Soft Modes, Critical Fluctuations, and Optical Properties for a Two-Valley Model of Gunn-Instability Semiconductors

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The frequency- and field-dependent dielectric constant $\epsilon(\omega,E)$ has been calculated for the uniform-field state of a two-valley model of Gunn-instability semiconductors, using a phenomenological approach based on momentum balance and particle conservation. In this model, four plasma modes are obtained as solutions of $\epsilon(\omega_p,E)=0$, and the dependence of all four modes on the electric field has been determined. We show that in the neighborhood of each of the critical fields E_{c1} and E_{c2} defined by $\sigma_0(E_{c1,2})=0$, where σ_0 is the dc differential conductivity, the frequency of one of the plasma modes is purely imaginary and goes to zero as $E\to E_{c1}$ and $E\to E_{c2}$, the mode becoming unstable for $E>E_{c1}$ and $E<E_{c2}$, respectively. Critical fluctuations associated with the soft modes have been investigated. The density, current, and field fluctuations become temporally long-range as $E\to E_{c1,2}$. The field (voltage) noise spectrum becomes sharply peaked at $\omega=0$ in these limits. We have also calculated the field dependence of the optical properties for frequencies in the neighborhood of the zero-field plasma frequency and in the low-frequency region. At the critical fields the medium becomes nonabsorptive in the zero-frequency limit.

I. INTRODUCTION

IN an equilibrium system undergoing a phase transition one often finds a dynamical mode whose frequency goes to zero at the transition, and the existence of such a soft mode is intimately related to the occurrence of critical fluctuations. In nonequilibrium systems transitions can occur, for instance, from one steady state to another, which have many of the characteristics of a phase transition. In semiconductors exhibiting negative differential conductivity such as GaAs, transitions occur from a dissipative state with a uniform field distribution to one which is highly nonuniform for $E_{c1}<E<E_{c2}$.[1] The current-field characteristic is shown schematically in Fig. 1. We have studied the occurrence of soft modes associated with the transitions at E_{c1} and E_{c2}. General considerations suggest that the modes related to these transitions are longitudinal excitations (plasma modes), the frequencies of which are obtained as the zeros of the field- and frequency-dependent dielectric constant. In Ref. 2, we showed that the frequency of a plasma mode goes to zero along the imaginary axis whenever the static differential conductivity goes to zero. For a two-valley model of Gunn-instability semiconductors, we find four plasma modes. In addition to the two modes which are oscillatory for sufficiently high carrier concentrations and low fields, at the onset of electron transfer two pure relaxation-type modes appear, one of which is the soft mode which goes to zero at $E\to E_{c1}$.[2] The poles corresponding to the former two modes will usually not reach the imaginary axis for $E\lesssim E_{c1}$. This may occur only for particular values of the carrier concentrations such that the zero-field values of these modes lie close to the imaginary axis. For sufficiently low carrier concentrations ($n_c\lesssim 10^{15}/\text{cm}^3$),[3] all four modes are purely imaginary for all values of the field, and the mode which lies closest to the real axis for $E=0$ is the soft mode which goes to zero as $E\to E_{c1}$.

These results differ from those obtained by Conwell[4] for the two-valley model. Only two modes were considered, and we find that the mode assumed by Conwell to be the soft mode is the correct soft mode only for very low carrier concentrations ($n_c\lesssim 10^{14}/\text{cm}^3$).[3] Furthermore, a soft mode was obtained only for a certain range of carrier concentrations in contradiction to general considerations[2] as well as to simple physical arguments.[4]

For $E\gg E_{c2}$, the frequency of all four modes will be purely imaginary for carrier concentrations $n_c\lesssim 10^{17}$ because of the low mobility of the upper valley.[3] In all cases the purely imaginary mode nearest to the real axis is the mode which goes to zero as $E\to E_{c2}$.

Associated with the soft modes we find that the current, density, and field fluctuations become temporally long-range. This differs from most phase transitions where typically some correlation function becomes spatially long-range.

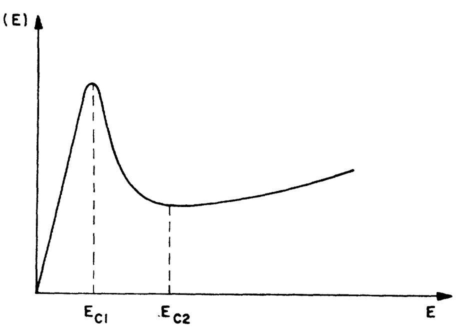

FIG. 1. Current-field characteristic (schematic).

[1] J. B. Gunn, Solid State Commun. 1, 88 (1963).
[2] E. Pytte and H. Thomas, Phys. Rev. Letters 20, 1167 (1968); 20, 1466 (1968).
[3] This value of n_c depends, of course, on the particular values assumed for the mobilities of the two valleys.
[4] E. Conwell, Phys. Rev. Letters 21, 288 (1968).

II. BEHAVIOR OF PLASMA MODES FOR $E=0$ AND $E\approx E_{c1,2}$

We consider a system which is in a steady state in the presence of a spatially uniform time-independent field $\mathbf{E}$, and study its linear response to an additional external field

$$\delta\mathbf{E}^{\mathrm{ext}}(\mathbf{r}t)=\delta\mathbf{E}^{\mathrm{ext}}(\mathbf{q}\omega)e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}. \tag{1}$$

The additional current induced by $\delta\mathbf{E}^{\mathrm{ext}}$ is given in terms of the linear response tensor $\boldsymbol{\kappa}$ by

$$\delta\mathbf{j}^{\mathrm{ind}}(\mathbf{q}\omega)=\boldsymbol{\kappa}(\mathbf{q}\omega,E)\cdot\delta\mathbf{E}^{\mathrm{ext}}(\mathbf{q}\omega). \tag{2}$$

The conductivity tensor $\boldsymbol{\sigma}$ is defined by the relation

$$\delta\mathbf{j}^{\mathrm{ind}}(\mathbf{q}\omega)=\boldsymbol{\sigma}(\mathbf{q}\omega,E)\cdot\delta\mathbf{E}^{\mathrm{tot}}(\mathbf{q}\omega), \tag{3}$$

where $\delta\mathbf{E}^{\mathrm{tot}}$ is the total additional field

$$\delta\mathbf{E}^{\mathrm{tot}}=\delta\mathbf{E}^{\mathrm{ext}}+\delta\mathbf{E}^{\mathrm{ind}}. \tag{4}$$

The induced field $\delta\mathbf{E}^{\mathrm{ind}}$ is the screening field produced by $\delta\mathbf{j}^{\mathrm{ind}}$. From Maxwell's equations, one obtains the relation between $\delta\mathbf{E}^{\mathrm{ind}}$ and $\delta\mathbf{j}^{\mathrm{ind}}$,

$$\omega^2\delta\mathbf{E}^{\mathrm{ind}}+c^2\mathbf{q}\times(\mathbf{q}\times\delta\mathbf{E}^{\mathrm{ind}})=-4\pi i\omega\delta\mathbf{j}^{\mathrm{ind}}, \tag{5}$$

which can be solved for $\delta\mathbf{E}^{\mathrm{ind}}$. The result can be expressed in terms of a dynamical screening tensor

$$\mathbf{s}(\mathbf{q}\omega)=\frac{4\pi}{\omega^2-c^2q^2}(\omega^2\mathbf{1}-c^2\mathbf{q}\mathbf{q}) \tag{6}$$

in the form

$$\delta\mathbf{E}^{\mathrm{ind}}(\mathbf{q}\omega)=-(i/\omega)\mathbf{s}(\mathbf{q}\omega)\cdot\delta\mathbf{j}^{\mathrm{ind}}(\mathbf{q}\omega). \tag{7}$$

The linear response tensor $\boldsymbol{\kappa}$ and the conductivity tensor $\boldsymbol{\sigma}$ are thus related by

$$\boldsymbol{\kappa}=[\mathbf{1}+(i/\omega)\boldsymbol{\sigma}\cdot\mathbf{s}]^{-1}\cdot\boldsymbol{\sigma}=\boldsymbol{\sigma}\cdot[\mathbf{1}+(i/\omega)\mathbf{s}\cdot\boldsymbol{\sigma}]^{-1}. \tag{8}$$

We assume that the field $\mathbf{E}$ is applied along a symmetry direction of the crystal, and consider only waves which propagate along the same direction ($\mathbf{q}\|\mathbf{E}$). This direction is then a principal axis of both tensors $\boldsymbol{\kappa}$ and $\boldsymbol{\sigma}$, and the corresponding principal values κ and σ describe the parallel longitudinal response ($\delta\mathbf{E}\|\mathbf{q}$, $\mathbf{E}$) of the crystal. From Eqs. (6) and (8) it follows that they are related by

$$\kappa=\sigma/[1+(4\pi i/\omega)\sigma]=\sigma/\epsilon, \tag{9}$$

where

$$\epsilon=1+(4\pi i/\omega)\sigma \tag{10}$$

is the longitudinal dielectric constant for the $\mathbf{E}$ direction. We shall be interested only in the parallel longitudinal response, and we restrict the discussion to the $q=0$ limit.

