Spectral Density Sum Rules, Current Algebra, and Zero-Mass Extrapolations*

J. DOOHER†
Adelphi University, Garden City, New York 11530
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The new spectral sum rule

$$\int_0^\infty \rho_\pi^{(0)}(\mu^2)d\mu^2=\int_0^\infty \frac{\rho_\pi^{(1)}(\mu^2)}{\mu^2}d\mu^2$$

is derived under the assumption that there exists a unique limit in which all masses are zero for the $A_1\rho\pi$ and $\rho\pi\pi$ decay amplitudes. In this sum rule, $\rho_\pi^{(0)}(\mu^2)$ is the pseudoscalar spectral function and $\rho_\pi^{(1)}(\mu^2)$ is the axial-vector spectral function, both of which occur in the Källén-Lehmann representation of the propagator of the π-type axial-vector current. An experimental test of this sum rule is proposed. The equivalent sum rule for K-type axial-vector currents is derived from considerations of the ϕKK amplitude, and applications of these sum rules in conjunction with the first Weinberg sum rules are discussed. In particular, it is found that $f_K/f_\pi=1$ for the pion and kaon decay amplitudes.

I. INTRODUCTION

THE results of using the hypothesis of partially conserved axial-vector currents (PCAC) and the algebra of currents to obtain coupling constants consist of empirical determinations of these coupling constants. That this is so, can be seen in the context of Lagrangian field theory where renormalized coupling constants are by definition experimentally determined parameters in the theory. The results of current algebra and PCAC are therefore simply approximate relationships among quantities which are accessible to experiment. If these relationships turn out to be valid experimentally, then a test is provided for the current algebra and the validity of the mass extrapolations employed. However, these methods cannot be considered to be a dynamical determination of the couplings as in the bootstrap theories, for example. This paper extends the concept of mass extrapolations in such a way as to provide a dynamical determination of some of the coupling constants. In particular, we will treat the well-known Kawarabayashi, Suzuki, Riazzudin, and Fayyuzudin (KSRF) relation for the ρ-meson coupling g_ρ.[1,2] Their result is

$$g_\rho=m_\rho/f_\pi\,, \qquad (1)$$

where f_π is the pion decay amplitude. The use of ρ-meson dominance of the pion electromagnetic form factor and Eq. (1) yields a ρ width of approximately 144 MeV. If the results of Das, Mathur, and Okubo are taken,[3] the derivation of which utilized a once subtracted dispersion relation for the amplitude for $\rho\to 2\pi$ and pole dominance of the pion electromagnetic form factor, then (1) is replaced by

$$g_\rho=(m_\rho/f_\pi)(1+\delta)\,, \qquad (2)$$

where $\delta<0$. Reasonable values for δ yield a width of about 140 MeV. However, if the results of the recent colliding-beam experiments are confirmed,[4] which yield a width of about 90 MeV, then we must conclude that neither Eq. (1) nor Eq. (2) is well satisfied.

We propose to use PCAC and current algebra to obtain vertex functions at a point where all the masses are zero rather than on the mass shell. To do this, we must formulate our method of zero-mass extrapolation in such a way as to yield results with dynamical content. In this paper, we first consider the extrapolation of the $A_1\rho\pi$ and $\rho\pi\pi$ decay amplitudes. Our dynamical assumption is that the zero-mass extrapolation using the field theoretical form of PCAC and the zero-mass extrapolation using pole dominance in the form of field current identities yield identical zero-mass limits for the relevant vector-meson decay amplitudes. This "loose" statement will become more specific when the applications are discussed. The result of this assumption when it is applied to the $A_1\rho\pi$ system will be a new sum rule for the spectral densities of the Källén-Lehmann representations of the axial-vector and vector-current two-point functions. When this new sum rule is added to the first Weinberg sum rule, which is on rather firm footing in the context of field-current identities,[5,6] a set of relations among vector-meson and axial-vector-meson coupling constants can be derived which is consistent with experiment. Such applications are discussed in Sec. VI. This new sum rule is shown to imply the equality of the two possible zero-mass extrapolations for the $\rho\pi\pi$ system and therefore the consistency of our dynamical assumption is demonstrated. Also, applying a recently derived sum rule involving the cross section for the hadronic yield of electron-positron annihilation,[7]

