate an extra factor of 2 with each of the $N\delta$-functions in Eq. (3) [since $\delta(S-1)+\delta(S+1)=2\delta(S^2-1)$]. This in turn gives rise, for $\nu=1$ only, to an additional factor of 2^N in Eqs. (6), (8), and (9) and to an additional factor of 2 in the argument of the logarithm in Eqs. (14) and (15). Thus, the only effect of this additional factor of 2^N in the Ising model partition function is to add a term $\delta\bar{\psi}^{(1)}=-t\lambda^{-1}\ln 2$ to the free energy and a term $\delta\bar{S}^{(1)}=\lambda^{-1}\ln 2$ to the entropy.

FIG. 1. Normalized internal energy $\bar{E}^{(\nu)}=-y_\nu$ (with $\lambda=\nu$) as a function of (a) $t\equiv kT/J$, and (b) $x\equiv J/kT$ for $\nu=1$ (Ising model), $\nu=2$ (planar model), $\nu=3$ (Heisenberg model), $\nu=4$, $\nu=6$, and $\nu=\infty$ (spherical model). Curves for other values of ν lie in between the values shown. The correlation function between two spins located R sites apart is $(y_\nu)^R$, and decreases monotonically with ν for every value of R. Since $y_\nu<1$ for $T>0$, the spontaneous magnetization $[\sim\lim_{R\to\infty}(y_\nu)^R]$ vanishes for $T>0$.



III. SPIN CORRELATION FUNCTIONS

The two-spin correlation function $\langle \mathbf{S}_0\cdot\mathbf{S}_R\rangle$ for spins localized on sites which are a distance R apart is obtained in exactly the same fashion as the partition function (by performing first the integration over $\mathbf{S}_N$, then $\mathbf{S}_{N-1},\cdots$) with the result

$$\lambda^{-1}\langle \mathbf{S}_0\cdot\mathbf{S}_R\rangle=(y_\nu)^R. \qquad (18)$$

From Eq. (11) (or Fig. 1) we note that $y_\nu<1$ for all finite values of x; hence, for $T>0$ the spontaneous magnetization per spin (or "infinite-range order") M is given by

$$M^2\propto \lim_{R\to\infty}\langle \mathbf{S}_0\cdot\mathbf{S}_R\rangle=0. \qquad (19)$$

Thus, although the correlation between two spins increases monotonically with decreasing temperature, $M=0$ for all $T>0$. However, for $T=0$, $y_\nu=1$ and M achieves its saturation value.

Although the absence of a spontaneous magnetization (alone) does not mathematically preclude the possibility that the normalized zero-field susceptibility per spin $\chi\equiv(\partial M/\partial H)_0=(-\partial^2\psi/\partial H^2)_0$ diverges to infinity at some nonzero temperature, it is not difficult to show that, for a ferromagnet, χ rises smoothly with decreasing temperature, approaching infinity as $T\to 0$. Now the free energy $\psi^{(\nu)}$ has been calculated in the presence of a magnetic field H only for the case $\nu=1$, and we have not succeeded in generalizing the calculation to arbitrary ν. However, one can easily obtain the zero-field susceptibility per spin directly from the zero-field two-spin correlation function of Eq. (18) with the result

$$\chi^{(\nu)}=(\lambda m^2/\nu kT)(1+y_\nu)/(1-y_\nu), \qquad (20)$$

where $m\equiv g\mu_B$ is the magnetic moment per spin. The temperature dependence of the susceptibility for the case $J<0$(antiferromagnet) is shown in Fig. 5 for various values of ν; for $J>0$ (ferromagnet), $\chi^{(\nu)}$ diverges to infinity as $T\to 0$ in a fashion which is described in Appendix B in terms of the low-temperature expansion of y_ν.

Higher-order correlation functions may also be calculated, with a corresponding increase in labor; for example, we obtained the four-spin correlation function. Knowledge of this function enables us to generalize to all ν a model of two- and three-dimensional lattices, considered to be composed of chains of classical spins, in which we treat the intrachain interactions exactly and the interchain interactions via a molecular-field approximation.[3]

FIG. 2. Normalized free energy $\bar{\psi}^{(\nu)}\equiv\psi^{(\nu)}/\lambda J$, as given by Eq. (14) (with $\lambda=\nu$), as a function of (a) $t\equiv kT/J$, and (b) $x\equiv J/kT$ for $\nu=2$, 3, 4, 6, and ∞. Note that the curvature is the same for *all* values of ν, reflecting the fact that $\partial S^{(\nu)}/\partial T=\partial^2\bar{\psi}^{(\nu)}/\partial T^2>0$. (See Fig. 3.)

