Pion Production in Nucleon-Nucleon Collisions at Threshold*

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We present a theory of soft-pion production by nucleons which provides a connection between the production amplitude and nucleon-nucleon scattering. This is done by applying the algebra of currents to the S-matrix element for the process $\gamma+2n \rightarrow 2N+\pi$. After reducing out both the photon and pion, the hypothesis of partially conserved axial-vector current is used to replace the pion field by the divergence of the axial-vector current. Using the current commutation relations, we then obtain the pion production amplitude in terms of the amplitude for photoproduction of a pion on two nucleons, plus a correction integral which vanishes in the soft-pion limit. Upon taking the soft-photon–soft-pion limit, only certain pole terms survive. In this way, pion production is directly related to off-shell nucleon-nucleon scattering. Results are presented for the reactions $pp \rightarrow np\pi^+$, $np \rightarrow pp\pi^-$, and $pp \rightarrow pp\pi^0$. The theoretical predictions are found to be in good agreement with the cross-section data near threshold for π^- and π^0 production, and with an extrapolation from higher-energy data for π^+ production.

I. INTRODUCTION

For some time now, soft-pion techniques have been applied to strong-interaction physics.[1,2] Much of the recent interest has been aroused by the success of calculations[3] using current commutation relations[4] in conjunction with the hypothesis of partial conservation of axial-vector current (PCAC).[5] Particular application of these methods to problems involving the production of pions has been discussed in a general way by Weinberg.[6] Along these lines, Chang has studied the reaction $\pi N \rightarrow \pi\pi N$ near threshold,[7] with numerical results in very good agreement with experimental data.

It is natural to expect that these techniques can also be applied to the production of pions in nucleon collisions $NN \rightarrow NN\pi$, where the pion is "soft." We treat this problem in this paper, restricting our considerations, for mathematical convenience, to the threshold region.[8]

Pion production by nucleons has previously been considered by many authors. At moderately high energies ($T_{\text{lab}} \approx 2$ GeV), the peripheral model[9] has been moderately successful. However, at these energies the pion is usually quite energetic, i.e., it is not soft. For somewhat lower energies ($T_{\text{lab}} \approx 700$ MeV), an isobar model proposed by Mandelstam[10] works nicely. Here, too, the pion, which is assumed to arise from the decay of a nucleon isobar produced in the collision, is too energetic to be considered soft. The only treatment, until recently, of the production of low-energy pions by nucleons is the phenomenological analysis (not a dynamical theory) of Gell-Mann and Watson.[11,12] Their expressions for production cross sections, fit to data at somewhat higher energies, can be extrapolated to threshold, where the pion has little kinetic energy.

We present here a theory of soft-pion production by nucleons which, in close analogy to soft bremsstrahlung, provides a connection between the production amplitude and nucleon-nucleon scattering. This is done by considering the algebra of currents as applied to the S-matrix element for the process $\gamma+2N \rightarrow 2N+\pi$. After performing the Lehmann-Symanzik-Zimmermann (LSZ) reduction on both the photon and pion, PCAC is used to replace the pion field operator by the divergence of the axial-vector current. After integrating by parts and using current commutation relations, we obtain the pion production amplitude in terms of the amplitude for photoproduction of a pion on two nucleons, plus a correction integral which vanishes in the soft-pion limit.

At this point, we take the soft-photon and soft-pion limits in the photoproduction amplitude. It turns out that only certain pole terms survive, in accordance with the Adler-Dothan theorem.[13] In this way, we finally obtain the pion production amplitude in terms of off-shell nucleon-nucleon scattering amplitudes. The formal results are precisely equivalent to those following from the direct application of the Adler-Dothan theorem to the axial-vector vertex in $NN \rightarrow NN\pi$.[14] The advantage of the present method, we believe, is that the results are in a form better adapted to a treatment of higher-

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[8] A brief summary of the method used here and a preliminary result for the reaction $pp \rightarrow np\pi^+$ has been published; see M. E. Schillaci, R. R. Silbar, and J. E. Young, Phys. Rev. Letters **21**, 711; **21**, 1030(E) (1968).
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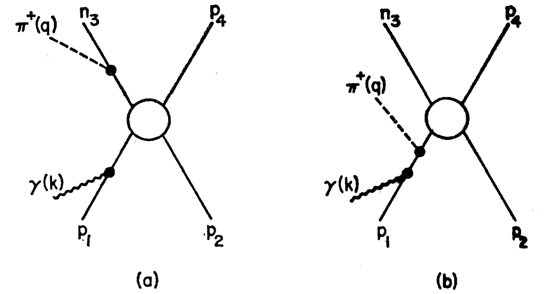

FIG. 1. Typical nucleon-pole graphs contributing to the process $\gamma+2N\rightarrow 2N+\pi$; (b) survives in the soft-photon–soft-pion limit while (a) does not.

order corrections, when and if desired. Furthermore, the connection between pion production and pion photoproduction is interesting in itself, but lack of data precludes testing this relation at this time.

