Anomalous Magnetic Moments of Nucleons and Sidewise Dispersion Relations

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A theoretical calculation of the anomalous magnetic moments of the nucleons is presented, based on Bincer's sidewise dispersion relations. Each of the relevant amplitudes is consistently derived in the approximation of keeping only intermediate πN states. The result is obtained in terms of the elastic S_{11} and P_{11} phase shifts. The isoscalar and isovector anomalous magnetic moments are calculated for three alternative models of the phase shifts involved.

1. INTRODUCTION

THE electromagnetic structure of the nucleon has been studied for many years, using dispersion relations in the photon mass.[1] Although some particular features are reasonably well understood, there are others, like the anomalous magnetic moments (a.m.m.), that are still puzzling us. A difficulty in the mentioned approach lies in evaluating the isoscalar form factors, since even the lowest mass intermediate state is a 3π state. Such a state is presumably dominated at low energy by the ω, whereas the ρ dominates the 2π (isovector) contribution. Experimentally, the isoscalar a.m.m. is very small as compared to the isovector one.

A different approach was developed by Bincer[2] using dispersion relations in the mass of one of the external nucleons. Here it is relatively easy to account for the lowest contribution, i.e., the πN state with $I=J=\frac{1}{2}$ (Fig. 2). However, there is no indication about the asymptotic behavior of the amplitudes, so that the use of unsubtracted dispersion relations remains at the moment an assumption.

A rough evaluation of the nucleon a.m.m. within this approach has been made by Drell and Pagels[3] in a paper mainly devoted to the magnetic moment of the electron. For evaluating the a.m.m. of the nucleon, they used the following approximations: (a) a point πNN vertex; (b) only Born terms for the pion photoproduction amplitude; where, however, (c) the nucleon, in the zeroth-order approximation, has no a.m.m.; (d) the introduction of an external cutoff. Point (c) expresses the philosophy that the a.m.m. can be constructed as an expansion in the low-energy terms, starting from a pure γ_μ coupling, in analogy to the purely electromagnetic problem of the electron's a.m.m. We will come back later to this point.

An important (but not unexpected) feature of Drell and Pagels's result is that the required cutoff turns out to be relatively small, corresponding to about 1.5 nucleon masses. Now the fact that the relevant structure is confined in the low-energy region gives, *a posteriori*, a justification for using approximations (a) and (b) above, as well as the saturation of unitarity with πN states only. At the same time, it gives a simple explanation for the dominance of the isovector part of the a.m.m., since the corresponding photoproduction amplitude is the one which dominates near the πN threshold.

In a more realistic calculation the cutoff should be replaced by a natural damping of the amplitudes at high energies. We can easily see that the unitarization of the πNN and photoproduction amplitudes, where we again retain the πN contribution and use the crude scattering-length approximation, is enough to make the dispersion integral of the magnetic moment convergent and provides an effective cutoff of the same magnitude of the one used by Drell and Pagels.

In this paper, we present a theoretical calculation of the a.m.m. of the nucleons, based on the Bincer sidewise dispersion relations and where each of the involved amplitudes is consistently obtained by saturating unitarity up to the πN intermediate state. In Sec. 2 we write down the dispersion relation for the a.m.m. and we express the absorptive part in terms of the πNN vertex form factor and the pion photoproduction ampli-

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[1] G. F. Chew, R. Karplus, S. Gasiorowicz, and F. Zachariasen, Phys. Rev. **110**, 265 (1958); P. Federbush, M. L. Goldberger, and S. B. Treiman, *ibid.* **112**, 642 (1958); W. R. Frazer and J. R. Fulco, Phys. Rev. Letters **2**, 365 (1959).
[2] A. M. Bincer, Phys. Rev. **118**, 855 (1960).
[3] S. D. Drell and H. R. Pagels, Phys. Rev. **140**, B397 (1965). More recently, S. D. Drell and D. J. Silverman have used Bincer's approach to discuss the difference of nucleon and pion electromagnetic radii [S. D. Drell and D. J. Silverman, Phys. Rev. Letters **20**, 1325 (1968)].

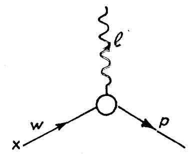

FIG. 1. Electromagnetic vertex of the nucleon.

tudes. The πNN vertex is treated in detail in Sec. 3, while Sec. 4 is devoted to photoproduction. At the end of this treatment, everything is expressed in terms of the πN elastic S_{11} and P_{11} phase shifts. These phase shifts are obtained in three different models valid at low energy. In the first model we use the scattering-length approximation for both S and P waves, while in the second model we use the scattering-length approximation for the S wave and a one-resonance formula for the P wave. These are very simple models which, however, enable us to obtain simple analytic expressions for the πNN vertex form factors. The third model is a best fit obtained from the experimental data on πN scattering up to 350-MeV pion lab kinetic energy. Finally, in Sec. 5 we report the numerical results. These are not very different for the three models, and give $\mu'^S \simeq -0.17$ and $\mu'^V \simeq 1.0$ for the isoscalar and isovector a.m.m., respectively.