The plasma frequencies are the poles of the response function $\kappa(\omega,E)$ which, on account of Eq. (9), are obtained as the solutions of

$$\epsilon(\omega_p,E)=0. \tag{11}$$

We first show that under certain regularity conditions, this equation has solutions which go to zero as $E\rightarrow E_{c1,2}$, independent of the properties of any particular model. We write $\epsilon(\omega,E)$ as a sum of an even and an odd function of ω in the form

$$\epsilon(\omega,E)=\epsilon'(\omega,E)+(4\pi i/\omega)\sigma'(\omega,E), \tag{12}$$

where $\epsilon'(\omega,E)$ and $\sigma'(\omega,E)$ are even in ω and regular at $\omega=0$. Because of the property

$$\epsilon(-\omega^*,\,E)=\epsilon^*(\omega,E), \tag{13}$$

the functions $\epsilon'(\omega,E)$ and $\sigma'(\omega,E)$ are real on both the real and the imaginary ω axes. On the real axis, they are equal to the real parts of the dielectric constant and the conductivity, respectively. We assume that these functions have Taylor expansions

$$\begin{aligned}\epsilon'(\omega,E)&=\epsilon_0(E)+\epsilon_2(E)\omega^2+\cdots,\\ \sigma'(\omega,E)&=\sigma_0(E)+\sigma_2(E)\omega^2+\cdots,\end{aligned} \tag{14}$$

with a finite radius of convergence, and that the coefficients are continuous functions of E for $E=E_{c1,2}$. The static differential conductivity $\sigma_0(E)$ vanishes for $E=E_{c1,2}$. We assume further that the static dielectric constant $\epsilon_0(E)$ is positive for these values of the electric field. These assumptions may be shown, from Eq. (55) below, to be satisfied for the two-valley model. Writing Eq. (11) in the form

$$\omega_p=-4\pi i\frac{\sigma'(\omega_p,E)}{\epsilon'(\omega_p,E)}, \tag{15}$$

we see that for E sufficiently close to $E_{c1,2}$ there exists a solution

$$\omega_p=-4\pi i\frac{\sigma_0(E)}{\epsilon_0(E)}+O\left[\left(\frac{\sigma_0(E)}{\epsilon_0(E)}\right)^2\right], \tag{16}$$

which is purely imaginary and goes to zero proportionally to $\sigma_0(E)$ as $E\rightarrow E_{c1}$ or $E\rightarrow E_{c2}$. Therefore, for E sufficiently close to either of these critical fields, there exists a plasma mode which shows a pure dielectric relaxation behavior with a relaxation time going to infinity as $E\rightarrow E_{c1,2}$. When the differential conductivity changes sign, these modes become unstable. These results were derived for $E\rightarrow E_{c1}$ in Ref. 2. It is clear, however, that these considerations are equally valid for $E\rightarrow E_{c2}$.

If we consider the case of n-type semiconductors, we can write the electronic susceptibility as a sum of interband and intraband contributions in the usual way. For frequencies such that $\omega\ll E_g$ where E_g is the bandgap, the bound electrons will contribute only to the static dielectric constant ϵ_l of the lattice, and Eq. (10) may be written[5]

$$\epsilon(\omega,E)=\epsilon_l+(4\pi i/\omega)\sigma_c(\omega,E), \tag{17}$$

where $\sigma_c(\omega,E)$ is the conduction-electron contribution, and where ϵ_l is assumed to be independent of E.

In the zero-field limit all the conduction electrons will be near the conduction-band minimum such that no

[5] In this paper, the interaction of the plasma modes with the optical phonons will be neglected.

transfer can take place to any other parts of the conduction band. Then assuming the Drude form for the conductivity

$$\sigma_c(\omega)=i\frac{n_c e^2}{m_1}\frac{1}{\omega+i\gamma_1}, \qquad (18)$$

the dielectric constant may be written

$$\epsilon(\omega,0)=\epsilon_l\left[1-\frac{\omega_0^2}{\omega(\omega+i\gamma_1)}\right]. \qquad (19)$$

Here

$$\omega_0=\left(\frac{4\pi n_c e^2}{\epsilon_l m_1}\right)^{1/2} \qquad (20)$$

is the plasma frequency in the limit $\gamma_1=0$, n_c is the number of electrons in the conduction band, m_1 their effective mass, and $1/\gamma_1$ a phenomenologically introduced relaxation time.

The solutions for the plasma modes obtained from Eq. (19) may be written

$$\omega_{p1,2}=\pm[\omega_0^2-\gamma_1^2/4]^{1/2}-i\gamma_1/2. \qquad (21)$$

For $\omega_0\gg\gamma_1$,

$$\omega_{p1,2}=\pm\omega_0-i\gamma_1/2, \qquad (22)$$

which is the usual expression for a weakly damped plasma frequency. However, when $\gamma_1\geq 2\omega_0$, the plasma frequencies $\omega_{p1,2}$ are both purely imaginary. For $\gamma_1\gg\omega_0$ we obtain

$$\omega_{p1}=-i\gamma_1, \quad \omega_{p2}=-i\omega_0^2/\gamma_1, \qquad (23)$$

where the mode with the largest imaginary part has been labeled ω_{p1}. This convention will be used throughout the paper. The ratio γ_1/ω_0 depends on the carrier concentration. The frequency ω_0 is proportional to $n_c^{1/2}$, while γ_1 depends much more weakly on n_c. The behavior of $\omega_{p1,2}$ for a fixed value of γ_1 and for decreasing values of ω_0 is shown in Fig. 2. For $\omega_0\to 0$ (carrier concentration going to zero), the plasma poles approach the limiting values

$$\omega_{p1}=-i\gamma_1, \quad \omega_{p2}=0, \qquad (24)$$

and the weight of the poles in the response function $\kappa(\omega,E)$ goes to zero.

In Ref. 2, a simple model for $\epsilon(\omega,E)$ was used to interpolate between the limits $E=0$ and $E\approx E_{c1}$. In this model it was assumed that the effect of the electric field and the various microscopic scattering mechanisms, the heating of the electrons and the transfer to the low-mobility local minima, could be described by a single phenomenological parameter chosen so as to reproduce the experimentally observed behavior of the differential conductivity.[6] Both the zero-frequency limit and the high-frequency limit[7] (apart from neglecting the field dependence of the total effective mass) were correctly given by this model, in addition to satisfying the required dispersion relation. However, as will be shown below using a more realistic two-valley model, a considerably more complicated dependence of $\epsilon(\omega,E)$ on ω is obtained for intermediate frequencies, and several field-dependent parameters need to be considered. For zero frequency and low fields and for high frequencies, the limiting forms considered by Conwell[4] may be obtained from this expression.

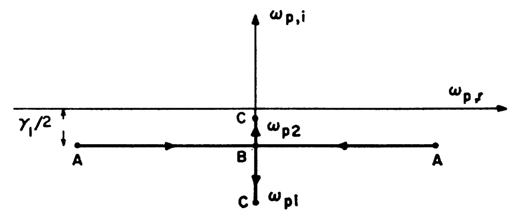

FIG. 2. Behavior of the zero-field plasma modes as a function of ω_0 for fixed γ_1.

A: $\gamma_1/\omega_0=0.4$, B: $\gamma_1/\omega_0=2.0$, C: $\gamma_1/\omega_0=3.0$.

III. TWO-VALLEY MODEL

Electron transfer can in general take place to a number of equivalent valleys. In GaAs the conduction-band minimum is at the center of the Brillouin zone. The next higher minima are along the ⟨100⟩ directions 0.36 eV above the bottom of the conduction band, and appear to be located at the edge of the Brillouin zone and to have X_1 symmetry.[8] Additional local minima along the ⟨111⟩ directions with L_1 symmetry are, based on experimental evidence, believed to lie at higher energies, whereas theoretical calculations place these minima either a $\frac{1}{10}$ V below or at about the same energy.[8] In agreement with other recent calculations of transport properties of GaAs, only the X_1 minima will be taken into account explicitly.[9–11] These minima will be formally treated as a single valley. Thus intervalley scattering among the equivalent X_1 valleys (which is the dominant mechanism determining the mobility of the upper valleys[10]) will formally be treated as intravalley scattering.

The dielectric constant $\epsilon(\omega,E)$ will be calculated using a phenomenological approach based on momentum balance and particle conservation. It will be expressed in terms of several field-dependent (but frequency-independent) parameters which have to be determined either experimentally or by microscopic calculations.

Calculations of transport properties in GaAs, apart from a recent calculation using the Monte Carlo

[6] Physically, therefore, this parameter does not refer to the relaxation time of the electrons in the central valley as assumed in Ref. 4.

[7] See, for example, V. L. Bonch-Bruevich, in *Proceedings of the International School of Physics "Enrico Fermi" Course XXXIV*, edited by J. Tauc (Academic Press Inc., New York, 1966), p. 331.

[8] See Ref. 10 and further references listed in this paper.

[9] P. N. Butcher and W. Fawcett, Phys. Letters **21**, 489 (1966).

[10] E. M. Conwell and M. O. Vassell, Phys. Rev. **166**, 797 (1968).

[11] A. D. Boardman, W. Fawcett, and H. D. Rees, Solid State Commun. **6**, 305 (1968).

method,[11] have been based on approximate solutions of coupled Boltzmann equations for the ⟨000⟩ and the ⟨100⟩ valleys. In these calculations it is assumed that the distribution function may either be represented by the first two terms of a spherical harmonic expansion[10] or by a displaced Maxwellian in each valley.[9] The values for the critical fields obtained by these two methods differ to a certain extent, with the approach based on the displaced Maxwellian distributions giving the larger values. Intermediate values are obtained by the Monte Carlo calculation. The higher values in the case of displaced Maxwellian distributions are attributed to underestimation of the number of high-energy carriers when polar scattering is dominant.[11] This allows less transfer to the ⟨100⟩ valleys. The lower values obtained using the two-term spherical harmonics approximation result from the underestimation of the inelasticity of the scattering processes allowing the carriers to heat up more rapidly and transfer to the ⟨100⟩ valleys. In the Monte Carlo calculation the distribution function in the central valley is found to be strongly asymmetric up to fields far in excess of the threshold field E_{c1}.[11]

The results obtained for field-dependent parameters such as the drift velocity by the different approaches are, however, in very good qualitative agreement. For the numerical calculations, we have made use of the results of Ref. 10 primarily because in this paper the field dependence of all the required parameters is given explicitly (in graphical form). These include the fractional occupation of the central valley, the mobilities in the central valley and in the X_1 minima, and the rate of change of these quantities with the electric field. These quantities are not all independent, but because they have all been calculated in the same approximation, the results should be internally consistent. We wish to emphasize, however, that our formal expressions are not dependent on the results of any particular approximation procedure.