The theoretical treatment described above is presented in Sec. II. In Sec. III, we begin with the amplitude derived in this way and calculate production cross sections near threshold. The mathematical simplifications resulting from the restriction to threshold are considerable.[14] Because of the off-shell nature of the nucleon-nucleon scattering involved, a correction must be introduced to account for the effective inelasticity. We first give the cross sections for the "unbound" processes, $pp\rightarrow np\pi^+$ and $np\rightarrow pp\pi^-$. Although the derivation in Sec. II does not strictly apply to neutral-pion production, e.g., $pp\rightarrow pp\pi^0$, we can, and do, treat it here in a similar manner.

Numerical results for these cross sections are presented and discussed in Sec. IV. The agreement between theoretical predictions and experimental data (or extrapolations of the data[12]) is very good.

II. DERIVATION OF PRODUCTION AMPLITUDE

As discussed in the Introduction, we begin by considering the S-matrix element for the photoproduction process

$$\gamma+N_1+N_2\rightarrow N_3+N_4+\pi\,, \tag{1}$$

namely,

$$S_{fi}=\langle \text{out}\,;qp_3p_4|kp_1p_2\,;\text{in}\rangle\,. \tag{2}$$

Here k, q, and p are the four-momenta for photon, pion, and nucleons, respectively. The standard LSZ reduction technique applied to both pion and photon yields the following T-matrix element:

$$T_{fi}=T_{NN\pi,\gamma NN}=T_{\beta\pi,\gamma\alpha}=[(2\pi)^6 2k_0 2q_0]^{-1/2}\epsilon^\nu M_\nu{}^i\,, \tag{3a}$$

$$M_\nu{}^i=-i\int d^4x(q^2-\mu^2)e^{iq\cdot x}\langle\beta|T[\varphi_i(x)j_\nu{}^{\text{EM}}(0)]|\alpha\rangle\,. \tag{3b}$$

We abbreviate the final (initial) two-nucleon out (in) state by $\beta(\alpha)$, indicate the charge state of the pion (of mass μ) by the subscript i, and denote the photon polarization vector by ϵ^ν. Equation (3b) actually defines the matrix element when the pion is off shell. Also, various surface terms have been dropped in deriving this expression.

At this point, we use the PCAC hypothesis to replace the pion field in $M_\nu{}^i$;

$$\varphi_i(x)=C^{-1}\partial^\mu A_\mu{}^i(x)\,,\quad C=i\mu^2F_\pi/\sqrt{2}\,, \tag{4}$$

where $A_\mu{}^i$ is the axial-vector current and F_π is the pion-decay constant. Integrating by parts (and dropping surface terms) yields

$$\begin{aligned}M_\nu{}^i=iC^{-1}\int d^3x(q^2-\mu^2)e^{-iq\cdot x}&\\ \times\langle\beta|[A_0(0,\mathbf{x}),j_\nu{}^{\text{EM}}(0)]|\alpha\rangle&\\ -q^\mu C^{-1}\int d^4x(q^2-\mu^2)e^{iq\cdot x}&\\ \times\langle\beta|T[A_\mu{}^i(x)j_\nu{}^{\text{EM}}(0)]|\alpha\rangle&\,.\end{aligned} \tag{5}$$

The electromagnetic current $j_\nu{}^{\text{EM}}(0)$ is composed of two-vector currents $j_\nu{}^{\text{EM}}=V_\nu{}^{i=3}+(1/\sqrt{3})V_\nu{}^{i=8}$. The relevant current commutation relations entering the first term in Eq. (5) are, therefore,

$$[A_0{}^{(\pm)}(0,\mathbf{x}),V_\nu{}^3(0)]=\mp A_\nu{}^{(\pm)}(0,\mathbf{x})\delta^3(\mathbf{x})\,, \tag{6a}$$

$$[A_0{}^{(\pm)}(0,\mathbf{x}),V_\nu{}^8(0)]=0\,, \tag{6b}$$

and

$$[A_0{}^3(0,\mathbf{x}),V_\nu{}^3(0)]=[A_0{}^3(0,\mathbf{x}),V_\nu{}^8(0)]=0\,. \tag{6c}$$

Schwinger terms have been dropped from the above commutation relations since, assuming they are c numbers, it can be shown that they do not contribute to the final result.[15] We then rewrite Eq. (5), for the case of charged pion production, as

$$\begin{aligned}M_\nu{}^{(\pm)}=\mp iC^{-1}(q^2-\mu^2)\langle\beta|A_\nu{}^{(\pm)}(0)|\alpha\rangle&\\ -C^{-1}q^\mu\int d^4x(q^2-\mu^2)e^{iq\cdot x}&\\ \times\langle\beta|T[A_\mu{}^{(\pm)}(x)j_\nu{}^{\text{EM}}(0)]|\alpha\rangle&\,.\end{aligned} \tag{7}$$