2. DISPERSION RELATION FOR THE ANOMALOUS MAGNETIC MOMENT

Let us consider the electromagnetic vertex of the nucleon represented in Fig. 1, where the outgoing nucleon of momentum p and photon of momentum l are on the mass shell ($p^2=m^2$ and $l^2=0$), and the ingoing nucleon of momentum $w=p+l$ has the variable mass $W^2=w^2$. Here W will be our dispersion variable. This vertex is described by six form factors[2] according to[4]

$$\begin{aligned} e\bar{u}(p,s)\Gamma_\mu(p,w) = e\bar{u}(p)\{&[\gamma_\mu F_1(W) \\ &-(i/2m)\sigma_{\mu\nu}l^\nu F_2(W)+l_\mu F_3(W)] \\ &\times(\boldsymbol{w}+W)/2W+[\gamma_\mu F_1(-W) \\ &-(i/2m)\sigma_{\mu\nu}l^\nu F_2(-W)+l_\mu F_3(-W)] \\ &\times(-\boldsymbol{w}+W)/2W. \end{aligned} \quad (1)$$

The normalization is such that $F_1(m)$ is the nucleon charge, in units of e, and $F_2(m)=\mu'$ is the anomalous magnetic moment, a.m.m., in units of $e/2m$. Conservation of the electromagnetic current implies the generalized Ward identity[2]

$$(m\mp W)F_1(\pm W,l^2)-l^2F_3(\pm W,l^2)=(m\mp W)e, \quad (2)$$

but no conditions are implied for $F_2(\pm W)$. In the following we shall split the form factors in the usual way, i.e., $F_i=F_i{}^S+\tau_3 F_i{}^V$, where the superscripts S and V denote the isoscalar and isovector parts, respectively.

The form factors $F_2{}^{S,V}(\pm W)$ can be obtained from (1), using the formula

$$\sum_{s,\tau} e\bar{u}(p,s,\tau)\Gamma_\mu(p,w)\nu_2{}^{\mu S,V}(\pm W)u(p,s,\tau) = -(e/2m)F_2{}^{S,V}(\pm W), \quad (3)$$

where the sum is over the spin and isospin components of the nucleon, and the projectors $\nu_2{}^{\mu S,V}(\pm W)$ are given by

$$\begin{aligned} \nu_2{}^{\mu S}(\pm W) &= \mp\frac{mW}{(W^2-m^2)^2}\,\frac{\pm\boldsymbol{w}+W}{2W}(-i\sigma^{\mu\nu}l_\nu), \\ \nu_2{}^{\mu V}(\pm W) &= \tfrac{1}{2}\tau_3\nu_2{}^{\mu S}(\pm W). \end{aligned} \quad (4)$$

In his paper,[2] Bincer proved that the form factors F_i satisfy dispersion relations in W. Here we assume that F_2 satisfies a dispersion relation without subtractions, that is,

$$F_2(\pm W)=\frac{1}{\pi}\int_{m+\mu}^{\infty} dW'\left[\frac{\mathrm{Im}F_2(W')}{W'\mp W}+\frac{\mathrm{Im}F_2(-W')}{W'\pm W}\right], \quad (5)$$

This also shows that $F_2(\pm W)$ can be considered as a unique function of W, for either positive or negative values of W.

In the one-pion approximation, the absorptive part of F_2 corresponds to the graph of Fig. 2 and can be evaluated in terms of the πNN vertex, where one nucleon is offshell, and of the usual photoproduction amplitude. For the πNN vertex we use the representation

$$\begin{aligned} \bar{u}(p,s)V_\pi{}^\alpha(p,w) = g\bar{u}(p,s)i\gamma_5\tau^\alpha &\times[F_\pi(W)(\boldsymbol{w}+W)/2W \\ &+F_\pi(-W)(-\boldsymbol{w}+W)/2W], \end{aligned} \quad (6)$$

where α refers to the isospin of the pion, g is the standard pseudoscalar coupling constant ($g\simeq 13.6$), and F_π is normalized such that $F_\pi(m)=1$.

For the photoproduction, we use the standard Chew-Goldberger-Low-Nambu (CGLN)[5] notation. Since the quantum numbers of the πN channel are $J^P=\frac{1}{2}^+$, only the multipole M_{1-} will contribute to the process.

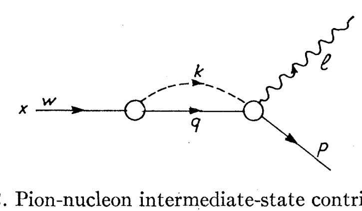

FIG. 2. Pion-nucleon intermediate-state contribution to the absorptive part of the nucleon current.

[4] Our notations are those of J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill Book Co., New York, 1965).

[5] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

The unitarity relation, using Eq. (3) for projecting out $\mathrm{Im}F_2$, reads

$$-\frac{e}{2m}\mathrm{Im}F_2{}^{S,V}(\pm W)=\frac{1}{8\pi^2}\sum_{s,\tau}\int d^4q\,\delta(q^2-m^2)\theta(q_0)$$
$$\times\delta[(w-q)^2-\mu^2]\theta(w_0-q_0)\bar{u}(p,s,\tau)$$
$$\times A_\mu{}^\alpha(W)(\not{q}+m)V_\pi{}^{\alpha\dagger}(q,W)\nu_2{}^{\mu S,V}(\pm W)u(p,s,\tau)\,,\quad(7)$$