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[1] K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters **16**, 255 (1966); **16**, 384 (E) (1966).
[2] Riazuddin and Fayyazudin, Phys. Rev. **147**, 1071 (1966).
[3] T. Das, V. S. Mathur, and S. Okubo, Phys. Rev. Letters **19**, 1085 (1967).
[4] The most recent data on $e^+ + e^- \to 2\pi$ are not conclusive on this point. See V. L. Auslander *et al.*, Phys. Letters **25B**, 433 (1967); J. E. Augustin *et al.*, Phys. Rev. Letters **20**, 129 (1968). These two groups have reported different cross sections and widths. The Novosibirsk group has since increased their width to 105 ± 15 MeV. However, there is still disagreement.
[5] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967).
[6] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).
[7] J. Dooher, Phys. Rev. Letters **19**, 600 (1967).

a test is proposed of this new sum rule. The method is then extended to the ϕKK system in Sec. V. Finally, the results are discussed and compared with more conventional calculations of vector-meson coupling constants.

II. ZERO-MASS EXTRAPOLATIONS FOR THE $A_1\rho\pi$ SYSTEM

These extrapolations are performed by employing the Lehmann-Symanzik-Zimmermann (LSZ) reduction technique. The field-theoretical form of the PCAC hypothesis consists in the proportionality of the pion field and the divergence of the axial-vector current. Different extrapolations are defined by the order in which the integration by parts of the divergence of the axial-vector current is done, i.e., whether before or after the decaying vector or axial-vector meson is taken out of the state vector. The decaying particle is always the last particle to be taken out of the state vector. The extrapolation by PCAC, which we will consider, is defined by doing the integration by parts first. The extrapolation using pole dominance is defined by first reducing the amplitude to its final form as the Fourier transform of a three-point function and then using the Ward identity to extrapolate to zero mass. It is possible to use the Ward identity because the field-current identity for the ρ^0 meson introduces the isovector part of the electromagnetic current into the three-point function.

A. PCAC Extrapolation for $A_1\rho\pi$

In general, there are two form factors for the $A_1\rho\pi$ decay

$$\langle\pi^+(k)\rho^0(q)|A_1^+(p)\rangle = -\frac{i(2\pi)^4\delta^4(p-q-k)}{[8p_0\omega q_0(2\pi)^9]^{1/2}}[g_{A\rho\pi}(-m_A^2,-m_\rho^2,-m_\pi^2)\epsilon^A\cdot\epsilon^\rho + h_{A\rho\pi}(-m_A^2,-m_\rho^2,-m_\pi^2)(\epsilon^A\cdot k)(\epsilon^\rho\cdot p), \quad (3)$$

where $\epsilon_\mu{}^A$ and $\epsilon_\mu{}^\rho$ are polarization vectors.

We next perform the LSZ reduction as described above. We do not, in the interest of simplicity, exhibit the wave-function renormalization constants explicitly. Since we will derive results based on the equality of completely reduced amplitudes, these wave-function renormalization constants which occur as multiplicative factors will cancel and do not enter into our final results. Therefore, we write the following expression for the decay amplitude:

$$\langle\pi^+(k)\rho^0(q)|A_1^+(p)\rangle = -\frac{(k^2+m_\pi^2)(q^2+m_\rho^2)}{m_\pi^2 f_\pi m_\rho^2}g_\rho\epsilon_\nu{}^\rho\int d^4x d^4y\, f_k{}^*(x)f_q{}^*(y) \times\langle 0|T(V_{3\nu}(y),\partial_\mu A_\mu{}^-(x))|A_1{}^+(p)\rangle, \quad (4)$$

where $f_k=[(2\pi)^3 2\omega]^{-1/2}e^{ik\cdot x}$, and we have used PCAC in the form

$$\partial_\mu A_\mu{}^-(x) = -m_\pi^2 f_\pi\phi_\pi{}^-(x) \quad (5)$$

and the field-current identity[8]

$$\rho_\mu{}^0(x) = -g_\rho/m_\rho^2 V_{3\mu}(x). \quad (6)$$

Next, we integrate the divergence of the axial-vector current by parts, throwing out the surface term, and use the following commutation relation:

$$\delta(x_0-y_0)[V_{3\nu}(y),A_0{}^-(x)] = -\delta^4(x-y)A_\nu{}^-(y)+\text{S.T.} \quad (7)$$