We now contract Eq. (7) with $i\kappa^\nu\equiv i(k-q)^\nu$, and use the PCAC relation

$$\begin{aligned}i\kappa^\nu\langle\beta|A_\nu{}^{(\pm)}(0)|\alpha\rangle&=\langle\beta|\partial^\nu A_\nu{}^{(\pm)}(0)|\alpha\rangle\\ &=[C/(\mu^2-\kappa^2)]\langle\beta|J_{\pi^\pm}(0)|\alpha\rangle\,,\end{aligned} \tag{8}$$

where the matrix element of the pion source current $J_{\pi^\pm}$ is related to the pion production amplitude. In this

[15] Alternatively, we could take the limit $\mathbf{q}\rightarrow 0$, carry out the integration over d^3x to get the axial charge $Q^{(\pm)}$ and use the commutation relations between charge and current corresponding to Eq. (6), which do not involve any Schwinger terms.

way, Eq. (7) becomes

$$\kappa^\nu M_\nu^{(\pm)} = \pm(q^2-\mu^2)(\kappa^2-\mu^2)^{-1}\langle\beta|J_{\pi^\pm}(0)|\alpha\rangle - C^{-1}q^\mu\kappa^\nu\int d^4x\,(q^2-\mu^2)e^{iq\cdot x} \times\langle\beta|T[A_\mu^{(\pm)}(x)j_\nu^{\mathrm{EM}}(0)]|\alpha\rangle. \quad (9)$$

We propose to study this equation in the soft-photon–soft-pion limit. First, we show that the term with the integral vanishes in this limit. Let $\mathbf{q}\to 0$ and $\boldsymbol{\kappa}\to 0$; this can be done without difficulty. Then, by inserting a complete set of states in the matrix element in the integrand, one can see that there is no possibility of the integral behaving as $1/q_0\kappa_0$ (or worse) as κ_0 and q_0 both tend to zero. Thus, in this limit, the last term in Eq. (9) vanishes.

On the other hand, the remaining two terms in Eq. (9) do not vanish in this limit, and this yields a relation between pion production and photoproduction on two nucleons. Although this relation is interesting, because of the lack of photoproduction data, we do not pursue this further at this time.

In view of the discussion above, we can rewrite Eq. (9) as

$$\langle\beta|J_{\pi^\pm}(0)|\alpha\rangle = \pm\lim_{q_0\to 0}\lim_{\kappa_0\to 0}\kappa_0 M_0^{(\pm)}. \quad (10)$$

In order to extract the dominant contributions to the right-hand side of this equation, we make use of the Adler-Dothan theorem,[13] which determines the leading behavior in the low-energy limit for processes involving the coupling of particles to vector and axial-vector currents. This theorem, as it applies to this problem, states that the leading terms in κ_0 and q_0 arise from those graphs which involve nucleon poles, with pseudovector pion coupling in accordance with PCAC. All such graphs fall into either of two classes: those for which the photon and pion are attached to different external nucleon lines and those for which they are attached to the same line. For the case of π^+ production in p-p collisions, typical examples of these graphs are illustrated in Fig. 1. All graphs of the class typified by Fig. 1(a) do not survive in the soft-photon–soft-pion limit since their contribution to $M_0^{(\pm)}$ does not contain a pole at $\kappa_0=0$. Graphs such as in Fig. 1(b) will survive, but only if the limits are taken in the order $\kappa_0\to 0$ first, then $q_0\to 0$.

We illustrate the limiting procedure in detail by examining the graph in Fig. 1(b). The contribution to the production amplitude of this graph is

$$\kappa^\mu\frac{g}{2M}\sum_\alpha F_\alpha^{np}\bar{u}_4 t_\alpha u_2\cdot\bar{u}_3 t^\alpha\frac{1}{\not p_1+\not k-\not q-M} \times\tau_-\not q\gamma_5\frac{1}{\not p_1+\not k-M}\Gamma_\mu^{p}u_1. \quad (11)$$

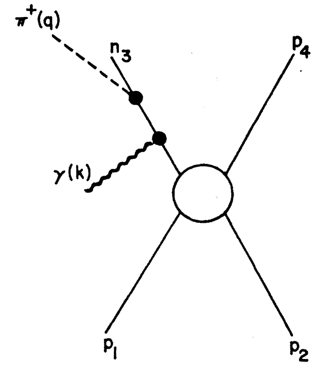

FIG. 2. A nucleon-pole graph contributing to the process $\gamma+2N\to 2N+\pi$ which survives in the soft-photon–soft-pion limit but vanishes at threshold.

As mentioned previously, the pion-nucleon coupling is pseudovector, with coupling constant $g/2M$ ($g^2/4\pi = 14.6$ and $M\equiv$ nucleon mass). The quantities t_α are the Dirac matrices 1, γ_μ, $(1/\sqrt{2})\sigma_{\mu\nu}$, $i\gamma_\mu\gamma_5$, and γ_5, which, together with the five invariant functions, F_α^{np},[16] describe the off-shell neutron-proton scattering occurring after the pion emission. The F_α^{np} have arguments $\nu'=\nu+\kappa\cdot p_2$ and $\Delta'=\Delta+\kappa\cdot p_3$, where $\nu=p_1\cdot p_2+p_3\cdot p_4$ and $\Delta=p_1\cdot p_3+p_2\cdot p_4$. Finally, Γ_μ^N is the electromagnetic vertex function which reduces, in the soft-photon limit, to γ_μ for protons and zero for neutrons.