The particle density, effective electron mass, and the drift velocity will be denoted by n_1, m_1, and $\mathbf{v}_1$ for the central valley, and by n_2, m_2, and $\mathbf{v}_2$ for the upper valley where $n_1+n_2=n_0$. Particle and momentum balance then lead to the following four equations:

$$(\partial n_1/\partial t)+\mathrm{div}(n_1\mathbf{v}_1)=-R_{12}, \tag{25}$$

$$(\partial n_2/\partial t)+\mathrm{div}(n_2\mathbf{v}_2)=-R_{21}, \tag{26}$$

$$m_1[\partial(n_1\mathbf{v}_1)/\partial t+\nabla\cdot(n_1\mathbf{v}_1\mathbf{v}_1)]+\nabla\cdot\mathbf{p}_1 = n_1 e\mathbf{E}^{\mathrm{tot}}-\mathbf{S}_{11}-\mathbf{S}_{12}, \tag{27}$$

$$m_2[\partial(n_2\mathbf{v}_2)/\partial t+\nabla\cdot(n_2\mathbf{v}_2\mathbf{v}_2)]+\nabla\cdot\mathbf{p}_2 = n_2 e\mathbf{E}^{\mathrm{tot}}-\mathbf{S}_{22}-\mathbf{S}_{21}. \tag{28}$$

In these equations,

$$R_{12}=R_{1\to 2}-R_{2\to 1}=-R_{21}, \tag{29}$$

where $R_{1\to 2}$ and $R_{2\to 1}$ are the intervalley scattering rates. We assume that the scattering rates are proportional to the number of electrons. Thus,

$$R_{1\to 2}=n_1 r_1(\mathbf{v}_1), \quad R_{2\to 1}=n_2 r_2(\mathbf{v}_2). \tag{30}$$

The transfer rate r_1 from the central valley to the upper valley will depend strongly on the drift velocity $\mathbf{v}_1$. It is almost zero below a certain threshold because the transfer can take place only when the electrons in the central valley have been heated sufficiently by the field so that the tail of their distribution extends to energies higher than the energy of the X_1 minima. Above this threshold, r_1 increases very rapidly with $\mathbf{v}_1$. The dependence of the reverse transfer rate r_2 on the drift velocity is much weaker and will be neglected in our numerical calculations.

Further, $\mathbf{S}_{11}$ and $\mathbf{S}_{22}$ are the momentum sinks due to intravalley scattering. We write

$$\mathbf{S}_{11}=n_1 m_1\boldsymbol{\gamma}_1(\mathbf{v}_1)\cdot\mathbf{v}_1, \quad \mathbf{S}_{22}=n_2 m_2\boldsymbol{\gamma}_2(\mathbf{v}_2)\cdot\mathbf{v}_2, \tag{31}$$

where the relaxation frequency tensor $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ will, in general, be velocity-dependent. Finally, $\mathbf{S}_{12}$ and $\mathbf{S}_{21}$ are the momentum losses due to intervalley scattering. We assume that the average momentum loss in an intervalley scattering process is equal to the average electron momentum in the respective valley (although the actual momentum loss will be somewhat larger). Thus,

$$\mathbf{S}_{12}=m_1\mathbf{v}_1 R_{1\to 2}=n_1 m_1\mathbf{v}_1 r_1(\mathbf{v}_1), \quad \mathbf{S}_{21}=m_2\mathbf{v}_2 R_{2\to 1}=n_2 m_2\mathbf{v}_2 r_2(\mathbf{v}_2). \tag{32}$$

In the general case, we would also need expressions for the partial pressure tensors $\mathbf{p}_1$ and $\mathbf{p}_2$, but we shall restrict ourselves to the spatially uniform case ($q=0$) for which the momentum diffusion terms vanish.

We consider first the stationary state in the presence of a uniform field $\mathbf{E}$. Then the intervalley scattering rates $R_{1\to 2}$ and $R_{2\to 1}$ must be equal such that

$$n_1 r_1=n_2 r_2 \tag{33}$$

or

$$n_1=\frac{r_2}{r_1+r_2}n_c, \quad n_2=\frac{r_1}{r_1+r_2}n_c. \tag{34}$$

From Eqs. (27) and (28) we obtain, by using Eq. (33),

$$\mathbf{v}_1=(e/m_1)(\boldsymbol{\gamma}_1+r_1\mathbf{1})^{-1}\cdot\mathbf{E}=\boldsymbol{\mu}_1(\mathbf{v}_1)\cdot\mathbf{E}, \quad \mathbf{v}_2=(e/m_2)(\boldsymbol{\gamma}_2+r_2\mathbf{1})^{-1}\cdot\mathbf{E}=\boldsymbol{\mu}_2(\mathbf{v}_2)\cdot\mathbf{E}, \tag{35}$$

where we have introduced the mobility tensors $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$ of the two valleys. Using Eqs. (34), the current density

$$\mathbf{j}=e(n_1\mathbf{v}_1+n_2\mathbf{v}_2) \tag{36}$$

can be expressed as a function of the two drift velocities,

$$\mathbf{j}=en_c\frac{r_1\mathbf{v}_2+r_2\mathbf{v}_1}{r_1+r_2}. \qquad (37)$$

If we solve Eqs. (35) for $\mathbf{v}_1$ and $\mathbf{v}_2$ as functions of $\mathbf{E}$ and substitute into Eq. (37), we obtain the steady-state current-field characteristic.

In order to obtain the frequency-dependent conductivity we consider the changes induced by the additional electric field

$$\delta\mathbf{E}^{\text{ext}}(t)=\delta\mathbf{E}^{\text{ext}}(\omega)e^{-i\omega t}.$$

Then

$$n_1(t)=n_1+\delta n\,,$$
$$n_2(t)=n_2-\delta n\,,$$
$$\mathbf{v}_1(t)=\mathbf{v}_1+\delta\mathbf{v}_1\,,$$
$$\mathbf{v}_2(t)=\mathbf{v}_2+\delta\mathbf{v}_2\,.$$

The deviations from the steady-state values δn, $\delta\mathbf{v}_1$, and $\delta\mathbf{v}_2$ may be related linearly to the perturbing field $\delta\mathbf{E}^{\text{tot}}$ by means of the particle and momentum conservation laws. We obtain from Eqs. (25)–(28)

$$(\omega+iR)\delta n=-i(n_1\delta\mathbf{v}_1\cdot dr_1/d\mathbf{v}_1-n_2\delta\mathbf{v}_2\cdot dr_2/d\mathbf{v}_2)\,,$$
$$[n_1\delta\mathbf{v}_1\cdot(\mathbf{1}-\boldsymbol{\alpha}_{11})+n_2\delta\mathbf{v}_2\cdot\boldsymbol{\alpha}_{21}]\cdot(\omega\mathbf{1}+i\boldsymbol{\Gamma}_1)=i(n_1e/m_1)\delta\mathbf{E}^{\text{tot}}, \qquad (38)$$
$$[n_1\delta\mathbf{v}_1\cdot\boldsymbol{\alpha}_{12}+n_2\delta\mathbf{v}_2\cdot(\mathbf{1}-\boldsymbol{\alpha}_{22})]\cdot(\omega\mathbf{1}+i\boldsymbol{\Gamma}_2)=i(n_1e/m_1)\delta\mathbf{E}^{\text{tot}},$$

where $\boldsymbol{\alpha}_{kl}$ is defined by

$$\boldsymbol{\alpha}_{kl}=\frac{i\omega}{\omega+iR}\frac{dr_k}{d\mathbf{v}_k}\mathbf{v}_l\cdot(\omega\mathbf{1}+i\boldsymbol{\Gamma}_l)^{-1},\quad(k,l=1,2) \qquad (39)$$

and where we have introduced the parameters

$$\boldsymbol{\Gamma}_k=\frac{d}{d\mathbf{v}_k}[(\boldsymbol{\gamma}_k+r_k\mathbf{1})\cdot\mathbf{v}_k],\quad(k=1,2)$$
$$R=r_1+r_2. \qquad (40)$$

The change in current density is given by

$$\delta\mathbf{j}=e[n_1\delta\mathbf{v}_1+n_2\delta\mathbf{v}_2+(\mathbf{v}_1-\mathbf{v}_2)\delta n]. \qquad (41)$$

The first two terms are due to the velocity change of carriers in the lower and upper valley, respectively, and the last term describes the current contribution produced by the intervalley transfer of carriers. When δn, $\delta\mathbf{v}_1$, and $\delta\mathbf{v}_2$ are eliminated in terms of $\delta\mathbf{E}^{\text{tot}}$ by means of Eqs. (38), we obtain the differential conductivity tensor $\boldsymbol{\sigma}_c(\omega,E)$ defined by

$$\delta\mathbf{j}(\omega)=\boldsymbol{\sigma}_c(\omega,E)\cdot\delta\mathbf{E}^{\text{tot}}(\omega). \qquad (42)$$

We assume that Eqs. (35) have unique solutions $\mathbf{v}_{1,2}$ as functions of $\mathbf{E}$. With $\mathbf{E}$ along a symmetry direction of the crystal, the drift velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ and the current density $\mathbf{j}$ will then be parallel to $\mathbf{E}$, and this direction will be a principal axis of the tensors $\boldsymbol{\gamma}_k$, $\boldsymbol{\mu}_k$, $\boldsymbol{\Gamma}_k$, $\boldsymbol{\alpha}_{kl}$, and $\boldsymbol{\sigma}_c$. The principal values for this direction will be denoted by the corresponding light face symbols. For the component $\sigma_c(\omega,E)$ describing the response to a parallel field $\delta E\|E$ we find

$$\sigma_c(\omega,E)=\frac{ie^2}{1-\alpha_{11}-\alpha_{22}}\left\{\frac{n_1}{m_1}\frac{1}{\omega+i\Gamma_1}\left(1-\frac{i\theta_1}{\omega+iR}-\alpha_{12}-\alpha_{22}\right)+\frac{n_2}{m_2}\frac{1}{\omega+i\Gamma_2}\left(1-\frac{i\theta_2}{\omega+iR}-\alpha_{11}-\alpha_{21}\right)\right\}. \qquad (43)$$

Here, we have introduced the parameters

$$\theta_1=(v_1-v_2)(dr_1/dv_1)\,,\qquad \theta_2=(v_2-v_1)(dr_2/dv_2)\,. \qquad (44)$$

The α_{kl} represent correction terms which vanish in the zero frequency and high frequency limits, as well as in the zero field and the high field limits. They are insignificant except for intermediate fields $E_{c1}\lesssim E\lesssim E_{c2}$.

As can be seen from Eqs. (38), Γ_1 and Γ_2 are essentially the drift velocity relaxation frequencies, and R is the charge transfer relaxation frequency. Further, θ_1 and θ_2 determine essentially the contributions to δj due to electron transfer from the lower to the upper valley and vice versa. The mechanism responsible for the Gunn instability is the steep increase of θ_1 as $E\rightarrow E_{c1}$. In fact, as a first approximation, the field dependence of all the parameters except θ_1 could be neglected for fields up to E_{c1}.