When the Schwinger terms (S.T.) exist, they will not contribute to the zero-momentum limit and henceforth are ignored. The resulting equation is

$$\langle\pi^+(k)\rho^0(q)|A_1^+(p)\rangle = -\frac{(k^2+m_\pi^2)(q^2+m_\rho^2)g_\rho}{m_\pi^2 f_\pi m_\rho^2}\epsilon_\nu{}^\rho ik_\mu \times\int d^4x d^4y\, f_k{}^*(x)f_q{}^*(y)\langle 0|T(V_{3\nu}(y),A_\mu{}^-(x))|A_1{}^+(p)\rangle - \frac{(k^2+m_\pi^2)(q^2+m_\rho^2)g_\rho(2\pi)^4}{m_\pi^2 f_\pi m_\rho^2[(2\pi)^6 4q_0\omega]^{1/2}}\delta^4(p-q-k)\epsilon_\nu{}^\rho \times\langle 0|A_\nu{}^-(0)|A_1{}^+(p)\rangle. \quad (8)$$

We next take $A_1{}^+$ meson out of the state vector and take the limit $p_\mu,\, q_\mu,\, k_\mu\to 0$. The result is, using the definition of $g_{A\rho\pi}$ given by Eq. (3),

$$g_{A\rho\pi}(0,0,0) = -(ig_\rho m_A{}^2/f_\pi)\tilde{\alpha}, \quad (9)$$

where the Fourier transform of the relevant two-point function is

$$\tilde{\Delta}_{\mu\nu}{}^A(p) = i\int d^4x\, e^{ip\cdot x}\langle 0|T(A_\nu{}^-(0),A_{1\nu}{}^+(x))|0\rangle\,; \quad (10)$$

$A_{1\nu}{}^+(x)$ is the $A_1{}^+$ field. The spectral representation for $\tilde{\Delta}_{\mu\nu}{}^A$ is

$$\tilde{\Delta}_{\mu\nu}{}^A(p) = \frac{\delta_{\mu\nu}}{(2\pi)}\int\frac{\tilde{\rho}_\pi{}^{(1)}(\mu^2)}{\mu^2+p^2-i\epsilon}d\mu^2 + \frac{p_\mu p_\nu}{(2\pi)}\int\frac{\tilde{\rho}_\pi{}^{(1)}(\mu^2)}{\mu^2(\mu^2+p^2-i\epsilon)}d\mu^2 + \frac{p_\mu p_\nu}{(2\pi)}\int\frac{\tilde{\rho}_\pi{}^{(0)}(\mu^2)}{\mu^2+p^2-i\epsilon}d\mu^2. \quad (11)$$

The definition of $\tilde{\alpha}$ is

$$\tilde{\alpha} = \frac{1}{2\pi}\int\frac{\tilde{\rho}_\pi{}^{(1)}(\mu^2)}{\mu^2}d\mu^2. \quad (12)$$

The next step is to relate $\tilde{\rho}_\pi{}^{(1)}(\mu^2)$ to the spectral func-

[8] T. D. Lee and B. Zumino, Phys. Rev. **163**, 1667 (1967).

tion $\rho_\pi^{(1)}(\mu^2)$, which occurs in the first Weinberg sum rule[5] which is

$$\frac{1}{2\pi}\int \frac{\rho_\pi^{(1)}(\mu^2)}{\mu^2}d\mu^2+\frac{1}{2\pi}\int \rho_\pi^{(0)}(\mu^2)d\mu^2=\frac{2m_\rho^2}{g_\rho^2}. \quad (13)$$

Field-current identities[8] can be used to relate $A_{1\mu}^+$ to A_μ^+.