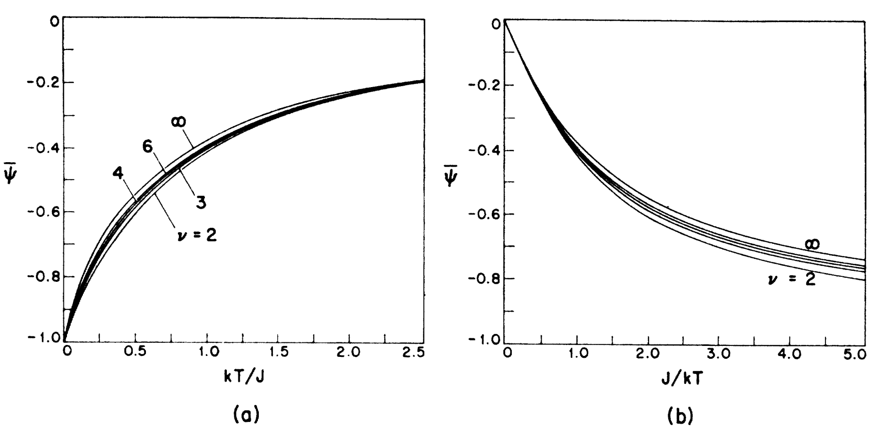


TABLE I. Thermodynamic formulas for selected spin dimensionalities ν.

	$z\equiv\lambda x\equiv\lambda J/kT;\quad t\equiv(1/x);\quad \rho\equiv[1+(2x)^2]^{1/2}$		
ν	$\bar{\psi}^{(\nu)}\equiv\psi^{(\nu)}/\lambda J$	$\bar{E}^{(\nu)}\equiv E^{(\nu)}/\lambda J$	$\bar{C}^{(\nu)}\equiv C^{(\nu)}/\lambda k$
1	$-t\lambda^{-1}\ln[2\cosh z]$	$-\tanh z$	$\lambda^{-1}[z\,\mathrm{sech} z]^2$
2	$-t\lambda^{-1}\ln[I_0(z)]$	$-I_1(z)/I_0(z)$	$\lambda^{-1}\{z^2+z\bar{E}-[z\bar{E}]^2\}$
3	$-t\lambda^{-1}\ln[(\sinh z)/z]$	$-[\coth z-1/z]$	$\lambda^{-1}\{1-[z\,\mathrm{csch} z]^2\}$
⋮			
∞	$-t\{-\frac{1}{2}+\frac{1}{2}\rho-\frac{1}{2}\ln[\frac{1}{2}(1+\rho)]\}$	$-2x/(1+\rho)$	$2x^2/(\rho^2+\rho)$
	$\bar{S}^{(\nu)}\equiv S^{(\nu)}/\lambda k=x(\bar{E}^{(\nu)}-\bar{\psi}^{(\nu)})$		
	$\chi^{(\nu)}\equiv\nu kT\chi^{(\nu)}/\lambda m^2=(1-\bar{E}^{(\nu)})/(1+\bar{E}^{(\nu)})$		