For the response to a perpendicular field $\delta E\perp E$ we obtain

$$\sigma_{c\perp}(\omega,E)=ie^2\left(\frac{n_1}{m_1}\frac{1}{\omega+i\Gamma_{1\perp}}+\frac{n_2}{m_2}\frac{1}{\omega+i\Gamma_{2\perp}}\right). \qquad (45)$$

The perpendicular conductivity does not depend on θ_1, and, therefore, does not show any anomalous behavior associated with the Gunn instability.

From Eq. (43) the static parallel differential conductivity is given by

$$\sigma_0(E)=e^2\left[\frac{n_1}{m_1\Gamma_1}\left(1-\frac{\theta_1}{R}\right)+\frac{n_2}{m_2\Gamma_2}\left(1-\frac{\theta_2}{R}\right)\right]. \qquad (46)$$

It has been plotted as a function of the electric field in Fig. 3 using values for the field-dependent parameters which will be discussed below. The calculated differential conductivity is in qualitative agreement with the measured velocity-field characteristics.

In the high-frequency limit we obtain the correct limiting form[7]

$$\sigma_c(\omega,E)\rightarrow\frac{in_ce^2}{m_{\text{eff}}}\frac{1}{\omega}, \qquad (47)$$

where

$$\frac{1}{m_{\text{eff}}}=\frac{n_1}{n_c}\frac{1}{m_1}+\frac{n_2}{n_c}\frac{1}{m_2}. \qquad (48)$$

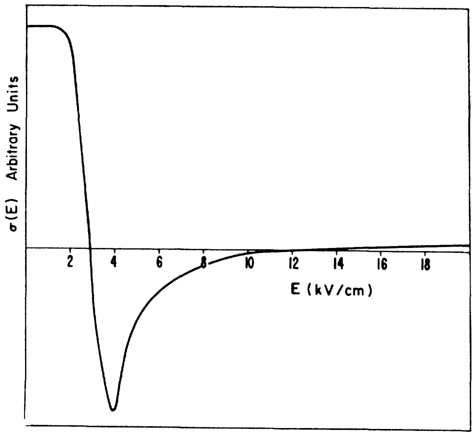

FIG. 3. Field dependence of the static differential conductivity.

The sum rule

$$\int \frac{d\omega}{\pi} \sigma_c'(\omega,E) = \frac{n_c e^2}{m_{\rm eff}} \tag{49}$$

is also satisfied.

We note, however, that for intermediate frequencies the expression for the conductivity given by Eq. (43) has considerably more structure than the simple model used for $\sigma_c(\omega,E)$ in Ref. 2.

In the subsequent calculations, we assume r_2 to be field-independent, such that $\theta_2=0$, and $\alpha_{21}=\alpha_{22}=0$. It will be convenient to rewrite the expressions for Γ_1, Γ_2, R, and $v_{12}(dr_1/dv_1)$. By means of the definitions Eqs. (35) we eliminate the velocities v_1, v_2 and the combinations (γ_1+r_1) and (γ_2+r_2) in terms of the mobilities. Then choosing the field as the independent variable, we obtain

$$\Gamma_{1,2} = \frac{e}{m_{1,2}} \frac{1}{dv_{1,2}/dE}, \qquad v_{1,2}\frac{dr_1}{dv_1} = \mu_{1,2}\frac{E dr_1/dE}{dv_1/dE}, \tag{50}$$

and

$$\frac{dv_{1,2}}{dE} = \mu_{1,2} + E\frac{d\mu_{1,2}}{dE} = \mu_{1,2}\left(1+\frac{d\ln\mu_{1,2}}{d\ln E}\right).$$

Further, from the steady-state condition Eq. (34) we may write

$$r_1+r_2 = n_c r_2/n_1. \tag{51}$$

By assumption, r_2 is independent of the field such that

$$\frac{dr_1}{dE} = -\frac{n_c r_2}{n_1^2}\frac{dn_1}{dE}. \tag{52}$$

Collecting these results, we obtain

$$\begin{aligned} &\Gamma_1 = \frac{e}{m_1}\frac{1}{\mu_1(1+z_1)}, \quad \Gamma_2 = \frac{e}{m_2}\frac{1}{\mu_2(1+z_2)}, \\ &R = n_c r_2/n_1, \\ &\phi_1 \equiv v_1(dr_1/dv_1) = -yR/(1+z_1), \\ &\phi_2 \equiv v_2(dr_1/dv_1) = -(\mu_2/\mu_1)yR/(1+z_1), \\ &\theta_1 = \phi_1 - \phi_2 = -(\mu_1-\mu_2/\mu_1)yR/(1+z_1). \end{aligned} \tag{53}$$

Here we have introduced the logarithmic derivatives

$$y = \frac{d\ln n_1}{d\ln E}, \qquad z_{1,2} = \frac{d\ln\mu_{1,2}}{d\ln E}, \tag{54}$$

which measure the strength of the field dependence of the number of electrons in the central valley and of the mobilities μ_1 and μ_2. The choice of the parameters n_1, $\mu_{1,2}$, y, and $z_{1,2}$ is somewhat arbitrary and was motivated to a large extent by the fact that these particular parameters could be obtained most directly from existing calculations of static transport properties.[10] We note, however, that the transfer rate r_1 which is very difficult to estimate directly has been eliminated from these expressions.

For the numerical calculations we make use of the results of Figs. 12–14 of Ref. 10, where n_1, μ_1, and μ_2 have been plotted as a function of the electric field. We use the values calculated for $D_{12}=5\times10^8$ eV/cm, $D_{jj'}=1\times10^9$ eV/cm with nonparabolicity (NP) of the central minimum included. (The parameters D_{12} and $D_{jj'}$ are defined in Ref. 10.) For the zero-field mobilities, we therefore use

$$\mu_1(0) = 5200 \text{ cm}^2/\text{V sec}, \qquad \mu_2(0) = 145 \text{ cm}^2/\text{V sec}.$$

The slopes of the curves which are needed to calculate y, z_1, and z_2 were estimated directly from the figures. For the additional field-independent parameters required we used the following values[8]:

$$m_1 = 0.072 m_0, \quad m_2 = 0.36 m_0, \quad r_2 = 4.0\times10^{11} \text{ sec}^{-1},$$

where m_0 is the free-electron mass. (The value for r_2 is discussed below.) The corresponding zero-field values for γ_1 and γ_2 are

$$\gamma_1(0) = 4.7\times10^{12} \text{ sec}^{-1}, \qquad \gamma_2(0) = 3.4\times10^{13} \text{ sec}^{-1}.$$

In Ref. 10, the mean drift velocity-field characteristic was calculated using the same distribution function and choice of parameters as in the calculation of n_1, μ_1, and μ_2. The calculation of the static differential conductivity from Eq. (46) using the values just discussed then serves as a consistency check. As shown in Fig. 3, we obtain for the critical fields

$$E_{c1}=2.9\text{ kV/cm}\,,\quad E_{c2}=12\text{ kV/cm}$$

in agreement with the values given in Ref. 10.[12] It should be emphasized, however, that the agreement with experimentally determined drift velocity - field characteristics is far from satisfactory.[8] In particular, the values obtained for E_{c2} are much larger than those calculated theoretically if a minimum is observed at all. In the high-field region, the explicit numerical calculations in this paper must therefore be regarded as preliminary model calculations.

Limiting forms of the dielectric constant for a two-valley model have been given by Conwell,[4] neglecting the field dependence of the mobilities μ_1 and μ_2 which is a reasonable approximation for $E\lesssim E_{c1}$. For zero frequency and for $\omega\approx\gamma_1\ll r_1$, the expressions we obtain differ from those given by Conwell by the appearance of Γ_1 and Γ_2 instead of γ_1 and γ_2 and by the correction terms α_{kl}. For low fields $r_1\ll\gamma_1$ and $\alpha_{kl}\approx 0$. Then because $r_2\ll r_2$ the results are essentially the same. For fields $E\approx E_{c1}$, Conwell assumes that $\gamma_1\ll r_1$. For this case the momentum loss due to intervalley scattering is the dominant process, and the results disagree. Furthermore, using the high-energy value[4] of the transfer time $\tau_{1\to 2}$ for r_1^{-1} overestimates the value of r_1. The condition $r_1\gg\gamma_1$ is, in fact, difficult to realize experimentally. The transfer rate r_1 is an average of the energy-dependent transfer rate $\tau_{1\to 2}^{-1}$ over the electron distribution. It is difficult to estimate because of the rapid decrease of $\tau_{1\to 2}$ after the onset of electron transfer. We have instead estimated the value of r_1 from the values of n_1 and r_2, using the steady-state condition

$$n_1r_1=(n_c-n_1)r_2\,.$$

The transfer rate $\tau_{2\to 1}^{-1}$ is a very slowly varying function of the energy, and for r_2 we have used the value of $\tau_{2\to 1}^{-1}$ at the bottom of the X_1 minima.[13] This gives the value for r_2 quoted above. The value of r_1 for $E=E_{c1}$ obtained in this way,

$$r_1(E_{c1})=1.0\times 10^{11}\text{ sec}^{-1}\,,$$

is more than two orders of magnitude smaller than the value assumed by Conwell.[4] For $E=20$ kV/cm we obtain

$$r_1=4.6\times 10^{12}\text{ sec}^{-1}\,,$$

which is still only of the same order of magnitude as γ_1.

The high-frequency form assumed by Conwell is identical to our Eq. (47).