where $A_\mu{}^\alpha$ is the photoproduction amplitude.[6] Explicitly, we get

$$\mathrm{Im}F_2{}^{S,V}(W)=g(W/2m)[(W-m)^2-\mu^2]^{1/2}\times F_\pi{}^*(W)M_{1-}{}^{S,V}(W)\,,\quad(8)$$

$$\mathrm{Im}F_2{}^{S,V}(-W)=g(W/2m)[(W+m)^2-\mu^2]^{1/2}\times F_\pi{}^*(-W)E_{0+}{}^{S,V}(W)\,,\quad(9)$$

where $W>m+\mu$ and we have used the reciprocity relation[7] $M_{1-}(-W)=E_{0+}(W)$. Also, note the relation of the amplitudes $M_{1-}{}^{S,V}$ to the CGLN amplitudes:

$$M_{1-}{}^S=3M_{1-}{}^{(0)}\,,\qquad M_{1-}{}^V=M_{1-}{}^{(+)}+2M_{1+}{}^{(-)}\,.\quad(10)$$

The πNN vertex and the photoproduction amplitudes are studied in detail in Secs. 3 and 4.

3. TREATMENT OF THE PION-NUCLEON VERTEX

In this section we want to obtain an expression for the πNN vertex form factor $F_\pi(\pm W)$ defined in Eq. (6). In the approximation of elastic unitarity, the absorptive part of F_π corresponds to the graph of Fig. 3 and is given by

$$\mathrm{Im}F_\pi(\pm W)=e^{-i\alpha(\pm W)}\sin\alpha(\pm W)F_\pi(\pm W)\,,\quad(11)$$

for $W\geqslant m+\mu$. In Eq. (11), $\alpha(+W)=\delta_P(W)$ is the P_{11} phase shift for πN scattering, and $\alpha(-W)=\delta_S(W)$ is the S_{11} phase shift, according to the MacDowell[8] reciprocity relation $\delta_{l+}(-W)=\delta_{(l+1)-}(W)$ ($l=0$ in our case).

The solution of the homogeneous Hilbert problem associated with Eq. (11) is[9]

$$F_\pi(W)=P(W)\exp Q(W)\,,\quad(12)$$

where $Q(W)$ is of the form

$$Q(W)=\frac{1}{\pi}\int_{m+\mu}^{\infty}dW'\left[\frac{\alpha(W')}{W'-W}+\frac{\alpha(-W')}{W'+W}\right],\quad(13)$$

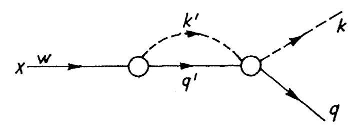

FIG. 3. Pion-nucleon intermediate-state contribution to the absorptive part of the πN vertex.

and $P(W)$ is an arbitrary polynomial. This expression of F_π is valid, provided

$$\left|\int_{m+\mu}^{\infty}dW[\alpha(W)-\alpha(-W)]\right|<\infty\,.\quad(14)$$

The structure of the S- and P-wave phase shifts is rather complicated by the presence of several resonances. The lowest of them are[10] $S_{11}(1548)$ (probable); $S_{11}(1709)$, $P_{11}(1466)$ (Roper resonances); $P_{11}(1751)$ (not confirmed). In order to treat the πN scattering in the low-energy region, we use the following three models.

Model I. We use the scattering-length approximation for both S and P waves. We take the following numerical values[11]: $a_0=0.171\mu^{-1}$ and $a_1=-0.101\mu^{-3}$. Here the approximation for P wave breaks down at very low energy, much below the 1466 resonance.

Model II. Here we use the scattering-length approximation for S wave, which gives a reasonable agreement below the 1548 resonance, and a resonant form for P wave, corresponding to the 1466 Roper resonance.

Model III. Here the S_{11} and P_{11} phase shifts are taken directly from experiment. Specifically, we use the fit by Roper *et al.*[12] for elastic scattering, obtained from data up to 350-MeV pion lab kinetic energy (corresponding to 1350-MeV total c.m. energy).

In each of these models the phase shifts seem to tend to a constant value, so that the condition (14) is not satisfied. Therefore, we have to take, for $Q(W)$, a subtracted form and we get for F_π the following expression:

$$F_\pi(W)=\exp Q(W)\,,\quad(15)$$

$$Q(W)=\frac{W-m}{\pi}\int_{m+\mu}^{\infty}dW'\left[\frac{\alpha(W')}{(W'-W)(W'-m)}-\frac{\alpha(-W')}{(W'+W)(W'+m)}\right],\quad(16)$$

[6] The correspondence with the CGLN notation is the following: $\epsilon^\mu A_\mu{}^\alpha=H^\alpha=H^{(+)}\delta_{\alpha 3}+H^{(-)}\frac{1}{2}[\tau_\alpha,\tau_3]+H^{(0)}\tau_\alpha$.

[7] J. S. Ball, Phys. Rev. **124**, 2014 (1961).

[8] S. W. MacDowell, Phys. Rev. **116**, 774 (1960).

[9] See, for instance, N. I. Muskhelishvili, *Singular Integral Equations* (P. Nordhoff Ltd., Groningen, The Netherlands, 1953).

[10] C. Lovelace, *Proceedings of the Heidelberg International Conference on Elementary Particles* (North-Holland Publishing Co., Amsterdam, 1968), p. 79; CERN Report No. Th837 (unpublished); A. Donnachie, R. G. Kirsopp, and C. Lovelace, CERN Report No. Th838 and Addendum (unpublished).