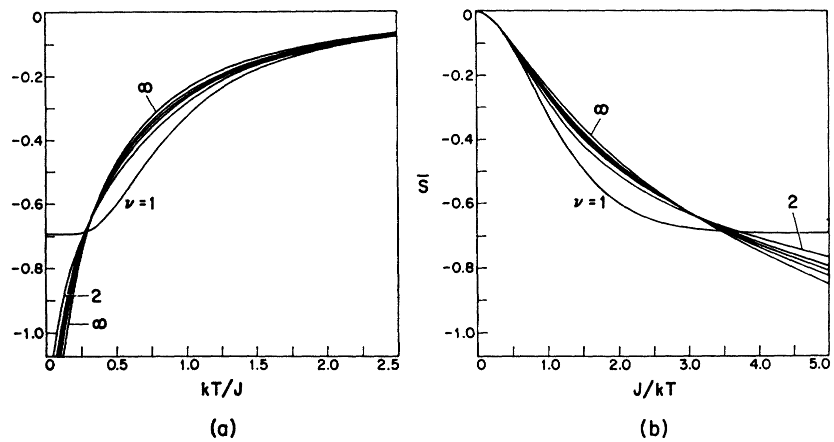


FIG. 3. Normalized entropy $\bar{S}^{(\nu)}\equiv S^{(\nu)}/\lambda k$, as given by Eq. (14), as a function of (a) $t\equiv kT/J$, and (b) $x\equiv J/kT$ for $\nu=1$, 2, 3, 4, 6, and ∞. The plot for $\nu=1$ is $\bar{S}^{(1)}-\ln 2$. At low temperatures $S^{(\nu)}$ diverges logarithmically to $-\infty$ as $T\to 0$ for $\nu>1$ (diverging "faster" the larger ν is). Hence as $T\to 0$, $C^{(\nu)}[=-T\partial S^{(\nu)}/\partial T]$ approaches a nonzero constant (which increases with ν) for $\nu>1$. From the fashion which $S^{(1)}\to$ const as $T\to 0$, we have that $C^{(1)}\to 0$. (See Fig. 4.)

IV. DISCUSSION

We have seen (cf. Figs. 1–5) that the various thermodynamic quantities are monotonic functions of spin dimensionality ν. This leads us to ask if anything particularly interesting occurs in the limit $\nu\to\infty$. We shall see that the limiting expressions for all of the normalized thermodynamic quantities are identical to those calculated by Berlin and Kac for the spherical model.[10] Consider first the normalized free energy $\bar{\psi}^{(\nu)}$, as given by Eq. (14), with the choice $\lambda=\nu$.[5] The relevant asymptotic expansion for Bessel functions when both the argument and order are large is developed in Watson's treatise.[12] Thus the leading term in the asymptotic expansion of $\bar{\psi}^{(\nu)}$ for large ν is

$$\bar{\psi}^{(\infty)}=-t\{-\tfrac{1}{2}+\tfrac{1}{2}\rho-\tfrac{1}{2}\ln[\tfrac{1}{2}(1+\rho)]\}\,, \qquad (21)$$

where $\rho\equiv[1+(2x)^2]^{1/2}$. Equation (21) agrees precisely with Eq. (C18) of Ref. 10 for the free energy of the spherical model.[13] Similarly, the first term in the asymptotic expansion of the spin correlation function is, from Eqs. (11) and (18),

$$(y_\infty)^R=[2x/(1+\rho)]^R\,, \qquad (22)$$

which agrees with Eq. (39) of Ref. 10. (It appears, moreover, that these limiting properties of $\mathcal{H}^{(\nu)}$ are generalizable to lattices of higher dimensionality.[14]) This result is extremely useful, especially since the spherical model is exactly soluble for two- and three-dimensional lattices, thereby providing an "anchor point" in the hierarchy $\mathcal{H}^{(\nu)}$ (which, with the important exception of Onsager's solution of the two-dimensional $\nu=1$ case, appears hopelessly insoluble for lattices of higher dimensionality than the linear chain). Moreover, the monotonicity found in the present linear-chain calculations is reflected in high-temperature expansions for lattices of higher dimensionality.[4] Hence the spherical model, which in the past has been interpreted as a soluble approximation to the Ising model, might in fact turn out to be a much better approximation to the much more "realistic" Heisenberg model.

In conclusion, we remark that all of the above results for the linear chain can be easily generalized to a Bethe lattice,[15] which displays a phase transition at a temperature determined by the equation

$$y_\nu=\sigma\,, \qquad (23)$$

where $q\equiv\sigma+1$ is the lattice-coordination number (number of nearest-neighbors). Note that Eq. (23) reduces to

[12] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, London, 1958).

[13] Note that the J of Berlin and Kac (Ref. 10) is identical to our J, so that our parameter $x(\equiv J/kT)$ is equal to twice their parameter $K[\equiv\frac{1}{2}J/kT]$.

[14] H. E. Stanley, in Proceedings of the 1968 IUPAP Conference on Statistical Mechanics, Kyoto, J. Phys. Soc. Japan (to be published); H. E. Stanley, Phys. Rev. **176**, 718 (1968).

[15] Note that the linear chain is a Bethe lattice of coordination number $q=2$. For a picture of a Bethe lattice of coordination number $q=4$, see Fig. 2 of K. Kawasaki [Phys. Rev. **145**, 224 (1966)].