IV. FIELD DEPENDENCE OF PLASMA MODES

From the definition of the dielectric constant Eq. (17) and the expression for the differential conductivity Eq. (43), we obtain

$$\epsilon(\omega,E)=\epsilon_l\left\{1-\frac{\omega_0^2}{\omega(1-\alpha_{11}-\alpha_{22})}\times\left[\frac{n_1}{n_c}\frac{1}{\omega+i\Gamma_1}\left(1-i\frac{\theta_1}{\omega+iR}-\alpha_{12}-\alpha_{22}\right)+\frac{n_2}{n_c}\frac{m_1}{m_2}\frac{1}{\omega+i\Gamma_2}\left(1-i\frac{\theta_2}{\omega+iR}-\alpha_{11}-\alpha_{21}\right)\right]\right\}\tag{55}$$

with ω_0 as defined in Eq. (20). We now assume r_2 to be field independent and set $\theta_2=\alpha_{21}=\alpha_{22}=0$. Then, the condition Eq. (11) determining the plasma frequencies takes the form

$$\begin{aligned}&\omega(\omega+i\Gamma_1)(\omega+i\Gamma_2)(\omega+iR)-i\phi_1\omega^2(\omega+i\Gamma_2)\\&-\omega_0^2(n_1/n_c)\{(\omega+i\Gamma_2)[\omega+i(R-\theta_1)]-i\phi_2\omega\}\\&-\omega_0^2(n_2/n_c)(m_1/m_2)\{(\omega+i\Gamma_1)(\omega+iR)-i\phi_1\omega\}=0\end{aligned}\tag{56}$$

with $\phi_{1,2}$ defined in Eq. (53). The physical nature of the plasma modes is characterized by the relative magnitude of the three terms contributing to the additional current, Eq. (41). If one of the first two terms predominates, the mode is essentially a plasma oscillation of the electrons in the corresponding valleys. If the third term is the dominant one, the mode consists mainly of a transfer of electrons between the two valleys.

We consider first some simple limits. For low fields, no electron transfer takes place, and $n_1=n_c$, $n_2=0$, $r_1=0$, $R=r_2$. Further, $dr_1/dv_1=0$ such that $\theta_1=\phi_1=\phi_2=0$. Then, the dielectric constant reduces to the form given by Eq. (19), and we obtain the two plasma modes for the electrons in the central valley. By expanding the expression (55) for $\epsilon(\omega,E)$ in the small quantities n_2 and dr_1/dv_1, we can study the emergence of the additional plasma modes. We find that at the onset of electron transfer two purely relaxation-type modes appear with frequencies

$$\begin{aligned}\omega_{p1}^{(2)}&=-i\Gamma_2\,,\\\omega_{p2}^{(2)}&=-ir_2\,,\end{aligned}\tag{57}$$

the first of which consists predominantly in a motion of the carriers in the upper valley, while the second has a strong transfer character. For $r_2=0$, we obtain the modes discussed in Sec. II for a single valley (here the upper valley) in the limit where the carrier concentration goes to zero.

In the high-field limit $E\gg E_{c2}$, the carrier concentrations n_1 and n_2 approach field-independent values.[14] Then, $y=0$ and $dr_1/dv_1=0$, and the dielectric constant takes the form

$$\epsilon(\omega,E)=\epsilon_l\left[1-\frac{\omega_0^2}{\omega}\left(\frac{n_1}{n_c}\frac{1}{\omega+i\Gamma_1}+\frac{n_2}{n_c}\frac{m_1}{m_2}\frac{1}{\omega+i\Gamma_2}\right)\right].\tag{58}$$

[12] See Fig. 9, Ref. 10.

[13] E. M. Conwell and M. O. Vassell, Trans. IEEE **Ed-13**, 22 (1966).

[14] See Fig. 14, Ref. 10.

This is equivalent to a two-valley model with zero interband transfer and with effective momentum relaxation frequencies Γ_1 and Γ_2, which in general has three plasma modes. By expanding the expression for $\epsilon(\omega,E)$ Eq. (55) in the small quantity dr_1/dv_1 we find that the residue of the fourth pole vanishes in the limit $dr_1/dv_1 \to 0$ at a frequency

$$\omega_{p1}^{(1)} = -i(r_1+r_2), \tag{59}$$

and that it is predominantly a transfer-type mode. Since in addition $n_1 \ll n_2$ in this limit, two of the other three modes are essentially upper-valley modes with frequencies approximately given by

$$\omega_{p1,2}^{(2)} \approx \pm(\omega_0^2-\Gamma_2^2/4)^{1/2}-i\Gamma_2/2, \tag{60}$$

and the last one a lower-valley mode with frequency

$$\omega_{p2}^{(1)} \approx -i\Gamma_1. \tag{61}$$

Note, however, that the limit $n_1=0$ is not reached even for very high fields. According to Eq. (33), this would require either $r_2 \to 0$ or $r_1 \to \infty$, whereas the intervalley transfer times $\tau_{1\to 2}$ and $\tau_{2\to 1}$ both tend to finite values for high energies.[13] For the highest field value considered below, $E=20$ kV/cm, the value of n_1/n_c is still as large as 0.08.[14]

In the general case, the plasma frequencies have to be found as the roots of the expression (56) which is a fourth-degree polynomial with complex coefficients. We have determined these roots numerically on a computer using a subroutine developed for this purpose.[15] Figures 4(a)–4(f) show the results obtained for the behavior of the plasma frequencies as the field is increased continuously from $E=0$ to $E=20$ kV/cm for various carrier concentrations. In these calculations, all the parameters are assumed to be independent of the carrier concentration except for ω_0 which is proportional to $n_c^{1/2}$. The behavior of the plasma modes for a given value of n_c depends, of course, on the particular values of the field-dependent parameters used in this calculation. The values of these parameters may differ from sample to sample. Thus for the low-field mobility $\mu_1(0)$ there is considerable scatter in the reported values, even for a given carrier concentration. For the modes $\omega_{p1,2}^{(1)}$, because the ratio γ_1/ω_0 is the relevant parameter at least for low fields $E \lesssim E_{c1}$, different values of $\mu_1(0)$ may be approximately accounted for by scaling n_c such that γ_1/ω_0 remains constant. Because of the assumption that only ω_0 depends on n_c, the static differential conductivity will go to zero at $E_{c1}=2.9$ kV/cm and $E_{c2}=12$ kV/cm, when we use the field dependence of the parameters discussed in Sec. III, independently of the carrier concentration. Accordingly, soft plasma modes are obtained at these fields for all values of the carrier concentration.[16]

We first consider the region $E \lesssim E_{c1}$. For sufficiently large carrier concentrations such that $\gamma_1/\omega_0 < 2$, the zero-field values of the plasma modes $\omega_{p1,2}^{(1)}$ will have both a real and an imaginary part. The initial increase of the real part and corresponding decrease of the imaginary part is due to the initial rise[17,18] in the mobility $\mu_1(E)$, which decreases the ratio Γ_1/ω_0. For low fields, this effect dominates over the decrease in n_1 due to the transfer to the upper valley. For carrier concentrations such that $\Gamma_1/\omega_0 \approx 2.0$ as in Figs. 4(c) and 4(d), this effect is particularly large. For sufficiently low carrier concentrations, all four modes will be purely imaginary in the low-field limit. In all cases, the purely imaginary mode which lies nearest to the real axis for $E=0$ is the soft mode which goes to zero as $E \to E_{c1}$. Except for the lowest carrier concentration considered ($n_c=10^{14}/\mathrm{cm}^2$) this is always the transfer-type mode $\omega_{p2}^{(2)}$ with the initial value $\omega_{p2}^{(2)}=-ir_2$. For $n_c \lesssim 10^{14}/\mathrm{cm}^3$ the mode $\omega_{p2}^{(1)}$, which is predominantly a lower-valley mode, lies nearest to the real axis. In the zero-field limit this is the mode ω_{p2} discussed in Sec. II. For $n_c=10^{14}/\mathrm{cm}^3$, it differs from its limiting form $\omega_{p2}^{(1)}=-i\omega_0^2/\gamma_1$ only by a few percent. This is also the low-frequency mode considered by Conwell.[4] It is the soft mode, however, only for very low carrier concentrations such that $\omega_0^2/\gamma_1 \lesssim r_2$.

In the region $E_{c1}<E<E_{c2}$, the results (dashed portions of the curves) have physical significance only for the case where a uniform-field state is produced by suitable initial conditions, and only as long as the field stays approximately uniform. It is interesting to note that this uniform-field state is unstable with respect to one mode only, no other mode becoming soft in this region. The large negative imaginary value of $\omega_{p1}^{(1)}$ for intermediate fields is due to the fact that for these fields $z_1 \approx -1$,[17] so that Γ_1 becomes very large. When Γ_1 becomes comparable to Γ_2, the modes $\omega_{p1}^{(1)}$ and $\omega_{p2}^{(2)}$ repell each other and change character: The strongly field-dependent lower-valley mode $\omega_{p1}^{(1)}$ changes into the weakly field-dependent upper-valley mode $\omega_{p2}^{(2)}$, and vice versa, as Γ_1 becomes larger than Γ_2.

For $E \gtrsim E_{c2}$, all the modes are purely imaginary due to the low value of the mobility in the upper valley, except for the highest carrier concentration considered ($n_c=10^{18}/\mathrm{cm}^3$). Only then is the ratio Γ_2/ω_0 less than 2. In this case, the soft mode associated with the upper critical field E_{c2} is the transfer-type mode $\omega_{p2}^{(1)}$. For lower carrier concentrations, the upper-valley mode $\omega_{p2}^{(2)}$ is the soft mode at E_{c2}.

Thus, for $n=10^{18}/\mathrm{cm}^3$, the soft mode remains of predominant transfer character as the field is increased from $E \lesssim E_{c1}$ to $E \gtrsim E_{c2}$. For intermediate carrier concentra-

[15] We used a subroutine written by I. Gargantini based on an algorithm developed by P. Henrici and I. Gargantini, in *Proceedings of the Symposium on Constructive Aspects of the Fundamental Theorem of Algebra, Rüschlikon, 1967*, edited by B. Dejon and P. Henrici (Wiley-Interscience, Inc., New York, to be published). The calculations were performed on an IBM 360/40 computer.

[16] Gunn oscillations have, however, been observed only for relatively low carrier concentrations, $n_c \lesssim 10^{16}/\mathrm{cm}^3$.

[17] See Fig. 13, Ref. 10.

[18] As discussed in Ref. 10, an improved calculation of the mobility μ_1 may, however, not show an initial increase.