$$A_{1\mu}^+(x)=-\frac{g_\rho}{\sqrt{2}m_\rho^2}A_\mu^+(x)\,, \quad (14)$$

therefore,

$$\tilde{a}\delta_{\mu\nu}=-(g_\rho/m_\rho^2\sqrt{2})\Delta_{\mu\nu}^A(0)\,, \quad (15)$$

where

$$\Delta_{\mu\nu}^A(p)=\frac{\delta_{\mu\nu}}{2\pi}\int\frac{\rho_\pi^{(1)}(\mu^2)}{\mu^2+p^2-i\epsilon}d\mu^2 +\frac{p_\mu p_\nu}{2\pi}\int\frac{\rho_\pi^{(1)}(\mu^2)}{\mu^2(\mu^2+p^2-i\epsilon)}d\mu^2 +\frac{p_\mu p_\nu}{2\pi}\int\frac{\rho_\pi^{(0)}(\mu^2)}{p^2+\mu^2-i\epsilon}d\mu^2. \quad (16)$$

Thus, Eq. (9) becomes

$$g_{A\rho\pi}(0,0,0)=\frac{ig_\rho^2 m_A^2}{\sqrt{2}m_\rho^2 f_\pi}\frac{1}{2\pi}\int\frac{\rho_\pi^{(1)}(\mu^2)}{\mu^2}d\mu^2. \quad (17)$$

B. Pole-Dominance Extrapolation for $A_1\rho\pi$

For this method of extrapolation, we first reduce the amplitude to the Fourier transform of a three-point function

$$\langle\pi^+(k)\rho^0(q)|A_1^+(p)\rangle=-\frac{i(k^2+m_\pi^2)(q^2+m_\rho^2)(p^2+m_A^2)}{m_\pi^2 f_\pi m_\rho^2(2\pi)^{3/2}(8p_0\omega q_0)^{1/2}} \times g_\rho\epsilon_\nu{}^\rho W_\nu(k,p,q)(2\pi)^4\delta^4(p-q-k)\,, \quad (18)$$

where

$$W_\nu(k,p,q)=\epsilon_\alpha{}^A\int d^4x\, e^{-iq\cdot x-ik\cdot y} \times\langle 0|T(\partial_\mu A_\mu^-(y),V_{3\nu}(x),A_1^+(0))|0\rangle. \quad (19)$$

We may now write W_ν in terms of the form factors $g_{A\rho\pi}$ and $h_{A\rho\pi}$. Comparing Eq. (18) and Eq. (3), we obtain

$$\frac{(k^2+m_\pi^2)(q^2+m_\rho^2)(p^2+m_A^2)}{m_\pi^2 f_\pi m_\rho^2}g_\rho W_\nu(k,p,q) =\epsilon_\nu{}^A g_{A\rho\pi}(-m_A^2,-m_\rho^2,-m_\pi^2)+(\epsilon^A\cdot q) \times p_\nu h_{A\rho\pi}(-m_A^2,-m_\rho^2,-m_\pi^2). \quad (20)$$

We now use the following identity:

$$\partial/\partial x_\nu\langle 0|T(\partial_\mu A_\mu^-(y),V_{3\nu}(x),A_{1\alpha}^+(0))|0\rangle =\langle 0|T(\delta(x_0-y_0)[V_{30}(x),\partial_\mu A_\mu^-(y)],A_{1\alpha}^+(0))|0\rangle +\langle 0|T(\delta(x_0)[V_{30}(x),A_{1\alpha}^+(0)],\partial_\mu A_\mu^-(y))|0\rangle. \quad (21)$$

The relevant commutators are

$$\delta(x_0-y_0)[V_{30}(x),\partial_\mu A_\mu^-(y)]=-\partial_\mu A_\mu^-(y)\delta^4(x-y)\,, \quad (22)$$

$$\delta(x_0)[V_{30}(x),A_{1\alpha}^+(0)]=A_{1\alpha}^+(x)\delta^4(x)\,. \quad (23)$$

These relations follow from the fact that $\int d^3x\, V_{30}(\mathbf{x},0)$ is the isospin generator. Therefore, Eq. (21) becomes

$$\partial/\partial x_\nu\langle 0|T(\partial_\mu A_\mu^-(y),V_{3\nu}(x),A_{1\alpha}^+(0))|0\rangle =\langle 0|T(A_{1\alpha}^+(x),\partial_\mu A_\mu^-(y))|0\rangle\delta^4(x) -\langle 0|T(A_{1\alpha}^+(0),\partial_\mu A_\mu^-(x))|0\rangle\delta^4(x-y). \quad (24)$$