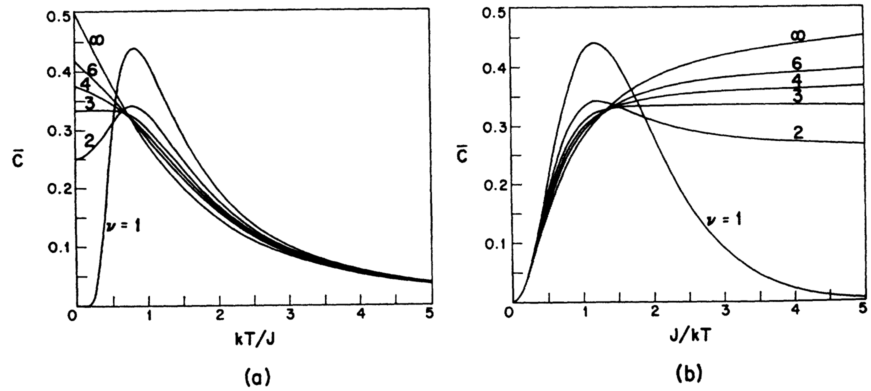
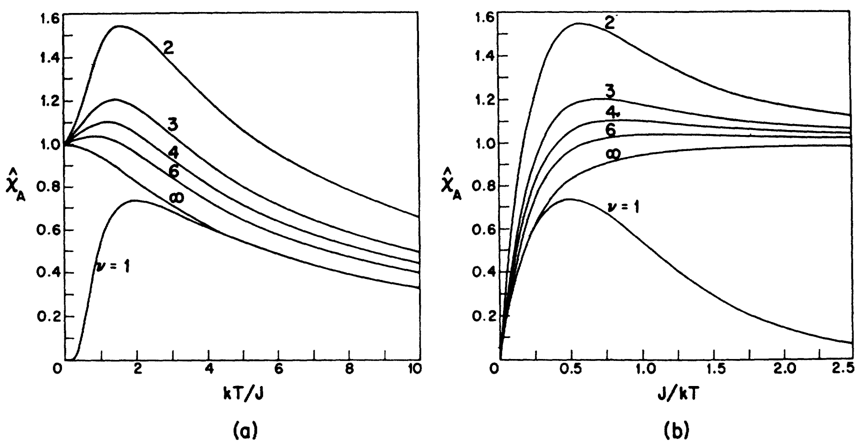


FIG. 4. Normalized specific heat $\bar{C}^{(\nu)}\equiv C^{(\nu)}/\lambda k$ as given by Eq. (16) (for $\lambda=\nu$) as a function of (a) $t\equiv kT/J$, and (b) $x\equiv J/kT$. As $T\rightarrow 0$, $\bar{C}^{(\nu)}\rightarrow\frac{1}{2}(\nu-1)/\lambda$ [i.e., $C^{(\nu)}\rightarrow\frac{1}{2}k(\nu-1)$], just as one expects from classical statistical mechanics. Note that the rather sharp specific-heat maximum (or "Schottky anomaly") found for $\nu=1$ becomes more rounded and occurs at a lower temperature for $\nu=2$, becomes a broad flat region for $\nu=3$, and disappears altogether for $\nu>3$. That $C^{(\nu)}(\equiv\partial E^{(\nu)}/\partial T)$ increases as the system is heated from $T=0$ for $\nu=1$ and 2 *only* is reflected in the initial upward curvature of the corresponding energy curves in Fig. 1(a). Note from(b) that the characteristic $1/T^2$ temperature dependence of $C^{(\nu)}$ at high T is found for all ν.

FIG. 5. Reduced antiferromganetic ($J<0$) susceptibility $\hat{\chi}_A^{(\nu)}T$ (with $\nu=\lambda$) as a function of (a) $t\equiv kT/J$ and (b) $x\equiv J/kT$. Here $\hat{\chi}_A^{(\nu)}(T)\equiv\chi_A^{(\nu)}(T)/\chi_A^{(\nu)}(0)$ for $\nu>1$, with $\hat{\chi}_A^{(1)}(T)\equiv\chi_A^{(1)}(T)$ [since $\chi_A^{(1)}(0)=0$]; hence, the relative heights of the peaks can be compared only when $\nu>1$. The maximum in $\hat{\chi}_A^{(\nu)}$ occurs at $t=2$ for $\nu=1$, "moving in" to $t\cong 1.7$, 1.4, 1.2, and 0.8 for $\nu=2$, 3, 4, and 6, respectively, and approaching $t=0$ as $\nu\rightarrow\infty$.