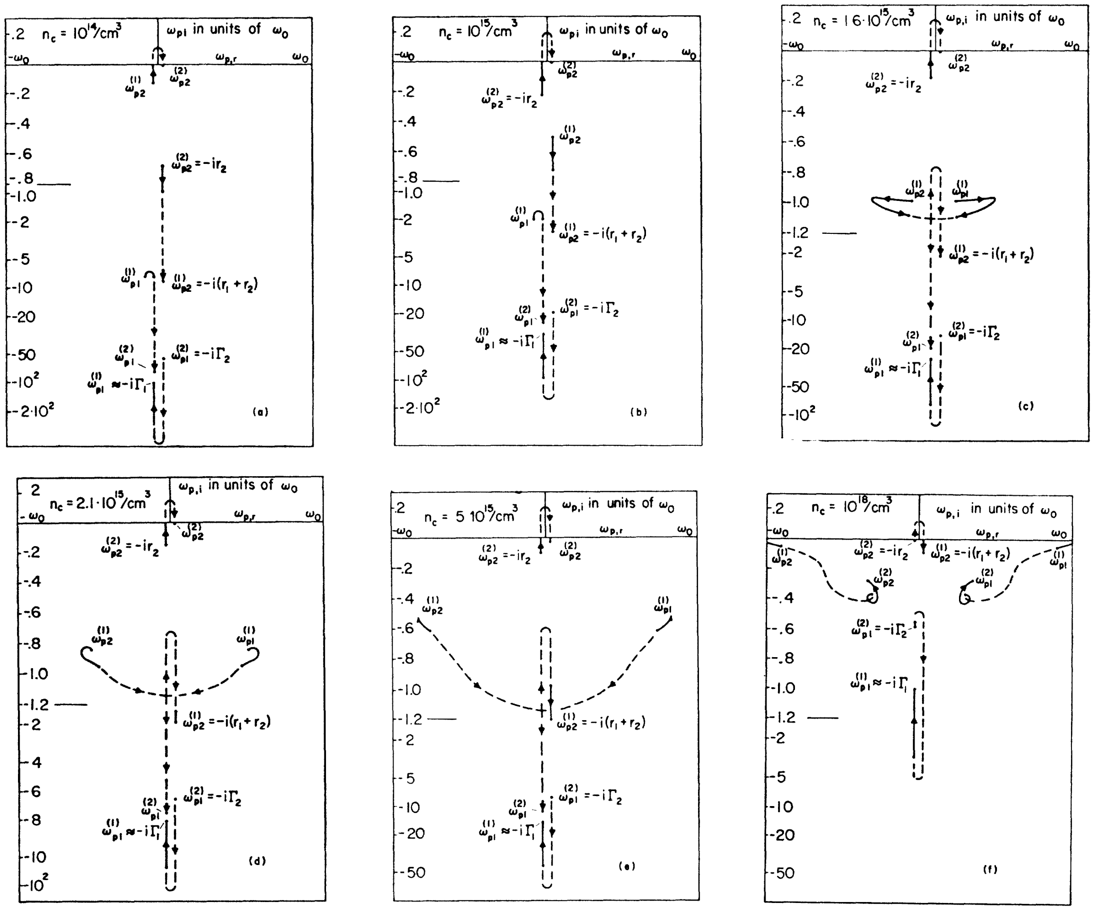

FIG. 4. Field dependence of the plasma modes for different carrier concentrations. For clarity the purely imaginary modes have been drawn slightly off the axis. The broken line shows the behavior of the modes for $E_{c1}<E<E_{c2}$. Note the change in the ordinate from a linear to a logarithmic scale.

tions, the soft mode $\omega_{p2}^{(2)}$ changes from transfer-type for $E\lesssim E_c$ to an upper-valley mode for $E\gtrsim E_c$. For $n_c=10^{14}/\text{cm}^3$, it changes from the lower-valley mode $\omega_{p2}^{(1)}$ for $E\lesssim E_c$ to the upper-valley mode $\omega_{p2}^{(2)}$ for $E\gtrsim E_c$.

For carrier concentrations in the range 1×10^{17} to $3\times10^{18}/\text{cm}^3$ the plasma modes $\omega_{p1,2}^{(1)}$ in GaAs interact strongly with the longitudinal optical phonons.[19] This interaction may be included by replacing ϵ_l in the expression Eq. (17) for the dielectric constant by the frequency-dependent quantity[20]

$$\epsilon_l(\omega)=\epsilon_{l\infty}+\frac{(\epsilon_{l0}-\epsilon_{l\infty})\omega_t^2}{\omega_t^2-\omega^2}, \tag{62}$$

where ω_t is the transverse optical phonon frequency.

[19] A. Mooradian and G. B. Wright, Phys. Rev. Letters **16**, 999 (1966); A. Mooradian and A. L. McWhorter, *ibid.* **19**, 899 (1967).

[20] See, for example, F. Stern, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press Inc., New York, 1963), Vol. 15, p. 300.

V. OPTICAL PROPERTIES

Because for small q there is no difference between the longitudinal and transverse dielectric constant, the transverse response of a system and, therefore, its optical properties will strongly depend on the behavior of the plasma poles. The optical properties are most conveniently expressed in terms of the index of refraction n and the extinction coefficient k, defined for real ω by

$$(n+ik)^2=\epsilon(\omega,E). \tag{63}$$

By making use of Eq. (12), we can write for the real and imaginary parts

$$\begin{aligned} n^2-k^2&=\epsilon'(\omega,E),\\ 2nk&=(4\pi/\omega)\sigma'(\omega,E). \end{aligned} \tag{64}$$

We have calculated n and k together with the reflectivity

$$\mathcal{R}=\frac{(n-1)^2+k^2}{(n+1)^2+k^2} \tag{65}$$

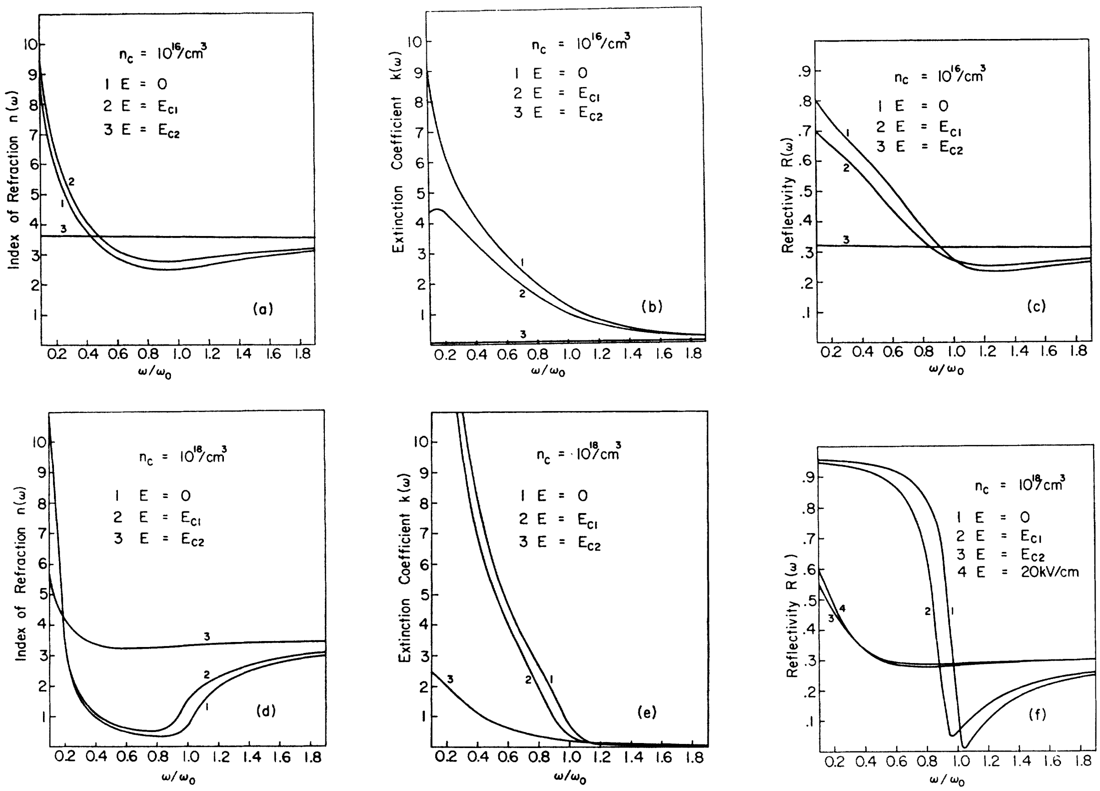

FIG. 5. The index of refraction, the extinction coefficient, and the reflectivity for frequencies $\omega\approx\omega_0$ for several values of the field, for two differeni carrier concentrations.

as a function of frequency for various values of the electric field for the two-valley model discussed in Sec. III. Figures 5(a)–5(f) show the results for frequencies in the neighborhood of ω_0 for two different carrier concentrations. In the zero-field limit, $\epsilon(\omega)$ is given by the single-band Drude formula Eq. (19), and the corresponding frequency dependence of n, k, and $\mathcal{R}$ is well known.[20] For high carrier concentrations such that $\gamma_1\ll\omega_0$, the reflectivity which is close to 1 for $\omega<\omega_0$ drops sharply at $\omega\approx\omega_0$, and for $\omega\gg\omega_0$ approaches the limiting value

$$\mathcal{R}_\infty=\frac{\sqrt{\epsilon_l}-1}{\sqrt{\epsilon_l}+1} \tag{66}$$

from below. For GaAs, $\epsilon_l=12.5$ and $\mathcal{R}_\infty=0.31$. When γ_1/ω_0 increases, the drop in R becomes flatter, and for $\gamma_1\gg\omega_0$, the reflectivity is approximately independent of frequency equal to its limiting value $\mathcal{R}_\infty$. In the high-field limit, $n_1\ll n_c$, and $\epsilon(\omega)$ is again given approximately by a single-band Drude formula, now for the upper valley. Here, however, Γ_2/ω_0 is not small compared to two because of the low mobility in the upper valley, and the frequency dependence of the reflectivity is consequently very flat. For $\omega\gg\omega_0$ (but still smaller than the band gap), the reflectivity approaches the limiting value $\mathcal{R}_\infty$ for all fields.

In order to observe a large effect due to the electric field, it is necessary to have a high carrier concentration and to go to very high fields. We see from Figs. 4(e) and 4(f) that for the high carrier concentrations, the poles $\omega_{p1,2}^{(1)}$ move only a relatively short distance towards the imaginary axis as the field increases from $E=0$ to $E=E_{c1}$. The corresponding change in the reflectivity curve shown in Figs. 5(c) and 5(f) is then also small. However, for $E\geq E_{c2}$ all the curves are nearly flat in the neighborhood of ω_0.