By first taking the limits $\boldsymbol{\kappa}$, $\mathbf{q}\to 0$, Eq. (11) reduces to

$$\kappa_0\frac{g}{2M}\sum_\alpha F_\alpha^{np}\bar{u}_4 t_\alpha u_2\cdot\bar{u}_3 t^\alpha\frac{\not p_1+\kappa_0\gamma_0+M}{2p_{10}\kappa_0+\kappa_0^2} \times\tau_-\gamma_0 q_0\gamma_5\frac{\not p_1+k_0\gamma_0+M}{2p_{10}k_0+k_0^2}\gamma_0 u_1. \quad (12)$$

Now, taking the limit $\kappa_0\to 0$ (which implies $k_0\to q_0$) we obtain, after some Dirac algebra,

$$\frac{g}{2M}\left(\frac{-2p_{10}+q_0}{2p_{10}+q_0}\right)\sum_\alpha F_\alpha^{np}\bar{u}_4 t_\alpha u_2\cdot\bar{u}_3 t^\alpha \times\left(1+\frac{M}{p_{10}}\gamma_0\right)\gamma_5\tau_- u_1. \quad (13)$$

Finally, taking the limit $q_0\to 0$ yields

$$-\frac{g}{2M}\sum_\alpha F_\alpha^{np}\bar{u}_4 t_\alpha u_2\cdot\bar{u}_3 t^\alpha\left(1+\frac{M}{p_{10}}\gamma_0\right)\gamma_5\tau_- u_1. \quad (14)$$

Since the coupling of the photon with the neutron vanishes in the soft-photon limit, only one other such graph contributes to the production amplitude, namely, the graph shown in Fig. 2. The resulting contribution is

$$+\frac{g}{2M}\sum_\alpha F_\alpha^{np}\bar{u}_4 t_\alpha u_2\cdot\bar{u}_3\gamma_5\tau_-\left(1+\frac{M}{n_{30}}\gamma_0\right)t^\alpha u_1. \quad (15)$$

It is worth noting at this point that identical results can

[16] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. 120, 2250 (1960).

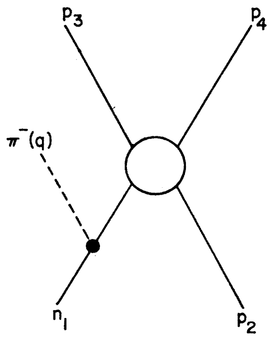

FIG. 3. The only nucleon-pole graph contributing to the process $np \to pp\pi^-$ at threshold.

be obtained from the direct application of the Adler-Dothan theorem to the pion production amplitude.[14]

Near production threshold, the final pair of nucleons is taken at rest (in the over-all c.m. frame) or nearly so. The contribution of the graph of Fig. 2 vanishes at threshold, as can be seen directly from Eq. (15) [since $\bar{u}_3(1-\gamma_0)=0$]. Thus, the threshold amplitude is given by Eq. (14), appropriately antisymmetrized. A similar expression is obtained for $np \to pp\pi^-$. Because of the commutation relation in Eq. (6c), this technique does not apply to neutral pion production. However, direct use of the Adler-Dothan theorem here yields expressions for the neutral pion production amplitude similar to Eqs. (14) and (15).

III. PION-PRODUCTION CROSS SECTIONS

The derivation of the production amplitude in the last section required taking the pion mass to zero. We assume that the amplitude for the production of a pion with physical mass μ extrapolates smoothly from the nonphysical amplitude, so that expressions such as Eq. (14) will be used for physical pions. In fact, if the pion were massless, the nucleon momentum at threshold would be zero and the production amplitude would, therefore, vanish [since $(1-\gamma_0)u_1=0$]. This is in accordance with the Adler consistency condition.[2]

Since internal emission has been ignored in the derivation in the last section, we must keep only the lowest-order terms in μ/M. At threshold, then, where $p_{10}=M+\frac{1}{2}\mu\approx M$, the amplitude for π^+ production in p-p collisions is

$$T_{\pi^+}=\frac{g}{2M}\sqrt{2}\sum_\alpha F_\alpha{}^{np}[\bar{u}_4 l_\alpha u_2\cdot\bar{u}_3 l^\alpha(1+\gamma_0)\gamma_5 u_1 - \bar{u}_4 l_\alpha u_1\cdot\bar{u}_3 l^\alpha(1+\gamma_0)\gamma_5 u_2]. \quad (16)$$

Here we have antisymmetrized in the spin and momentum labels of the initial protons.[17] Also, we used the fact that the arguments of the $F_\alpha{}^{np}$ near threshold [$p_3\approx p_4\approx(M,0)$] and for $\mathbf{q}=0$ are invariant under the interchange of p_1 and p_2: $\nu'=E_i^2+p_i^2+M^2-q_0E_i$ and $\Delta'=2E_iM-q_0M$, in the c.m. frame.