$y_\nu=1$ or $T_c=0$ for the linear chain.[15] As one would expect, the critical exponents γ, $\bar{\nu}$, and η have the same values for the Bethe lattice (1, $\frac{1}{2}$, and 0, respectively) as predicted by the molecular field theory.[16]

The above results can also be generalized to a closed "ring" of spins,[17] the relevant integral equation being solved using the Funk-Hecke theorem.[18] Moreover, the onset of infinite-range order as $T\rightarrow 0$ is characterized mathematically by the collapse of the discrete eigenvalue spectrum of an appropriate linear operator into a single value. Thus it is interesting to observe that the same "mathematical mechanism" appears to characterize the phase transition in the present model $\mathcal{H}^{(\nu)}$ (with strong, short-range forces) as in the Kac model (with weak, long-range forces.)[19]

[16] The exponent $\bar{\nu}$ describes the approach to zero as $T\rightarrow T_c^+$ of the inverse correlation range $\kappa\sim(T-T_c)^{\bar{\nu}}$, and is obtained from the second moment of the two-spin correlation function $\mu_2=2y_\nu(1+y_\nu)/(1-y_\nu)(1-\sigma y_\nu)^2$. The exponent η is obtained from the relation $\gamma=(2-\eta)\bar{\nu}$.

[17] By a ring we mean a linear chain with periodic boundary conditions imposed: $S_{N+1}=S_1$. The results of the chain and ring problems are identical in the thermodynamic limit $N\rightarrow\infty$.

[18] See, e.g., Bateman Manuscript Project, *Higher Transcendental Functions* (McGraw-Hill Book Co. Inc., New York, 1953), Vol. II, p. 247. We wish to thank Dr. M. Blume for pointing out this result.

ACKNOWLEDGMENTS

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[19] See M. Kac, in *Brandeis 1966 Summer Institute for Theoretical Physics*, edited by M. Chrétien (Gordon and Breach Science Publishers, Inc., New York, 1968) and references contained therein.

APPENDIX A: DERIVATION OF EQ. (5)

From the definition of $\mathfrak{z}^{(\nu)}(x)$ in Eq. (4),

$$\mathfrak{z}^{(\nu)}(x)=\frac{x}{2\pi i}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}d\sigma_1\cdots d\sigma_\nu\int_{-i\infty}^{i\infty}du \times\exp\Big[ux(\lambda-\sum_{n=1}^{\nu}\sigma_n{}^2)\Big]\exp\Big[x\sum_{n=1}^{\nu}c_n\sigma_n\Big], \quad \text{(A1)}$$

where we have suppressed the site index N and set $c_n\equiv\sigma_n(N-1)$. To interchange the order of the $d\sigma_n$ and du integrations we insert into the integrand the factor

$$\exp[x\alpha(\lambda-\sum_{n=1}^{\nu}\sigma_n{}^2)]$$

(which has the value unity for any value of α because of the constraint), and choose α to be a sufficiently large positive real number that $(u+\alpha)\sigma_n{}^2-c_n\sigma_n$ is positive. Then

$$\mathfrak{z}^{(\nu)}(x)=\frac{x}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}dv\, e^{vx\lambda}\prod_{n=1}^{\nu}\int_{-\infty}^{\infty}d\sigma_n \times\exp[-x(v\sigma_n{}^2-c_n\sigma_n)], \quad \text{(A2)}$$

where $v\equiv u+\alpha$. On completing the square,

$$\int_{-\infty}^{\infty}d\sigma_n\exp[-x(v\sigma_n{}^2-c_n\sigma_n)] =(\pi/vx)^{1/2}\exp[xc_n{}^2/4v], \quad \text{(A3)}$$

and on applying the constraint (1b) to eliminate $c_n\equiv\sigma_n(N-1)$, we obtain

$$\mathfrak{z}^{(\nu)}(x)=\frac{x}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}dv\exp\Big[\lambda x(v+1/4v)\Big]\Big(\frac{\pi}{vx}\Big)^{\nu/2}. \quad \text{(A4)}$$

Finally on setting $w\equiv 2v$ we obtain an integral representation of the modified Bessel function of the first kind,

$$\mathfrak{z}^{(\nu)}(x)=\tfrac{1}{2}x\Big(\frac{2\pi}{x}\Big)^{\nu/2}\frac{1}{2\pi i}\int_{2\alpha-i\infty}^{2\alpha+i\infty}dw \times\exp[\tfrac{1}{2}\lambda x(w+1/w)]w^{-\nu/2}, \quad \text{(A5)}$$

from which Eq. (5) follows at once.