Because of the large amount of heat generated for the high values of n_c and E required such experiments may be difficult to perform in GaAs. Gunn oscillations due to the electron transfer mechanism have, however, recently been observed in strained n-type Ge,[21] and here the conditions are much more favorable. Because the mobility in the central valley is much larger than in GaAs, particularly at low temperatures, much smaller values of n_c are needed in order to satisfy the condition $\gamma_1/\omega_0\ll1$. The critical field E_{c1} is also much less than in GaAs.

It follows from the above discussion that the field dependence of the optical properties for the carrier concentrations and the frequency range shown in Fig. 5 reflect mainly the field dependence of the plasma modes $\omega_{p1,2}^{(1)}$. The mechanism of electron transfer gives rise to

[21] John E. Smith, Jr., Appl. Phys. Letters 12, 233 (1968).

an additional field-dependent structure at the much lower frequencies $\omega \approx R$. For a qualitative discussion of this effect in the field range $0 \leq E \leq E_{c1}$ it is sufficient to consider the frequency range $\omega \ll \Gamma_1$, because even at $E = E_{c1}$, R is still almost ten times smaller than Γ_1. Further, because $n_2/(m_2\Gamma_2) \ll n_1/(m_1\Gamma_1)$ for $E \lesssim E_{c1}$, the contribution of the carriers in the upper valley may be neglected. We shall also neglect the correction terms α_{kl}.

Before the onset of electron transfer, $\sigma'(\omega)$ and $\epsilon'(\omega)$ are then given by their static values σ_0 and $\epsilon_0 = \epsilon_l - 4\pi\sigma_0/\Gamma_1$ in the whole frequency range considered, and the solution of Eqs. (64) shows the well-known frequency dependence of a medium with constant σ and ϵ. In the low-frequency region $\omega \ll 4\pi\sigma_0/|\epsilon_0|$, one obtains the usual skin-effect behavior

$$n = k = (2\pi\sigma_0/\omega)^{1/2}. \tag{67}$$

When the field exceeds the threshold for electron transfer, the parameter θ_1 rises very steeply with E. From the expression for the dielectric constant Eq. (55) we then find a frequency dependence of $\epsilon'(\omega)$ and $\sigma'(\omega)$, which is given by

$$\begin{aligned} \epsilon'(\omega,E) &= \epsilon_0{}^0 + 4\pi\theta_1\sigma_0{}^0 \frac{\Gamma_1+R}{\Gamma_1} \frac{1}{\omega^2+R^2}, \\ \sigma'(\omega,E) &= \sigma_0{}^0 - \theta_1 \frac{\sigma_0{}^0}{R}\left(1 - \frac{\Gamma_1+R}{\Gamma_1} \frac{\omega^2}{\omega^2+R^2}\right), \end{aligned} \tag{68}$$

where $\epsilon_0{}^0$ and $\sigma_0{}^0$ are the static dielectric constant and the static conductivity for $\theta_1 = 0$,

$$\epsilon_0{}^0 = \epsilon_l\left(1 - \frac{\omega_0{}^2}{\Gamma_1{}^2}\right), \quad \sigma_0{}^0 = \frac{\epsilon_l}{4\pi} \frac{\omega_0{}^2}{\Gamma_1}. \tag{69}$$

Thus, as ω decreases from values $\omega \gg R$ to $\omega = 0$, $\epsilon'(\omega,E)$ increases from $\epsilon_0{}^0$ to

$$\epsilon_0(E) = \epsilon_0{}^0 + 4\pi\theta_1\sigma_0{}^0 \frac{\Gamma_1+R}{\Gamma_1 R^2} \tag{70}$$

and $\sigma'(\omega)$ decreases from $\sigma_0{}^0(1+\theta_1/\Gamma_1)$ to

$$\sigma_0(E) = \sigma_0{}^0(1 - \theta_1/R). \tag{71}$$

At the critical field, $\theta_1 = R$. In addition to lowering the value of $\sigma_0(E)$, the mechanism of electron transfer has also a very drastic effect on the static dielectric constant. It increases from $\epsilon_0(0) = \epsilon_l(1 - \omega_0{}^2/\gamma_1{}^2)$ to $\epsilon_0(E_{c1}) = \epsilon_l(1 + \omega_0{}^2/\Gamma_1 R)$ as the field increases from $E = 0$ to $E = E_{c1}$. For the high carrier concentrations $\epsilon_0(E)$ changes from a large negative to a large positive value. Thus for $n_c = 10^{18}/\text{cm}^3$, we find $\epsilon_0(0) = -2 \times 10^3$ while $\epsilon_0(E_{c1}) = +1.5 \times 10^4$.

At the critical field E_{c1} Eqs. (68) take the form

$$\begin{aligned} \epsilon'(\omega,E_{c1}) &= \epsilon_0(E_{c1}) - 4\pi\sigma_0{}^0 \frac{\Gamma_1+R}{\Gamma_1 R} \frac{\omega^2}{\omega^2+R^2}, \\ \sigma'(\omega,E_{c1}) &= \sigma_0{}^0 \frac{\Gamma_1+R}{\Gamma_1} \frac{\omega^2}{\omega^2+R^2}. \end{aligned} \tag{72}$$

For low frequencies we then obtain from Eqs. (64)

$$\begin{aligned} n &= [\epsilon_0(E_{c1})]^{1/2}, \\ k &= \frac{1}{2} \frac{\omega}{R} \frac{\epsilon_0(E_{c1}) - \epsilon_0{}^0}{[\epsilon_0(E_{c1})]^{1/2}}, \end{aligned} \tag{73}$$

which shows that n goes to a constant while k goes to zero in the limit $\omega \to 0$. The same limiting behavior for n and k is also obtained at $E = E_{c2}$.

In Fig. 6, we have plotted the index of refraction and the extinction coefficient of the two-valley model for the interesting frequency range and for the two field values $E = 0$ and $E = E_{c1}$. (Note that r_2 which is equal

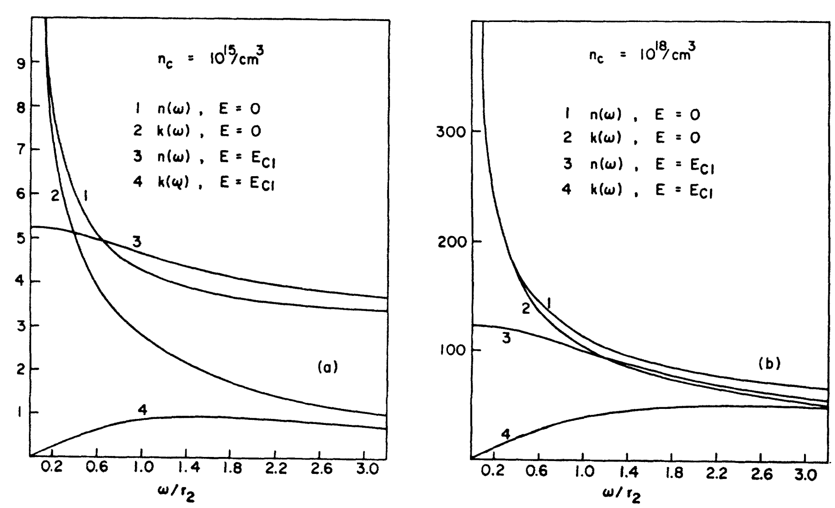

FIG. 6. The index of refraction and the extinction coefficient for frequencies $\omega \approx r_2$ for $E = 0$ and $E = E_{c1}$ for two different carrier concentrations.

to R for $E=0$, deviates from R only by 20% at $E=E_{c1}$.) The results show clearly the transition from the skin-effect behavior at $E=0$ to a nonabsorptive behavior at $E=E_{c1}$ in the low-frequency region.

VI. CRITICAL FLUCTUATIONS

The electric-field distribution changes from a uniform distribution to one which is highly nonuniform at the critical fields $E_{c1,2}$. Consequently we expect any critical fluctuations to be most pronounced for the field-field correlation function. The correlation function is defined in terms of the expectation value of the symmetrized product (anticommutator) of the field density operators

$$S_{\alpha\beta}{}^{(E)}(\mathbf{r}t,\mathbf{r}'t')=\tfrac{1}{2}\langle\{E_\alpha(\mathbf{r}t),E_\beta(\mathbf{r}'t')\}\rangle -\langle E_\alpha(\mathbf{r}t)\rangle\langle E_\beta(\mathbf{r}'t')\rangle. \quad (74)$$

Because the fluctuations of the applied field are assumed to be zero, E in this expression may be taken to refer either to the total field or the induced field. For a spatially uniform time-independent state of the system, these correlation functions depend only on the differences $\mathbf{r}-\mathbf{r}'$ and $t-t'$, and we introduce the space and time Fourier transforms $S_{\alpha\beta}{}^{(E)}(\mathbf{q}\omega)$ for real $\mathbf{q}$ and ω. Of interest are also the current- and charge-density correlation functions $S^{(j)}(\mathbf{r}t,\mathbf{r}'t')$ and $S^{(\rho)}(\mathbf{r}t,\mathbf{r}'t')$ defined analogously.