Squaring T_{π^+} and taking the spin average, we find that[18]

$$(|T_{\pi^+}|^2)_{\rm av}\equiv\frac{1}{4}\sum_{\rm spins}|T_{\pi^+}|^2=8g^2\mu M[3|F_S+F_V|^2 -2\,\mathrm{Re}(F_S+F_V)(F_T+F_A)^*+11|F_T+F_A|^2] \equiv 8g^2\mu M|\mathfrak{F}_{\pi^+}|^2, \quad (17)$$

where we have suppressed the np superscript on the invariant functions. The right-hand side of this equation can be evaluated by using the expressions for the invariant functions given in terms of the nucleon-nucleon phase shifts by Nyman.[19] (But see discussion below.)

The total production cross section is

$$\sigma_{\pi^+}=\int\frac{(|T_{\pi^+}|^2)_{\rm av}}{4E_1E_2v_{\rm rel}}\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4}\frac{d^3q}{(2\pi)^3 2\omega}\times(2\pi)^4\delta^4(P_f-P_i)$$
$$=\frac{1}{4\sqrt{2}\pi^2}\left(\frac{g^2}{4\pi}\right)\frac{\mu}{M}\int_0^\epsilon|\mathfrak{F}_{\pi^+}|^2[\eta(\epsilon-\eta)]^{1/2}d\eta\,, \quad (18)$$

where

$$W=2M+\mu+\epsilon=2M(1+T_{\rm lab}/2M)^{1/2}. \quad (19)$$

Here ϵ is the total kinetic energy of the three final particles in the over-all c.m. frame, and η is the kinetic energy of the final nucleons in their c.m. frame.[20] In obtaining Eq. (18), we have made various threshold approximations, such as $E_3=E_4=M$.

A similar calculation can be carried out for the reaction $np\to pp\pi^-$. Since, at threshold, the pion is emitted from initial nucleon lines, the only graph contributing here is that shown in Fig. 3. There is no interference occurring between graphs as in the last example. Because of the large momentum transfer involved in the off-shell proton-proton scattering, Coulomb contributions can be ignored there. We find that

$$\sigma_{\pi^-}=\frac{1}{8\sqrt{2}\pi^2}\left(\frac{g^2}{4\pi}\right)\frac{\mu}{M}\int_0^\epsilon|\mathfrak{F}_{\pi^-}|^2[\eta(\epsilon-\eta)]^{1/2}d\eta\,, \quad (20)$$

where

$$|\mathfrak{F}_{\pi^-}|^2=|F_S+F_V|^2+3|F_T+F_A|^2. \quad (21)$$

[17] An alternative way of antisymmetrizing is to take advantage of the isospin symmetries of the invariant functions (see Ref. 16). Here, one adds to the amplitude given by Eq. (14), the amplitude obtained from it by exchanging the spin, isospin, and momentum labels of the initial nucleons and of the final nucleons. This results in some simplification in the trace-taking which follows. The approximations involving the invariant functions that one is naturally led to make differ from those used in the text, but at threshold they are equivalent.

[18] It is convenient to take advantage of the $1+\gamma_0$ structure in the amplitude and to do the Dirac traces by first reducing them to Pauli traces. We use the convention that $\bar{u}u=2M$.

[19] E. Nyman, Phys. Rev. **170**, 1628 (1968).

[20] Actually, the $|\mathfrak{F}_{\pi^+}|^2$ in Eq. (18) is the angle average of the quantity appearing in Eq. (17). A similar comment applies to $|\mathfrak{F}_{\pi^-}|^2$ and $|\mathfrak{F}_{\pi^0}|^2$.

Again we have dropped the pp superscript. The invariant functions are to be evaluated using nuclear phase shifts only.

The remaining unbound process $pp\rightarrow pp\pi^0$ can be treated in the same way. In this case, the π^0 can be emitted from either initial proton and again there is interference between two graphs. We find the cross section for this case is given by an expression formally identical to Eq. (18), except for an over-all factor of one-fourth. The invariant functions in $|\mathfrak{F}_{\pi^0}|^2$ are those for pp scattering without Coulomb contributions.

The $F_\alpha(\nu')$ which appear in the integrals in Eqs. (18) and (20) correspond, as we have said, to off-shell scattering. We simply assume that these functions are given by the physical on-shell functions evaluated at the same value of ν',

$$|\mathfrak{F}^{(\mathrm{off})}(\nu')|^2\approx|\mathfrak{F}^{(\mathrm{on})}(\nu')|^2, \tag{22a}$$

where in terms of the integration variable η,

$$\nu'\approx 2M^2+\mu M+4p_f^2=2M^2+\mu M+4M\eta. \tag{22b}$$

The corresponding squared c.m. energy for the elastic scattering is $s=\nu'+2M^2$.