APPENDIX B: LOW-TEMPERATURE EXPANSIONS

At low temperatures the relevant asymptotic expansion for Bessel functions with large argument ($z\equiv\lambda x\equiv\lambda J/kT\equiv\lambda/t$) is[12]

$$I_{\nu/2-1}(z)\sim\frac{e^z}{(2\pi z)^{1/2}}\Big[\sum_{n=0}^{\infty}L_n{}^{(\nu)}(-z)^{-n} +i^{-(\nu-1)}e^{-2z}\sum_{n=0}^{\infty}L_n{}^{(\nu)}z^{-n}\Big], \quad \text{(B1)}$$

where the coefficients $L_n{}^{(\nu)}$ are defined by means of the recursion relation

$$L_n{}^{(\nu)}\equiv\frac{(\nu-2)^2-(2n-1)^2}{8n}L_{n-1}{}^{(\nu)} \quad \text{(B2)}$$

and $L_0{}^{(\nu)}\equiv 1$. At first sight it might appear unnecessary to include the second summation in Eq. (B1) due to the presence of the prefactor e^{-2z}. However, from the definition of the coefficients $L_n{}^{(\nu)}$ in (B2) we see that for both $\nu=1$ and 3 (the Ising and Heisenberg models) $L_n{}^{(\nu)}=0$ if $n>0$, for $\nu=5$ $L_n=0$ if $n>1$, $\cdots$, so that to determine the asymptotic behavior of $I_{\nu/2-1}(z)$ for small values of ν, we cannot ignore the second summation in (B1).

The first few terms of the low-temperature expansion of the free energy are readily obtained by combining Eqs. (14) and (B1),[11]

$$\bar{\psi}^{(\nu)}\sim-1-(kT/\lambda J)\{\ln[\Gamma(\tfrac{1}{2}\nu)/2\pi^{1/2}]+\tfrac{1}{2}(\nu-1) \times\ln(2kT/\lambda J)-L_1{}^{(\nu)}(kT/\lambda J)+\cdots\}. \quad \text{(B3)}$$

For $\nu=1$ and 3, $L_1{}^{(\nu)}=0$ and the term $(-)L_1{}^{(\nu)}(kT/\lambda J)$ in (B3) should be replaced by $(\pm)\exp(-2\lambda J/kT)$ for $\nu=1$ and 3, respectively.

The leading terms in the entropy $S^{(\nu)}=-\partial\psi^{(\nu)}/\partial T$ are then [from (15), or directly from (B3)]

$$\bar{S}^{(\nu)}\sim\lambda^{-1}\{\ln[\Gamma(\tfrac{1}{2}\nu)/2\pi^{1/2}]+\tfrac{1}{2}(\nu-1)[1+\ln(2kT/\lambda J)] -2L_1{}^{(\nu)}(kT/\lambda J)\}, \quad \text{(B4)}$$

which, for $\nu>1$, approaches $-\infty$ as $T\to 0$. For $\nu=1$ the "$T\ln T$ term" is not present in the free energy, and, therefore there is no logarithmic divergence in the entropy; rather,

$$\bar{S}^{(1)}\sim\lambda^{-1}[e^{-2\lambda J/kT}(2\lambda J/kT+1)], \quad \text{(B5)}$$

which approaches a constant as $T\to 0$.

The low-temperature behavior of the energy is easily obtained from the above expansions of $\psi^{(\nu)}$ and $S^{(\nu)}$ [or by expanding Eq. (11)],

$$\bar{E}^{(\nu)}\sim-[1-\tfrac{1}{2}(\nu-1)(kT/\lambda J)+L_1{}^{(\nu)}(kT/\lambda J)^2]. \quad \text{(B6)}$$

For $\nu=1$ and 3, the $L_1{}^{(\nu)}$ term is replaced by $\mp 2\times\exp(-2\lambda J/kT)$. The specific heat $C^{(\nu)}=\partial E^{(\nu)}/\partial T=T\partial S^{(\nu)}/\partial T$ is then

$$\bar{C}^{(\nu)}\sim\lambda^{-1}[\tfrac{1}{2}(\nu-1)-2L_1{}^{(\nu)}(kT/\lambda J)]. \quad \text{(B7)}$$

We notice that if $\lambda=1$, then $C^{(\nu)}(T=0)=\tfrac{1}{2}k(\nu-1)$, just as one would expect from classical statistical mechanics. The $L_1{}^{(\nu)}$ term becomes $\pm 4(\lambda J/kT)^2\exp(-2\lambda J/kT)$ for $\nu=1$ and 3, respectively.