We first find the relations between the different correlation functions imposed by Maxwell's equations. Because of gauge invariance, the operators $\rho(\mathbf{r}t)$ and $\mathbf{j}(\mathbf{r}t)$ satisfy the continuity equation

$$\partial\rho/\partial t+\mathrm{div}\mathbf{j}=0, \quad (75)$$

which gives rise to the relation

$$(\partial/\partial t)(\partial/\partial t')S^{(\rho)}(\mathbf{r}t,\mathbf{r}'t')=\boldsymbol{\nabla}\cdot\mathbf{S}^{(j)}(\mathbf{r}t,\mathbf{r}'t')\cdot\boldsymbol{\nabla}', \quad (76)$$

where the second $\boldsymbol{\nabla}$ operator acts on $\mathbf{S}^{(j)}$ from the right. We thus obtain for the Fourier transforms

$$\omega^2S^{(\rho)}(\mathbf{q}\omega)=\mathbf{q}\cdot\mathbf{S}^{(j)}(\mathbf{q}\omega)\cdot\mathbf{q}. \quad (77)$$

Further, from Maxwell's equations it follows that

$$(\partial^2\mathbf{E}/\partial t^2)+c^2\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{E})\equiv\boldsymbol{\Pi}\cdot\mathbf{E}=-4\pi(\partial\mathbf{j}/\partial t), \quad (78)$$

where

$$\boldsymbol{\Pi}=\left(\frac{\partial^2}{\partial t^2}-c^2\nabla^2\right)\mathbf{1}+c^2\boldsymbol{\nabla}\boldsymbol{\nabla} \quad (79)$$

is the vectorial wave propagation operator. This leads to the relation

$$\boldsymbol{\Pi}\cdot\mathbf{S}^{(E)}(\mathbf{r}t,\mathbf{r}'t')\cdot\boldsymbol{\Pi}'=(4\pi)^2(\partial/\partial t)(\partial/\partial t')\mathbf{S}^{(j)}(\mathbf{r}t,\mathbf{r}'t'), \quad (80)$$

where the second $\boldsymbol{\Pi}$ operator acts on $\mathbf{S}^{(E)}$ from the right. For the Fourier transforms we obtain

$$\boldsymbol{\Pi}(\mathbf{q}\omega)\cdot\mathbf{S}^{(E)}(\mathbf{q}\omega)\cdot\boldsymbol{\Pi}(\mathbf{q}\omega)=(4\pi\omega)^2\mathbf{S}^{(j)}(\mathbf{q}\omega) \quad (81)$$

with

$$\boldsymbol{\Pi}(\mathbf{q}\omega)=(\omega^2-c^2q^2)\mathbf{1}+c^2\mathbf{q}\mathbf{q}. \quad (82)$$

The dynamical screening tensor $\mathbf{s}(\mathbf{q}\omega)$ introduced in Eq. (6) is proportional to $\boldsymbol{\Pi}^{-1}$,

$$\boldsymbol{\Pi}(\mathbf{q}\omega)\cdot\mathbf{s}(\mathbf{q}\omega)=4\pi\omega^2\mathbf{1}. \quad (83)$$

Thus, Eq. (81) can be written in the form

$$\mathbf{S}^{(E)}(\mathbf{q}\omega)=(1/\omega^2)\mathbf{s}(\mathbf{q}\omega)\cdot\mathbf{S}^{(j)}(\mathbf{q}\omega)\cdot\mathbf{s}(\mathbf{q}\omega). \quad (84)$$

Equations (77) and (84) express $S^{(\rho)}$ and $\mathbf{S}^{(E)}$ in terms of $\mathbf{S}^{(j)}$.

At equilibrium ($E=0$), the fluctuation $\mathbf{S}^{(j)}$ can be obtained from the external response tensor $\boldsymbol{\kappa}$ introduced in Eq. (2) by means of the fluctuation-dissipation theorem, which in the classical limit $\hbar\omega\ll kT$ has the Nyquist form

$$\mathbf{S}^{(j)}(\mathbf{q}\omega)=2kT\ \mathrm{Re}\boldsymbol{\kappa}(\mathbf{q}\omega). \quad (85)$$

Away from equilibrium ($E\neq 0$), one may use a corresponding relation to define a symmetric second-order noise temperature tensor $\mathbf{T}_N$. In the most general case when $\mathbf{S}^{(j)}$ and $\boldsymbol{\kappa}$ do not have the same principal axes, we write the right-hand side as a symmetrized tensor product,

$$\mathbf{S}^{(j)}(\mathbf{q}\omega,\mathbf{E})=[k\mathbf{T}_N\cdot\mathrm{Re}\boldsymbol{\kappa}(\mathbf{q}\omega)+\mathrm{Re}\boldsymbol{\kappa}(\mathbf{q}\omega)\cdot k\mathbf{T}_N]. \quad (86)$$

In the case that both $\mathbf{E}$ and $\mathbf{q}$ are parallel to a symmetry direction of the crystal, only the correlation function for the current components parallel to $\mathbf{E}$ is expected to show critical fluctuations. For this fluctuation we obtain

$$S^{(j)}(q\omega,E)=2kT_N\ \mathrm{Re}\kappa(q\omega,E)=2kT_N\ \mathrm{Re}\frac{\sigma(q\omega,E)}{\epsilon(q\omega,E)}, \quad (87)$$

where $\kappa=\sigma/\epsilon$ is the parallel longitudinal component of $\boldsymbol{\kappa}$ defined in Sec. II, and T_N is the parallel longitudinal noise temperature. Equations (77) and (84) expressing the field- and charge-density fluctuations in terms of the current-density fluctuations take the form

$$S^{(E)}(q\omega,E)=(4\pi/\omega)^2S^{(j)}(q\omega,E),$$
$$S^{(\rho)}(q\omega,E)=(q/\omega)^2S^{(j)}(q\omega,E). \quad (88)$$

As before, we take the $q\to 0$ limit, i.e., we consider only the long-range behavior of the correlation functions. The noise temperature T_N will in general be a function of ω and E. In order to study the critical fluctuations, we assume that $T_N(\omega,E)$ has a Taylor expansion

$$T_N(\omega,E)=T_N{}^0(E)+\omega^2T_N{}^{(2)}(E)+\cdots \quad (89)$$

with a finite radius of convergence. Then under the same assumptions as in Sec. II, we obtain with the help of Eqs. (12) and (14) the low-frequency behavior for the fluctuations in the critical regions $E \approx E_{c1,2}$

$$S^{(j)}(\omega,E)=2kT_N{}^0(E)\frac{\omega^2\sigma_0(E)}{\omega^2\epsilon_0{}^2(E)+(4\pi)^2\sigma_0{}^2(E)}, \quad (90)$$

$$S^{(E)}(\omega,E)=2kT_N{}^0(E)\frac{(4\pi)^2\sigma_0(E)}{\omega^2\epsilon_0{}^2(E)+(4\pi)^2\sigma_0{}^2(E)}, \quad (91)$$

$$S^{(\rho)}(\omega,E)=2kT_N{}^0(E)\frac{q^2\sigma_0(E)}{\omega^2\epsilon_0{}^2(E)+(4\pi)^2\sigma_0{}^2(E)}, \quad (92)$$

where in the denominators we have kept the second-order terms in ω since $\sigma_0(E)\to 0$ for $E\to E_{c1,2}$. The corresponding long-time behavior of these correlation functions is given by

$$S^{(j)}(t-t',\,E)=-\frac{kT_N{}^0(E)}{4\pi\epsilon_0(E)}\frac{1}{\tau^2(E)}e^{-|t-t'|/\tau(E)}, \quad (93)$$

$$S^{(E)}(t-t',\,E)=\frac{4\pi kT_N{}^0(E)}{\epsilon_0(E)}e^{-|t-t'|/\tau(E)}, \quad (94)$$

$$S^{(\rho)}(t-t',\,E)=q^2\frac{kT_N{}^0(E)}{4\pi\epsilon_0(E)}e^{-|t-t'|/\tau(E)}, \quad (95)$$

where $\tau(E)$ is the dielectric relaxation time

$$\tau(E)=\frac{\epsilon_0(E)}{4\pi\sigma_0(E)}. \quad (96)$$

Equation (16) shows that in the critical region it is related to the soft-mode frequency $\omega_p(E)$ by

$$\tau(E)=-1/\mathrm{Im}\,\omega_p(E). \quad (97)$$

Equation (94) shows that the lifetime of the field fluctuations goes to infinity as $E\to E_{c1,2}$ and $\sigma(E)\to 0$. Correspondingly, the field noise spectrum Eq. (91) approaches a δ function at $\omega=0$. The current fluctuations, Eq. (93), also become long lived. However, because of the factor $1/\tau^2(E)$, the magnitude of the fluctuations goes to zero as $E\to E_{c1,2}$. The current noise spectrum goes to zero as $\omega\to 0$. The density fluctuations Eq. (95) vanish in the $q=0$ limit, since density correlations must have a finite range. They show the same long-time behavior for $E\to E_{c1,2}$ as the field fluctuations.

The above results apply directly only to the stable regions $E\leq E_{c1}$ and $E\geq E_{c2}$. The unstable region $E_{c1}\leq E<E_{c2}$ requires a more careful treatment of the temporal Fourier transforms and their inversions than in the derivation above. We expect that in this region the fluctuations will increase exponentially for large times, signaling the onset of the Gunn instability.

We have expressed the fluctuations in terms of the noise temperature. It is of interest to note that when the Coulomb interaction of the charge carriers is treated in the random-phase approximation (RPA), the same value is obtained for the longitudinal noise temperature as when this interaction is neglected altogether. The longitudinal current fluctuations in the RPA may be written in the following form:

$$S^{(j)}(\omega,E)=\frac{S_0{}^{(j)}(\omega,E)}{|\epsilon(\omega,E)|^2}, \quad (98)$$

where $S_0{}^{(j)}(\omega,E)$ are the fluctuations in the absence of the Coulomb interaction. In RPA the true conductivity, with the Coulomb interaction included, is approximated by the external response conductivity in the absence of the Coulomb interaction, $\kappa_{c,0}(\omega,E)$, such that[22]

$$\epsilon(\omega,E)=1+(4\pi i/\omega)\kappa_{c,0}(\omega,E). \quad (99)$$

The definition of the longitudinal noise temperature Eq. (87) then takes the form

$$kT_N=\frac{1}{2}\frac{S^{(j)}}{\mathrm{Re}\,\kappa}=\frac{1}{2}\frac{S_0{}^{(j)}}{|\epsilon|^2}\frac{1}{\mathrm{Re}(\kappa_{c,0}/\epsilon)}, \quad (100)$$

which may be rewritten

$$kT_N=\frac{1}{2}\frac{S_0{}^{(j)}}{\mathrm{Re}(\epsilon^*\kappa_{c,0})}. \quad (101)$$

Then making use of the expression for ϵ given by Eq. (99), we obtain

$$kT_N=\frac{1}{2}\frac{S_0{}^{(j)}}{\mathrm{Re}\,\kappa_{c,0}}=kT_N{}^0, \quad (102)$$

where $T_N{}^0$ is the noise temperature in the absence of the Coulomb interaction.

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[22] Equation (98) has been derived only for a nonpolarizable lattice, in which case $\epsilon_l=1$.