An alternative way of accounting for the off-shell scattering is to relate $|\mathfrak{F}(\nu')|^2$ to the same combination of physical F_α's at an energy near to the elastic threshold. More exactly,

$$|\mathfrak{F}^{(\mathrm{off})}(\nu')|^2=(p_f/p_i)|\mathfrak{F}^{(\mathrm{on})}(\nu_0)|^2, \tag{23}$$

where $\nu_0=2M^2+4p_f^2$ is the energy variable corresponding to elastic scattering at c.m. momentum p_f of the final nucleons. Equation (23) can be derived as follows: For $\mathbf{q}=0$,

$$\begin{aligned}\nu'&=(p_1-q)\cdot p_2+p_3\cdot p_4\\&=2M^2+2p_i^2+2p_f^2-q_0E_i\\&\approx 2M^2+p_i^2+3p_f^2\approx\nu_0+q_0M,\end{aligned} \tag{24}$$

where, by conservation of energy,

$$q_0\approx(p_i^2-p_f^2)/M, \tag{25}$$

assuming p_i, p_f, and q_0 are all $\ll M$. The pion energy q_0 is a measure of how far off shell the F_α are. For, if $q_0=0$, the F_α are certainly to be evaluated near the elastic threshold. Thus, we can estimate the off-shell correction by expanding in a Taylor's series in q_0 and, eventually, setting $q_0=\mu$.

$$|\mathfrak{F}(\nu')|^2\approx|\mathfrak{F}(\nu_0)|^2+q_0M\left(\frac{\partial}{\partial\nu}|\mathfrak{F}|^2\right)_{\nu_0}. \tag{26}$$

To estimate this derivative, we note that, in the energy range 10–300 MeV, both the np and pp total cross sections behave like $1/p_f^2$. Since we are interested in the case where $p_i\neq p_f$ (a kind of "inelastic" two-body reaction), these cross sections are related to the F_α by

$$\sigma\propto(p_f/p_i)|\mathfrak{F}_{\mathrm{el}}|^2, \tag{27}$$

where $|\mathfrak{F}_{\mathrm{el}}|^2$ is the combination of F_α which occurs in elastic scattering. We conclude that

$$|\mathfrak{F}_{\mathrm{el}}|^2\propto 1/p_f^3, \tag{28}$$

and it is reasonable to assume that this also holds for the $|\mathfrak{F}|^2$.[21] Then, keeping p_i fixed,

$$\frac{\partial|\mathfrak{F}|^2}{\partial\nu}=\frac{1}{(\partial\nu/\partial p_f^2)}\frac{\partial}{\partial p_f^2}|\mathfrak{F}|^2=-\tfrac{1}{2}|\mathfrak{F}|^2/p_f^2, \tag{29}$$

and Eq. (26) becomes

$$\begin{aligned}|\mathfrak{F}(\nu')|^2&\approx|\mathfrak{F}(\nu_0)|^2(1-q_0M/2p_f^2+\cdots)\\&\approx|\mathfrak{F}(\nu_0)|^2(1+q_0M/p_f^2)^{-1/2},\end{aligned} \tag{30}$$

which, with Eq. (25), is just Eq. (23).

We can now use the relation given in Eq. (23) to simplify the integral in Eq. (18). We first express the np invariant functions in terms of the isospin functions

$$F_\alpha{}^{np}=\tfrac{1}{2}F_\alpha{}^{I=0}+\tfrac{1}{2}F_\alpha{}^{I=1}=\tfrac{1}{2}F_\alpha{}^t+\tfrac{1}{2}F_\alpha{}^s, \tag{31}$$

where, in the last step, we have identified the S wave, $I=0$ functions with triplet scattering and the S wave, $I=1$ functions with singlet scattering. The invariant amplitudes are not all independent at the elastic threshold, where the following relations hold[22]:

$${}^1S_0:\ F_S{}^s=F_V{}^s=-F_T{}^s=-F_A{}^s, \tag{32a}$$

$${}^3S_1:\ \tfrac{1}{5}F_S{}^t=F_V{}^t=F_T{}^t=F_A{}^t. \tag{32b}$$

In the effective-range approximation,

$$\begin{aligned}F_A{}^s&=-(\pi/M)p_f^{-1}e^{i\delta_s}\sin\delta_s\\&=-(\pi/M)[-a_s^{-1}+\tfrac{1}{2}r_sp_f^2+ip_f]^{-1},\end{aligned} \tag{33a}$$

$$\begin{aligned}F_A{}^t&=+(\pi/M)p_fe^{i\delta_t}\sin\delta_t\\&=+(\pi/M)[-a_t+\tfrac{1}{2}r_tp_f^2+ip_f]^{-1},\end{aligned} \tag{33b}$$

with $a_{s,t}$ the singlet (triplet) scattering length and $r_{s,t}$ the singlet (triplet) effective range. Using these rela-

[21] We will see later that the $|\mathfrak{F}|^2$ are related to the $|F_{\mathrm{el}}|^2$ near elastic threshold.