This implies the following equation:

$$iq_\nu W_\nu=\epsilon_\alpha{}^A\int d^4y\, e^{-ik\cdot y}\langle 0|T(A_{1\alpha}^+(0),\partial_\mu A_\mu^-(y))|0\rangle -\epsilon_\alpha{}^A\int d^4y\, e^{-ip\cdot y}\langle 0|T(A_{1\alpha}^+(0),\partial_\mu A_\mu^-(y))|0\rangle. \quad (25)$$

We now use the following spectral representation for the relevant time ordered products in Eq. (25):

$$i\int d^4y\, e^{-ip\cdot y}\langle 0|T(A_{1\alpha}^+(0),\partial_\mu A_\mu^-(y))|0\rangle =\frac{ig_\rho}{\sqrt{2}m_\rho^2}\frac{p_\alpha}{2\pi}\int d\mu^2\frac{\mu^2\rho_\pi^{(0)}(\mu^2)}{\mu^2+p^2-i\epsilon}. \quad (26)$$

Equation (26) can be derived from Eqs. (16) and (14) by differentiation in a straightforward manner. A derivation is presented in the Appendix. Equation (25) now takes the form

$$iq_\nu W_\nu=\frac{g_\rho\epsilon_\alpha{}^A}{\sqrt{2}m_\rho^2}\left(\frac{k_\alpha}{2\pi}\int d\mu^2\frac{\mu^2\rho_\pi^{(0)}(\mu^2)}{\mu^2+k^2-i\epsilon} -\frac{p_\alpha}{2\pi}\int d\mu^2\frac{\mu^2\rho_\pi^{(0)}(\mu^2)}{\mu^2+p^2-i\epsilon}\right). \quad (27)$$

Expanding the spectral denominators, we obtain

$$iq_\nu W_\nu=-\frac{g_\rho(\epsilon^A\cdot q)}{\sqrt{2}m_\rho^2}\left(\frac{1}{2\pi}\int d\mu^2\rho_\pi^{(0)}(\mu^2)+O(k^2,p^2) +\cdots\text{higher powers of } k^2,\, p^2\right). \quad (28)$$

We may now take the limit p_μ, q_μ, $k_\mu\to 0$. The result is

$$g_{A\rho\pi}'(0,0,0)=\frac{ig_\rho^2}{\sqrt{2}m_\rho^2}\frac{m_A^2}{f_\pi}\frac{1}{2\pi}\int d\mu^2\rho_\pi^{(0)}(\mu^2). \quad (29)$$

The prime denotes the fact that a different limiting procedure has been used.

Finally, we equate $g_{A\rho\pi}(0,0,0)$ and $g_{A\rho\pi}'(0,0,0)$ as dictated by our initial dynamical assumption. This leads to the following sum rule[9]:

$$\int \frac{\rho_\pi^{(1)}(\mu^2)}{\mu^2} d\mu^2 = \int \rho_\pi^{(0)}(\mu^2) d\mu^2. \tag{30}$$

If we combine this sum rule with Eq. (13) and approximate $\rho_\pi^{(0)}(\mu^2)$ by the pion pole, we obtain the KSRF relationship, Eq. (1). Equation (30) gives the KSRF relationship dynamical content.

III. ZERO-MASS EXTRAPOLATIONS FOR THE $\rho\pi\pi$ SYSTEM

Calculations for the $\rho\pi\pi$ system, using procedures similar to those discussed in Sec. II, have essentially been carried out.[10] Thus, we will only briefly review the steps involved and state the results. To perform the PCAC extrapolation, the pions are taken out of the state vector, partial integration is performed, and then the current commutation relations are used. Finally, the ρ meson is taken out of the state vector. When all the masses are taken to zero, the following result is obtained[10]:

$$f_{\rho\pi\pi}(0,0,0) = m_\rho^2 / f_\pi^2 g_\rho. \tag{31}$$

The Weinberg sum rule for the vector spectral function is[5]

$$\frac{1}{2\pi} \int \frac{\rho_V^{(1)}(\mu^2)}{\pi^2} d\mu^2 = \frac{m_\rho^2}{g_\rho^2}, \tag{32}$$