[11] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).

[12] L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. **138**, B190 (1965). We use the solution 24 of these authors, obtained from 576 data and 37 parameters for S, P, and D waves. Fits up to 2-BeV pion kinetic energy are also available (Ref. 10), however, they are not free at present from serious ambiguities. See, e.g., M. Bander, P. W. Coulter, and G. L. Shaw, Phys. Rev. Letters **14**, 270 (1965).

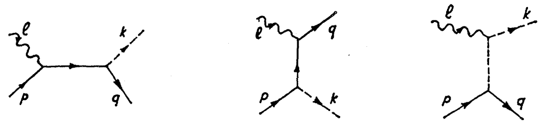

FIG. 4. Pole-term contribution to the photoproduction amplitude.

satisfying the condition $F_\pi(m)=1$. The arbitrary polynomial has been dropped from Eq. (15) for simplicity. If $\delta_S(\infty)=\alpha(+\infty)$ and $\delta_P(\infty)=\alpha(-\infty)$ are the asymptotic values of the phase shifts, the asymptotic behavior of the solution is

$$F_\pi(W) \xrightarrow[W\to\infty]{} \text{const}\, W^{-(1/\pi)[\delta_S(\infty)+\delta_P(\infty)]}. \tag{17}$$

We now consider the specific expression of the form factor in the above three models.

Model I. In the scattering-length approximation the relevant phase shifts are

$$\delta_S(W)=\arctan(a_0q), \tag{18}$$

$$\delta_P(W)=\arctan(a_1q^3). \tag{19}$$

The Omnès function of Eq. (15) takes the simple analytic form given by[13]

$$F_\pi(W)=(1-a_1q_t^3)/(1-ia_1q^3), \tag{20}$$

$$F_\pi(-W)=(1+a_0q_t)/(1-ia_0q), \tag{21}$$

where $W>0$, q is the πN c.m. three-momentum and

$$q_t=\mu[1-\mu^2/4m^2]^{1/2}. \tag{22}$$

Model II. In this model, the phase shift δ_S is the same as that of Eq. (18), while δ_P is given by

$$\delta_P(W)=\arctan[\gamma_rq^3/(W_r-W)], \tag{23}$$

where $W_r=1466$ MeV is the energy of the resonance and

$$\gamma_r=\Gamma_r/2q_r^3, \tag{24}$$

where $q_r=q(W_r)$ and $\Gamma_r\simeq 211$ MeV is the width of the resonance.[10] Expression (23) corresponds to the following resonant form for the $P_{11}\,\pi N$ amplitude:

$$f_{1-}(W)=(1/q)[\gamma_rq^3/(W_r-W-i\gamma_rq^3)]. \tag{25}$$

Even in this case, the vertex form factor takes a simple analytic form:

$$F_\pi(W)=(W_r-m-\gamma_rq_t^3)/(W_r-W-i\gamma_rq^3), \tag{26}$$

while $F_\pi(-W)$ is still the same as in Eq. (21).

Model III. In this case the form factors $F_\pi(\pm W)$ have to be numerically evaluated, starting from Eqs. (15) and (16), which can be rewritten in the more suitable form

$$F_\pi(\pm W)=\exp[Q_0(\pm W)+i\alpha(\pm W)], \tag{15'}$$

$$Q_0(\pm W)=\frac{\pm W-m}{\pi}P\int_{m+\mu}^{\infty}dW'\left[\frac{\alpha(W')}{(W'-W)(W'-m)}-\frac{\alpha(-W')}{(W'+W)(W'+m)}\right]. \tag{16'}$$

4. TREATMENT OF THE PHOTOPRODUCTION AMPLITUDE

The photoproduction amplitudes which are relevant to our problem are the multipoles M_{1-} and E_{0+}, as it appears from Eqs. (8) and (9). These amplitudes can be obtained from a unique analytic function $M(W)$, such that

$$M(W+i0)=((p_0+m)/(q_0+m))^{1/2}\times(ql)^{-1}M_{1-}(W), \tag{27}$$

$$M(-W-i0)=((p_0+m)/(q_0+m))^{1/2}E_{0+}(W), \tag{28}$$

where l is the photon momentum in the c.m. system. $M(W)$ satisfies the dispersion relation[14]

$$M(W)=B(W)+\frac{1}{\pi}\int_{m+\mu}^{\infty}dW'\times\left[\frac{\text{Im}M(W')}{W'-W}+\frac{\text{Im}M(-W')}{W'+W}\right], \tag{29}$$

where $B(W)$ is the contribution from the singularities not included in the region of integration.

A. Born Terms

The term B of Eq. (29) will be approximated by the Born terms of Fig. 4. These can be evaluated from the perturbative current of photoproduction,

$$\mathcal{J}_\mu{}^{\alpha\dagger}=-ieg\left[\Gamma_\mu\frac{p+l+m}{2(p\cdot l)}\gamma_5\tau^\alpha-\gamma_5\tau^\alpha\frac{q-l+m}{2(q\cdot l)}\Gamma_\mu-i\gamma_5\epsilon_{3\alpha\beta}\tau_\beta\frac{k_\mu}{(k\cdot l)}\right], \tag{30}$$

where

$$\Gamma_\mu=\tfrac{1}{2}(1+\tau_3)\gamma_\mu-(i/2m)\sigma_{\mu\nu}l^\nu(\mu'^S+\tau_3\mu'^V). \tag{31}$$

Explicitly, the contributions to the multipoles M_{1-} and E_{0+}, separating the isoscalar and isovector parts and the electric and magnetic (i.e., proportional to μ')

[13] See, e.g., G. Barton, *Dispersion Techniques in Field Theory* (W. A. Benjamin, Inc., New York, 1965).