[22] These relations can be extracted from Nyman's appendix (Ref. 19). See also Beder's appendix (Ref. 14); the triplet relations derived there are in error.

tions, Eq. (17) becomes, with[23]

$$p_i^2 \approx \mu M\,,$$

$$(|T_{\pi^+}|^2)_{\rm av} = 128 g^2 p_f (\mu M)^{1/2} (|F_A{}^s|^2 + 2|F_A{}^t|^2)$$

$$= 128\pi^2 g^2 \frac{p_f}{M}\left(\frac{\mu}{M}\right)^{1/2}\left[\frac{a_s^2}{1+a_s(a_s-r_s)p_f^2+(\frac{1}{2}a_s r_s)^2 p_f^4}+\frac{2a_t^2}{1+a_t(a_t-r_t)p_f^2+(\frac{1}{2}a_t r_t)^2 p_f^4}\right]$$

$$= 32\pi g^2 (p_f/M)(\mu/M)^{1/2}[\sigma_s(p_f^2)+2\sigma_t(p_f^2)]\,, \quad (34)$$

where $\sigma_{np} = \frac{1}{4}\sigma_s + \frac{3}{4}\sigma_t$. The cross section is

$$\sigma_{\pi^+} = \frac{1}{\sqrt{2}\pi}\left(\frac{g^2}{4\pi}\right)\frac{\mu}{M^3}(I_s + 2I_t)\,, \quad (35a)$$

$$I_{s,t} = \int_0^\epsilon \sigma_{s,t}(\eta)\eta[(\epsilon-\eta)/\mu]^{1/2} d\eta\,. \quad (35b)$$

IV. RESULTS AND DISCUSSION

We now evaluate the cross-section expressions derived in the last section and compare our predictions with experimental data. For all reactions the agreement is good, considering the sparse data in some cases. It is also of interest to juxtapose predictions for the same reaction, but for the different theoretical approximations employed. Here, too, the agreement is good, lending confidence in the numbers presented.

We use the reaction $pp \to np\pi^+$ as an example for making theoretical comparisons. In Fig. 4, we compare the cross section obtained from Eq. (18), where the off-shell combination $|\mathfrak{F}_{\pi^+}|^2$ is evaluated according to Eq. (23), with the results obtained from Eq. (35), which also uses Eq. (23) together with the effective-range approximation. The invariant functions $F_\alpha{}^{np}$, and thus $|\mathfrak{F}_{\pi^+}|^2$, are determined using only S-wave np phase shifts.[24] Also included in Fig. 4 is an extrapolation to threshold of higher-energy data, where we have assumed the final nucleons to be in a relative S state and the pion in an s state relative to the two-nucleon system[25] (this final state is designated as *Ss* in the notation of Rosenfeld[12]). The difference between the theoretical curves is small, and both can be considered to be in good agreement with the *Ss* extrapolation.

The use of the threshold relations and effective-range approximation enables one to write the production cross section in a more compact way, involving integrals

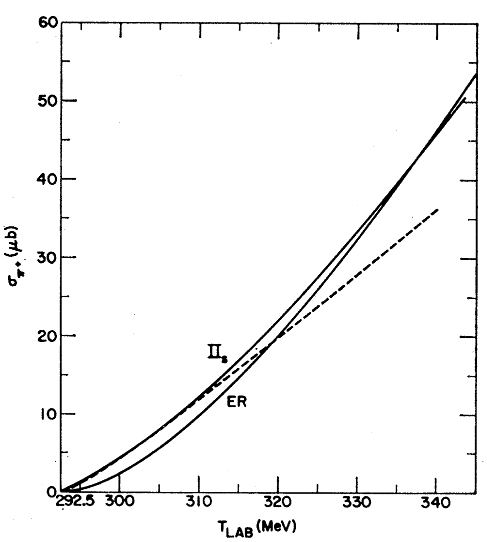

FIG. 4. Total cross section for the reaction $pp \to np\pi^+$ versus lab kinetic energy. The dashed curve is the *Ss* extrapolation from higher-energy data; (ER) corresponds to the use of the effective-range approximation; (II_s) corresponds to the use of $|\mathfrak{F}_{\pi^+}|^2$ with S-wave phase shifts only, evaluated according to Eq. (23).

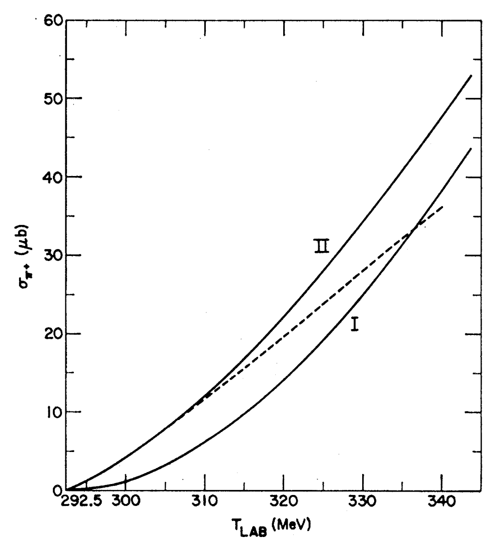

FIG. 5. Total cross section for the reaction $pp \to np\pi^+$ versus lab kinetic energy. The dashed curve is the *Ss* extrapolation from higher-energy data; (I) and (II) correspond to the approximations of Eqs. (22) and (23), respectively.

[23] It turns out that there is no interference between the singlet and triplet production amplitudes. This is to be expected because these states have different angular momentum.