where $\rho_V^{(1)}(\mu^2)$ for conserved currents is defined by

$$i\int d^4x\, e^{ip\cdot x}\langle 0|(V_\mu(0)V_\nu(x))|0\rangle = \frac{\delta_{\mu\nu}}{2\pi}\int \frac{\rho_V^{(1)}(\mu^2)}{\mu^2+p^2-i\epsilon}d\mu^2 + \frac{p_\mu p_\nu}{2\pi}\int \frac{\rho_V^{(1)}(\mu^2)}{\mu^2(\mu^2+p^2-i\epsilon)}d\mu^2 \tag{33}$$

and where $f_{\rho\pi\pi}$ is defined by

$$\langle \pi^+(k^+)\pi^-(k^-)|\rho^0(q)\rangle = -\frac{i(2\pi)^4\delta^4(q-k^+-k^-)}{[8q_0\omega^+\omega^-(2\pi)^9]^{1/2}} \times \epsilon_\alpha^\rho(k^- - k^+)_\alpha f_{\rho\pi\pi}(-m_\rho^2, -m_\pi^2, -m_\pi^2). \tag{34}$$

To extrapolate by using pole dominance, the decay amplitude is first reduced to the Fourier transform of a three-point function. Then, using the field-current identity, Eq. (6), this vertex function is related to the pion electromagnetic form factor. The pion electromagnetic form factor is then taken to the zero-mass limit by the use of the Ward identity. The result is[10]

$$f_{\rho\pi\pi}'(0,0,0) = g_\rho\left[1 + \frac{m_\pi^4}{2\pi}\int \frac{\sigma_\pi'(\mu^2)}{\mu^4}d\mu^2\right]. \tag{35}$$

Equating $f_{\rho\pi\pi}'(0,0,0)$ and $f_{\rho\pi\pi}(0,0,0)$, the following equation is obtained:

$$f_\pi^2\left(1 + \frac{m_\pi^4}{2\pi}\int \frac{\sigma_\pi'(\mu^2)}{\mu^4}d\mu^2\right) = \frac{m_\rho^2}{g_\rho^2}. \tag{36}$$

Using Eqs. (5), (13), and (32), it is a simple matter to show that Eq. (36) is equivalent to Eq. (30). We have demonstrated that our new sum rule is equivalent to the equality of zero-mass extrapolations for the $\rho\pi\pi$ system as well as the $A_1\rho\pi$ system. Therefore, our dynamical assumption is consistent in the context of the vector and axial-vector decays which are complicated to treat by more conventional approaches.[3,11,12] We now proceed to develop an experimental test of Eq. (30).

IV. TEST FOR EQ. (30)

To test this sum rule, we first show that the pion-pole dominates $\int \rho_\pi^{(0)}(\mu^2)d\mu^2$. Continuum contributions will be estimated to be smaller than the pole term by a factor of 10^{-5}.

A. Continuum Contributions to the Pseudoscalar Spectral Integral

We use the form of the pseudoscalar spectral integral which involves an integral over the pion spectral function, i.e., $\int[\sigma_\pi'(\mu^2)/\mu^4]d\mu^2$ which occurs in Eq. (36):

$$\sigma_\pi'(p^2) = (2\pi)^4 \sum_{n>1} \delta^4(p-p_n)|\langle 0|\phi_\pi(0)|n\rangle|^2. \tag{37}$$

Because of the μ^{-4} factor in the spectral integral, we expect the low-lying states to yield the most important contribution. We shall consider only the $\rho\pi$ contribution to Eq. (37). This contribution involves the matrix element, $\langle\pi(k)\rho(q)|\phi_\pi(0)|0\rangle$. This element may be related to the pion electromagnetic form factor corresponding to an off-shell pion of mass $-p^2$ interacting with a virtual photon with a four-momentum of q_μ and becoming an on-shell meson. Using standard LSZ reduction techniques and the translational invariance of the Wightman functions, the relationship is

$$\langle\pi(k)\rho(q)|\phi_\pi(0)|0\rangle = -\frac{(q^2+m_\rho^2)}{m_\rho^2(p^2+m_\pi^2)} \times g_\rho 2(\epsilon^\rho\cdot p)F_{\pi\pi}(q^2, -m_\pi^2, p^2), \tag{38}$$

where $q^2 = -m_\rho^2$ and the field-current identity, Eq. (6), has been used. We next neglect the off-shell variation of $F_{\pi\pi}$ and assume ρ-meson dominance. This is equivalent to the approximation

$$\frac{(q^2+m_\rho^2)}{m_\rho^2}F_{\pi\pi}(q^2, -m_\pi^2, p^2) \approx 1 \tag{39}$$

for $q^2 = -m_\rho^2$.