[14] See Ref. 7 and also N. Zagury, Phys. Rev. **145**, 1112 (1966).

parts, are the following:

$$
\begin{aligned}
M_e{}^S&=\frac{3}{4}\frac{g}{4\pi}\frac{1}{x^3(x+1)}\frac{1}{[(x-1)^2-\nu^2]^{1/2}}\left[3x-\frac{1-\nu^2}{x}+2\nu^2-\frac{x^2}{Q}\ln A\right],\\
E_e{}^S&=\frac{3}{4}\frac{g}{4\pi}\frac{1}{x^3(x-1)}\frac{1}{[(x+1)^2-\nu^2]^{1/2}}\left[3x-\frac{1-\nu^2}{x}-2\nu^2-\frac{x^2}{Q}\ln A\right],\\
M_m{}^S&=\frac{3}{4}\frac{g}{4\pi}\frac{1}{x^3}\frac{1}{[(x-1)^2-\nu^2]^{1/2}}\left[3-x^2+\nu^2-\frac{x}{Q}\ln A+\frac{2}{x}(x^2-1+\nu^2)\right],\\
E_m{}^S&=\frac{3}{4}\frac{g}{4\pi}\frac{1}{x^3}\frac{1}{[(x+1)^2-\nu^2]^{1/2}}\left[3-x^2+\nu^2-\frac{x}{Q}\ln A-\frac{2}{x}(x^2-1+\nu^2)\right],\\
M_e{}^V&=\frac{1}{4}\frac{g}{4\pi}\frac{1}{x^3(x+1)}\frac{1}{[(x-1)^2-\nu^2]^{1/2}}\left[x-\frac{3}{x}(1-\nu^2)+\frac{x^2}{Q}\ln A+2\nu^2\left(3+\frac{2x}{Q}\ln B\right)\right],\\
E_e{}^V&=\frac{1}{4}\frac{g}{4\pi}\frac{1}{x^3(x+1)}\frac{1}{[(x+1)^2-\nu^2]^{1/2}}\left[x-\frac{3}{x}(1-\nu^2)+\frac{x^2}{Q}\ln A-2\nu^2\left(3+\frac{2x}{Q}\ln B\right)\right],\\
M_m{}^V&=\frac{1}{4}\frac{g}{4\pi}\frac{1}{x^3}\frac{1}{[(x-1)^2-\nu^2]^{1/2}}\left[1-3x^2+3\nu^2+\frac{x}{Q}\ln A+\frac{2}{x}(x^2-1+\nu^2)\right],\\
E_m{}^V&=\frac{1}{4}\frac{g}{4\pi}\frac{1}{x^3}\frac{1}{[(x+1)^2-\nu^2]^{1/2}}\left[1-3x^2+3\nu^2+\frac{x}{Q}\ln A-\frac{2}{x}(x^2-1+\nu^2)\right],
\end{aligned}
\tag{32}
$$

where M and E refer to the multipoles M_{1-} and E_{0+}, respectively; the superscripts S and V, to the isoscalar and isovector parts; and the subscripts e and m, to the electric and magnetic parts. Also,

$$A=\left|\frac{x-\epsilon+Q}{x-\epsilon-Q}\right|, \tag{33}$$

$$B=\left|\frac{\epsilon-Q}{\epsilon+Q}\right|. \tag{34}$$

In the above expressions, $x=W/m$, $\epsilon=k_0/m$, $Q=q/m$, and $\nu=\mu/m$.

B. Estimate of the Multipoles M_{1-} and E_{0+}

This is the most delicate point of our analysis since the numerical results for the a.m.m. depend very strongly on the photoproduction process. On the other hand there are some general aspects to be kept present, which pose some restrictions to our treatment. First, we are essentially interested in the low-energy region since we believe, with Drell and Pagels[3] that the a.m.m. is mostly due to the low-energy structure. Then, according to the Kroll and Rudermann[15] theorem, the photoproduction amplitude near threshold is given essentially by the Born terms in the case of charged pions. For neutral pions no such a limit exists and the Born amplitude should be modified. In the CGLN notation the currents involved in π^0 photoproduction are $J_\mu{}^{(+)}$ and $J_\mu{}^{(0)}$. Our modification, however, will only affect $J_\mu{}^{(+)}$.

A second problem is the necessity of taking into account the effect of the 3-3 resonance in the crossed channel, as we know that this effect is important even at low energy. Therefore, we must add at least the 3-3 contribution to the Born terms and to the rescattering contribution.