The expansion for the ferromagnetic ($J>0$) susceptibility $\chi^{(\nu)}$ is readily obtained from the low-temperature expansion of $y_\nu\equiv-\bar{E}^{(\nu)}$ in Eq. (B6). For $\nu>1$,

$$y_\nu\sim 1-[(\nu-1)/(2\lambda J)]kT+L_1{}^{(\nu)}(kT/\lambda J)^2, \quad \text{(B8)}$$

and, on substituting into Eq. (20), $\chi^{(\nu)}$ diverges in the limit $T\to T_c(\equiv 0)$ as $(T-T_c)^{-\gamma}$ with $\gamma=2$.[20] However, $\chi^{(1)}$ diverges exponentially as $T\to 0$, since $y_1\sim 1-2\times\exp(-2\lambda J/kT)$.

To obtain the corresponding expansion of the antiferromagnetic susceptibility ($J<0$), we note that y_ν is an odd function of J so that, for $\nu>1$,

$$\chi_A{}^{(\nu)}\sim\frac{\lambda m^2}{\nu|J|}\frac{\nu-1}{4\lambda}\times\left\{1+\left[\frac{(\nu-1)}{4}-\frac{2L_1{}^{(\nu)}}{(\nu-1)}\right]\left(\frac{kT}{\lambda J}\right)+\cdots\right\}. \quad \text{(B9)}$$

For $\nu=1$,

$$\chi_A{}^{(1)}\sim\frac{\lambda m^2}{kT}\exp(-2\lambda|J|/kT)\left\{1+\exp(-2\lambda|J|/kT)+\exp(-4\lambda|J|/kT)+\cdots\right\}, \quad \text{(B10)}$$

so that $\chi_A{}^{(\nu)}\to 0$ as $T\to 0$ only for $\nu=1$.

APPENDIX C: HIGH-TEMPERATURE EXPANSIONS

At high temperatures the familiar Bessel-function expansion for small argument is

$$I_{\nu/2-1}(z)=\sum_{n=0}^{\infty}\frac{(\frac{1}{2}z)^{2n+(1/2)\nu-1}}{(n)!(n+\frac{1}{2}\nu-1)!}, \quad \text{(C1)}$$

from which all the various functions may be obtained exactly as was carried out in detail at low temperatures. Because no coefficients in (C1) are zero for $\nu=1$ or 3, we shall see that, in contrast to the low-temperature behavior, each of the functions has a similar algebraic structure for all values of ν.

Moreover, the coefficients in all the series except the susceptibility are so simply related to one another that we can express them all in terms of one basic set of coefficients $H_n{}^{(\nu)}$. We define the $H_n{}^{(\nu)}$ by the energy expansion

$$\bar{E}^{(\nu)}\equiv -y_\nu=\left(\frac{-\lambda x}{\nu}\right)\left[1+\sum_{n=1}^{\infty}H_n{}^{(\nu)}\left(\frac{\lambda x}{\nu}\right)^{2n}\right]; \quad \text{(C2)}$$

therefore, the free energy is[11]

$$\bar{\psi}^{(\nu)}=\left(\frac{-\lambda x}{\nu}\right)\left[\tfrac{1}{2}+\sum_{n=1}^{\infty}(2n+2)^{-1}H_n{}^{(\nu)}\left(\frac{\lambda x}{\nu}\right)^{2n}\right]. \quad \text{(C3)}$$

Note that both the internal energy and the free energy vary as J/kT at high temperatures. Hence, it follows that the entropy[11] and specific heat,

$$\bar{S}^{(\nu)}=\left(\frac{-\lambda x^2}{\nu}\right)\left[\tfrac{1}{2}+\sum_{n=1}^{\infty}\frac{2n+1}{2n+2}H_n{}^{(\nu)}\left(\frac{\lambda x}{\nu}\right)^{2n}\right] \quad \text{(C4)}$$

and

$$\bar{C}^{(\nu)}=\left(\frac{\lambda x^2}{\nu}\right)\left[1+\sum_{n=1}^{\infty}(2n+1)H_n{}^{(\nu)}\left(\frac{\lambda x}{\nu}\right)^{2n}\right], \quad \text{(C5)}$$

go to zero as $(J/kT)^2$ at high temperatures.