[24] M. H. MacGregor, R. A. Arndt, and R. M. Wright, Phys. Rev. **169**, 1128 (1968); **173**, 1272 (1968).

[25] This extrapolation can be carried out using the results contained in Refs. 11 and 12. In the isospin notation of these authors, $\sigma(pp \to np\pi^+) = \sigma_{10} + \sigma_{11}$. The *Ss* part of σ_{10} contains one parameter which is determined by the reaction $pp \to d\pi^+$. The *Ss* part of σ_{11} also contains only one parameter which is determined by the reaction $pp \to pp\pi^0$.

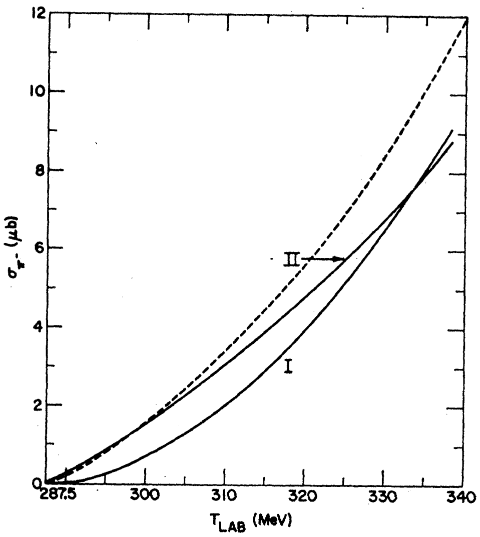

FIG. 6. Total cross section for the reaction $np \rightarrow pp\pi^-$ versus lab kinetic energy. The dashed curve is the data (Ref. 26); (I) and (II) correspond to the approximations of Eqs. (22) and (23), respectively.

that can be done analytically. On the other hand, using the exact combination $|\mathfrak{F}_{\pi^+}|^2$, yields results which are more accurate and which apply over a wider energy range. Thus, we will henceforth use cross section formulas of the form of Eq. (18).

The relative merits of the two methods used to correct for the off-shell scattering are shown in Fig. 5, again for the reaction $pp \rightarrow np\pi^+$. Both of the theoretical curves are obtained using Eq. (18). Curve I corresponds to the assumption that the off-shell scattering can be approximated as in Eq. (22) and curve II to the off-shell correction given by Eq. (23). For completeness, we again show the extrapolation from higher-energy data.

Both curves I and II agree rather well with the *Ss* extrapolation; however, at the lower energies, curve II provides a better fit. Here we have used all phase shifts[24] in determining $|\mathfrak{F}_{\pi^+}|^2$. Comparison of curve II with curve II_s of Fig. 4 shows that only S waves are important in this energy range.

We now discuss the other production reactions. In Fig. 6, we compare the theoretical predictions of Eq. (20) with the experimental data for the process $np \rightarrow pp\pi^-$. Again, curves I and II correspond to the off-shell corrections given by Eqs. (22) and (23), respectively. The data, as given by Handler,[26] are presented as a smooth continuous curve without errors.

[26] R. Handler, Phys. Rev. **138**, B1230 (1965).

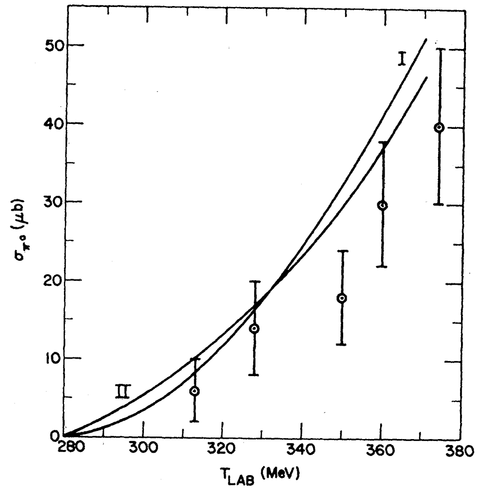

FIG. 7. Total cross section for the reaction $pp \rightarrow pp\pi^0$ versus lab kinetic energy. The data are taken from Ref. 27; (I) and (II) correspond to the approximations of Eqs. (22) and (23), respectively.

We have determined the approximate error, corresponding to one standard deviation, from his histogram of the number of events versus lab energy. Both theoretical curves agree, within error, with the data. As in Fig. 5, curve II provides a better fit at the lower energies.

The theoretical predictions for the reaction $pp \rightarrow pp\pi^0$ are shown in Fig. 7, together with experimental data.[27] Both theoretical curves are in good agreement with the lower-energy data points shown, and in fair agreement with the higher-energy data points. Thus, the theory seems to apply over a wider energy range here than that indicated by the reactions previously discussed.

In conclusion, it seems clear to us that the present techniques provide an accurate description of threshold π production in NN collisions.

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[27] A. F. Dunaitsev and Y. D. Prokoshkin, Zh. Experim. i Teor. Fiz. **36**, 1656 (1959) [English transl.: Soviet Phys.–JETP **9**, 1179 (1959)].