Third, it is important for our purpose to have separate expressions for the electric and magnetic parts of the amplitudes. However, this is not easy when dealing with the 3-3 resonance, unless we use a particular theory. For this we shall use the static theory of Chew and Low.[16]

In the static model we have for the magnetic part of the multipole $M_{1+}{}^{(3/2)}$ the expression[5]

$$M_{1+,m}{}^{(3/2)}=\frac{\mu^V}{f}\frac{l}{q}f_{1+}{}^{(3/2)}, \tag{35}$$

where μ^V is the total isovector magnetic moment of the nucleon, f is the isovector πN coupling constant ($f^2=0.082$), and

$$f_{1+}{}^{(3/2)}=(1/q)e^{i\delta_{33}}\sin\delta_{33} \tag{36}$$

is the resonant P_{33} amplitude. From here we can also

[15] N. M. Kroll and M. A. Ruderman, Phys. Rev. **93**, 233 (1954); A. Klein, *ibid.* **99**, 998 (1955).

[16] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1579 (1956); see also Ref. 5.

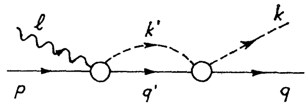

FIG. 5. Pion-nucleon intermediate-state contribution to the absorptive part of the photoproduction amplitude.

obtain the contributions of the 3-3 resonance to the other multipoles.[5]

For the multipoles $M_{1-}^{(+)}$ and $E_{0+}^{(+)}$ we have the following integral equations[5]:

$$\frac{M_{1-}^{(+)}(\omega)}{lq}=\frac{M_{1-}^{(+)B}(\omega)}{lq}+\frac{8}{9\pi}\int_\mu^\infty \frac{d\omega'}{\omega'+\omega}\,\frac{\mathrm{Im}M_{1+}^{(3/2)}(\omega')}{l'q'}+\frac{1}{\pi}\int_\mu^\infty \frac{d\omega'}{\omega'-\omega}\,\frac{\mathrm{Im}M_{1-}^{(+)}(\omega')}{l'q'}+\cdots, \quad (37)$$

$$\frac{E_{0+}^{(+)}(\omega)}{\omega}=\frac{E_{0+}^{(+)B}(\omega)}{\omega}+\frac{4}{3\pi}\int_\mu^\infty d\omega'\frac{\mathrm{Im}M_{1+}^{(3/2)}(\omega')}{l'q'}+\frac{1}{\pi}\int_\mu^\infty \frac{d\omega'}{\omega'-\omega}\,\frac{1}{\omega'}\,\mathrm{Im}E_{0+}^{(+)}(\omega')+\cdots, \quad (38)$$

where $\omega=W-m$ and we have neglected other terms with nonsingular integrands. Since the 3-3 resonance cannot contribute in the direct channel, it only contributes to the real part of the amplitudes, in the physical region $\omega>\mu$. In the approximation of keeping only a πN intermediate state, the imaginary part comes only from rescattering, corresponding to the graph of Fig. 5, and is given by

$$\mathrm{Im}M_{1-}(\omega)=e^{-i\alpha(W)}\sin\alpha(W)M_{1-}(\omega)\,, \quad (39)$$

$$\mathrm{Im}E_{0+}(\omega)=e^{-i\alpha(-W)}\sin\alpha(-W)E_{0+}(\omega)\,. \quad (40)$$

Equations (37) and (38), using also (35) and (36), have the following approximate solution[17] for the magnetic parts:

$$M_{1-,m}^{(+)}(\omega)=e^{i\alpha(W)}\cos\alpha(W)\times\left[M_{1-,m}^{(+)B}(\omega)+\frac{2}{3}\frac{\mu^V}{f}Z_{33}'(\omega)\frac{lq}{\omega}\right], \quad (41)$$

$$E_{0+,m}^{(+)}(\omega)=e^{i\alpha(-W)}\cos\alpha(-W)\times\left[E_{0+,m}^{(+)B}(\omega)+\frac{\mu^V}{f}Z_{33}\omega\right], \quad (42)$$

where

$$Z_{33}=\frac{4}{3\pi}\int_\mu^\infty d\omega\frac{\sin^2\delta_{33}(\omega)}{q^3}\,, \quad (43)$$

$$Z_{33}'(\omega)=\frac{4}{3\pi}\int_\mu^\infty d\omega'\frac{\omega}{\omega'+\omega}\,\frac{\sin^2\delta_{33}(\omega')}{q'^3}\,. \quad (44)$$

The 3-3 contribution is negligible for the electric parts of $M_{1-}^{(+)}$ and $E_{0+}^{(+)}$ and is absent in the other isospin components, so that we can assume for these amplitudes the following expressions:

$$M_{1-,e}^{(+)}(\omega)=e^{i\alpha(W)}\cos\alpha(W)\,M_{1-,e}^{(+)B}(\omega)\,, \quad (45)$$

$$E_{0+,e}^{(+)}(\omega)=e^{i\alpha(-W)}\cos\alpha(-W)\,E_{0+,e}^{(+)B}(\omega)\,, \quad (46)$$

$$M_{1-}^{(-,0)}(\omega)=e^{i\alpha(W)}\cos\alpha(W)\,M_{1-}^{(-,0)B}(\omega)\,, \quad (47)$$

$$E_{0+}^{(-,0)}(\omega)=e^{i\alpha(-W)}\cos\alpha(-W)\,E_{0+}^{(-,0)B}(\omega)\,. \quad (48)$$