The coefficients $H_n{}^{(\nu)}$ defined by Eq. (C2) are obtained directly from Eqs. (C1) and (11). For general ν and arbitrary order n we find

$$H_n{}^{(\nu)}=\nu^{2n+1}\sum_{m=0}^{n}(2m+\nu)^{-1}G_m{}^{(\nu)}F_{n-m}{}^{(\nu)}, \quad \text{(C6)}$$

where

$$G_m{}^{(\nu)}\equiv\Gamma(\tfrac{1}{2}\nu)/[\Gamma(m+1)\Gamma(m+\tfrac{1}{2}\nu)4^m], \quad \text{(C7)}$$

and the coefficients F_{n-m} are obtained by means of the recursion relation

$$F_l{}^{(\nu)}=-\sum_{k=1}^{l}G_k{}^{(\nu)}F_{l-k}{}^{(\nu)};\quad F_0\equiv 1. \quad \text{(C8)}$$

The first few coefficients are thereby found to be

$$H_1{}^{(\nu)}=-\nu/(\nu+2), \quad \text{(C9a)}$$

$$H_2{}^{(\nu)}=2\nu^2/(\nu+4)(\nu+2), \quad \text{(C9b)}$$

$$H_3{}^{(\nu)}=-\nu^3(5\nu+12)/(\nu+6)(\nu+4)(\nu+2)^2, \quad \text{(C9c)}$$

and

$$H_4{}^{(\nu)}=2\nu^4(7\nu+24)/(\nu+8)(\nu+6)(\nu+4)(\nu+2)^2. \quad \text{(C9d)}$$

The coefficients $K_n{}^{(\nu)}$ in the high-temperature series for the susceptibility

$$\chi^{(\nu)}=\frac{\lambda m^2}{\nu kT}\left[1+\sum_{n=1}^{\infty}K_n{}^{(\nu)}\left(\frac{\lambda x}{\nu}\right)^n\right] \quad \text{(C10)}$$

are not related in a particularly direct fashion to the $H_n{}^{(\nu)}$ because y_ν occurs in both the numerator and the denominator of Eq. (20); however, they may be obtained by the relation

$$K_{n+1}{}^{(\nu)}=\sum_{m=0}^{[n/2]}H_m{}^{(\nu)}K_{n-2m}{}^{(\nu)}, \quad \text{(C11)}$$

where $K_0{}^{(\nu)}\equiv 2$ for purposes of Eq. (C11), and the symbol $[\frac{1}{2}n]$ denotes the largest integer less than or equal to $\frac{1}{2}n$. Thus we find $K_1{}^{(\nu)}=K_2{}^{(\nu)}=2$,

$$K_3{}^{(\nu)}=4/(\nu+2), \quad \text{(C12a)}$$

$$K_4{}^{(\nu)}=-2(\nu-2)/(\nu+2), \quad \text{(C12b)}$$

$$K_5{}^{(\nu)}=-4(3\nu-4)/(\nu+4)(\nu+2), \quad \text{(C12c)}$$

and

$$K_6{}^{(\nu)}=4(\nu^3-2\nu^2-6\nu+8)/(\nu+4)(\nu+2)^2. \quad \text{(C12d)}$$

Since it is not difficult to obtain numerical expressions for the high-temperature coefficients $H_n{}^{(\nu)}$ and $K_n{}^{(\nu)}$ directly from Eqs. (C6) and (C11), we will not list specific values here. It is interesting to note, however, that a knowledge of even the first five $H_n{}^{(\nu)}$ and the first ten $K_n{}^{(\nu)}$ is sufficient to reproduce the curves in parts (b) of Figs. 1–5 down to fairly low temperatures ($J/kT\sim 1$).

[20] Hence the reduced susceptibility $\bar{\chi}\sim T\chi$ diverges as T^{-1} [or $\gamma=1$]. When we generalize from the linear chain to a Bethe lattice of arbitrary coordination number q (see Ref. 15), we have $\gamma=1$ for both χ and $\bar{\chi}$ since the phase transition occurs at a nonzero critical temperature.