We next need a further assumption in order to extend (41) and (42) away from the static limit. In fact, we see, e.g., from Eq. (42), that the contribution of the 3-3 resonance would be linearly divergent. Now, in the static limit, the Born terms, as from expressions (32), are

$$M_{1-,m}^{(+)B}(\omega)=-2f\mu^V\omega\,, \quad (49)$$

$$E_{0+,m}^{(+)B}(\omega)=-\tfrac{4}{3}f\mu^V(lq/\omega)\,, \quad (50)$$

so that Eqs. (41) and (42) can be written

$$M_{1-,m}^{(+)}(\omega)=e^{i\alpha(W)}\cos\alpha(W)\times M_{1-,m}^{(+)B}[1-Z_{33}'(\omega)/2f^2], \quad (51)$$

$$E_{0+,m}^{(+)}(\omega)=e^{i\alpha(-W)}\cos\alpha(-W)\times E_{0+,m}^{(+)B}[1-Z_{33}/2f^2]. \quad (52)$$

We will assume these expressions as approximately valid at all energies.

Concerning the numerical value of the parameter Z_{33} defined in (43), Höhler and Schmidt,[17] probably using an effective-range formula, give the value $Z_{33}=0.088$. The use of a resonant P_{33} amplitude analogous to Eq. (25), with $W_r=1236$ MeV and $\Gamma_r=125$ MeV, gives $Z_{33}=0.120$. Finally, using for δ_{33} the experimental fit by Roper *et al.*[12] up to 700-MeV pion kinetic energy, we have obtained $Z_{33}=0.091$. Since the contribution from higher energies is expected to be very small, the last is probably the most reliable value.

The factor Z_{33}' of Eq. (44) can be reduced to Z_{33} by making in the integrand the approximation $\omega/(\omega'+\omega)\simeq\omega/(\omega_r+\omega)$, where $\omega_r=298$ MeV is evaluated at the resonance. We thus obtain

$$Z_{33}'(\omega)=[\omega/(\omega_r+\omega)]Z_{33}\,. \quad (53)$$

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5. NUMERICAL RESULTS

We have now the ingredients to calculate the a.m.m. From (5), (8), and (9), we have the general expression

$$\mu'^{S,V}=\frac{g}{2m}\frac{1}{\pi}\int_{m+\mu}^{\infty}dW \times\left[\frac{W}{W-m}[(W-m)^2-\mu^2]^{1/2}F_\pi^*(W)M_{1-}^{S,V}(W) +\frac{W}{W+m}[(W+m)^2-\mu^2]^{1/2}F_\pi^*(-W)E_{0+}^{S,V}(W)\right], \quad (54)$$

where the form factors $F_\pi(\pm W)$ are given in Sec. 3 for three specific models of πN scattering and the photoproduction multipoles M_{1-} and E_{0+} are given in Sec. 4. These expressions are such that the integral in (54) is convergent.

In each case Eq. (54) is of the form

$$\mu'^S=I_e^S+\mu'^S I_m^S, \quad (55)$$

$$\mu'^V=I_e^V+\mu'^V I_m^V, \quad (56)$$

where $I_e^{S,V}$ and $I_m^{S,V}$ are the integrals containing, respectively, the electric and the magnetic part of the multipole amplitudes. These expressions, of course, can be immediately solved for the a.m.m.

Before giving the numerical results, we want to make a comment on the difference between the approach of Drell and Pagels[3] and the present one. The two approaches, besides the difference in the approximations used for the amplitudes and for the evaluation of the integral, differ for the method itself of calculating the a.m.m. In the Drell-Pagels philosophy, in analogy with the electron case, the a.m.m. of the nucleon is expanded in terms of the low-energy structure. In the first order, the a.m.m. is given by the graph of Fig. 2, with a πN intermediate state and where the nucleon is considered as having no anomalous moment. In our notation this would be

$$\mu'^{S,V}=I_e^{S,V}. \quad (57)$$

The second-order correction is then given by the correction to the same graph of Fig. 2 coming from the first order a.m.m. and by the graph with a $\pi\pi N$ intermediate state and normal moment, and so on. In other words, to pass from a given order in the expansion to the successive one corresponds to adding one more pion in the intermediate state. We observe that in the electrodynamic case there is a strict correspondence between this kind of expansion and the perturbative expansion since adding a new photon corresponds to increasing by one unit the order in α, while such a correspondence does not exist in the case of strong interactions. However, the Drell-Pagels philosophy is very interesting in order to define a perturbationlike expansion in terms of step-by-step saturation of unitarity. Our philosophy is different and, in a sense, more conventional. Our approximation also consists of partial saturation of unitarity, but we leave the magnetic moment as an unknown and we look for a consistent solution at every step.

TABLE I. Comparison among the three models described in Sec. 3 of the text.

	Model I	Model II	Model III
I_{e+}^S	−0.065	−0.103	−0.568
I_{e-}^S	−0.441	−0.441	−0.589
I_{e+}^V	+0.182	+0.390	+1.345
I_{e-}^V	+1.389	+1.389	+1.786
I_{m+}^S	−0.144	−0.156	−2.622
I_{m-}^S	−1.771	−1.771	−3.669
I_{m+}^V	−0.076	−0.094	−1.286
I_{m-}^V	−0.373	−0.373	−1.022
μ'^S	−0.173	−0.186	−0.159
μ'^V	+1.084	+1.211	+0.947

We remark that, in order to expect good results from Drell and Pagels's first approximation, both the two-pion contribution and the I_m term should be small. However, in our analysis we find that I_m is large and of the same order as I_e. Therefore, we expect that our approach should give a better result, provided the higher intermediate states are negligible. On the contrary, the fact that our result is not very good indicates that those contributions are not so small.

We finally report the numerical results of our analysis for each of the three models of Sec. 3.

Model I. The expressions for $F_\pi(\pm W)$ are given in Eqs. (20) and (21), while the multipole amplitudes are given in Eqs. (45)–(48), (51), and (52), where

$$e^{i\alpha(W)}\cos\alpha(W)=1/(1-ia_1q^3), \quad (58)$$

$$e^{i\alpha(-W)}\cos\alpha(-W)=1/(1-ia_0q), \quad (59)$$

the Born amplitudes are those of Eq. (32), and $Z_{33}=0.091$.

For the isoscalar and isovector a.m.m., we get

$$\mu'^S=-0.17, \quad \mu'^V=1.08. \quad (60)$$

Model II. We proceed here in a quite similar way. Now we have

$$e^{i\alpha(W)}\cos\alpha(W)=\frac{W_r-W}{W_r-W-i\gamma_rq^3} \quad (61)$$

and we obtain for the a.m.m. the following values:

$$\mu'^S=-0.18, \quad \mu'^V=1.21. \quad (62)$$

Model III. Here the πNN form factor has been numerically evaluated from Eqs. (15′) and (16′), where the phase shifts are those of Ref. 12 up to 350-MeV pion energy and have been taken as constant beyond that energy (they are almost constant at that energy and in the particular fit). The integrals of Eq. (54) have also been numerically calculated and we get

$$\mu'^S = -0.16\,, \quad \mu'^V = 0.95\,. \tag{63}$$

All the above results are summarized in Table I, where we also report the numerical values of the quantities I_e^S, I_e^V, I_m^S, and I_m^V of Eqs. (55) and (56).

We note that Models I and II give values of the integrals and of the anomalous moments which are rather close to each other. Model III uses an empirical fit and the problem arises whether the MacDowell symmetry is satisfied at least approximately by such fits. The prediction for μ'^S can be considered satisfactory in all three models, in view of the inaccuracies and approximations introduced. Presumably the small discrepancy comes from errors in the high-energy tail of the photoproduction amplitudes, and it should be possible to obtain better estimates by further refinements. Our general impression is that a better estimate (for μ'^S and especially for μ'^V) requires a better knowledge of photoproduction in the intermediate-energy region (beyond the region of the static model). On the other hand, much work is being devoted at this time in various laboratories to a better understanding of these amplitudes, and we hope that it will soon be possible substantially to improve the results obtained here.

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Signature of Regge Cuts Coupled to Spinning Particles

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It is proved, by perturbation theory and by the method of Sudakov, that if two Regge poles with trajectories α_1, α_2 and signatures τ_1, τ_2 are exchanged, the resulting Regge branch point at $j=\alpha_1+\alpha_2-1$ appears only in partial-wave amplitudes of signature $\tau_1\tau_2\eta$, where $\eta=-1$ if both Regge poles are fermions and $\eta=+1$ otherwise. An example is given from the case of proton-proton scattering.

I. INTRODUCTION

In situations where it is impossible to fit experimental data by assuming Regge-pole dominance, it is useful to investigate whether the discrepancy can be accounted for by contributions from branch cuts in the complex angular momentum plane. For this reason (as well as simply to satisfy one's theoretical curiosity), it is important to know which amplitudes receive contributions from particular Regge cuts.

A Regge pole has associated with it definite quantum numbers (e.g., isospin, G parity, parity, signature) and will affect only those amplitudes with an identical set of quantum numbers. It can be shown[1–5] that the exchange of two Regge poles with trajectories α_1 and α_2 will give rise to branch points in the complex j plane; of these branch points, the one lying furthest to the right has a trajectory

$$j=\alpha_1+\alpha_2-1\,, \tag{1}$$

where the arguments of α_1 and α_2 are given by definite rules.[2–4] In order to discover which amplitudes possess such a branch point, we need to know the quantum numbers associated with a two-Reggeon system.

It is clear that internal quantum numbers, such as isospin and G parity, will combine in exactly the same way as if the Reggeons were elementary particles; for example, the exchange of two Pomeranchukons will give a cut in an amplitude with $I=0$ and $G=1$, whereas the exchange of a Pomeranchukon and a pion Regge pole will give a cut in an amplitude with $I=1$ and $G=-1$. Gribov[6] pointed out that one would expect any particular Regge cut to appear in amplitudes of both parities because of the arbitrary orbital angular momentum associated with the two-Reggeon system. There remains the important question of signature.

Mandelstam[7] proved that a partial-wave amplitude involving a state of two elementary particles of spins σ_1 and σ_2 has a singularity in the j plane at

$$j=\sigma_1+\sigma_2-1\,, \tag{2}$$

provided this is a wrong-signature point. For positive-signature amplitudes, the wrong-signature points are the odd integers (or odd integers plus one-half in the case of boson-fermion amplitudes); for negative-signature amplitudes, the wrong-signature points are the even integers (or even integers plus one-half). Thus, if we put $\nu=\frac{1}{2}$ for a boson-fermion amplitude and $\nu=0$ otherwise, we see that the singularity at $j=\sigma_1+\sigma_2-1$

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