[9] R. J. Oakes, Phys. Rev. Letters **20**, 513 (1968).
[10] J. Dooher, Phys. Rev. **163**, 1852 (1967).
[11] D. A. Geffen, Phys. Rev. Letters **19**, 770 (1967).
[12] S. G. Brown and G. B. West, Phys. Rev. **168**, 1605 (1968).

Using Eq. (39) in Eq. (38) and applying this to Eq. (37), the contribution to $\sigma_\pi'(p^2)$ may be calculated. Summing over the vector-meson polarizations and integrating over the intermediate-state phase space, we obtain

$$\sigma_\pi'(\mu^2)=\frac{g_\rho^2\lambda^{3/2}(\mu^2,m_\rho^2,m_\pi^2)}{8\pi(\mu^2-m_\pi^2)^2\mu^2}, \tag{40}$$

where

$$\lambda(x,y,z)=(x-y-z)^2-4yz. \tag{41}$$

We may now evaluate the continuum integral in the left-hand side of Eq. (36). The result is

$$\frac{m_\pi^4}{2\pi}\int_{(3m_\pi)^2}^{\infty}\frac{\sigma_\pi(\mu^2)}{\mu^4}d\mu^2\approx\left(\frac{g_\rho^2}{4\pi}\right)\frac{(m_\pi/m_\rho)^4}{48\pi}\times[1+O((m_\pi/m_\rho)^2)]<10^{-5}. \tag{42}$$

Of course, there are other low-mass contributions which may be important such as the S-wave 3π continuum. We therefore consider Eq. (42) as only an indication that the μ^{-4} factor in Eq. (36) depresses the nonpole contributions. Equation (30) is now approximated by

$$f_\pi^2=\frac{1}{2\pi}\int\frac{\rho_V^{(1)}(\mu^2)}{\mu^2}d\mu^2. \tag{43}$$

B. Sum Rule for the Total Cross Section for the Hadronic Yield of Electron-Positron Annihilation

Such a sum rule has previously been derived and we will only review the steps involved and state the result.[7]

The total cross section for $e^+ + e^- \to$ hardons may be related to the spectral function for the Fourier transform of $\langle 0|T(j_\mu^{\text{em}}(0),\ j_\nu^{\text{em}}(x))|0\rangle$. If the field-current identities hold independent of the value of the unrenormalized vector meson mass, then the time-ordered product of the electromagnetic currents has the same spectral representation as the vector-meson propagator [see Eq. (33)]. However, if the field-current identities hold only in the limit of infinite unrenormalized mass, a spectral representation in the form of Eq. (33) will not exist. The relationship between the isovector part of $j_\mu^{\text{em}}(x)$ and the ρ-meson field in this latter situation is

$$(j_\mu^{\text{em}})_{I=1}=-\frac{m_\rho^2}{g_\rho}\rho_\mu+\frac{m_\rho^2}{m_0^2 g_\rho}\Box\rho_\mu. \tag{44}$$

When Eq. (44) is used in the Fourier transform of $\langle 0|T(j_\mu^{\text{em}}(0),j_\nu^{\text{em}}(x))|0\rangle$, the presence $\Box\rho_\mu$ as a factor in some terms will correspond to a factor of μ^2 in the spectral representation of these terms. Spectral integrals such as occur in Eq. (33) with these extra factors of μ^2 will not exist in general, and therefore, a spectral representation such as Eq. (33) will not exist even in the $m_0=\infty$ limit.[13] Instead, a spectral representation of the form

$$i\int d^4x\, e^{iq\cdot x}\langle 0|T(j_\mu^{\text{em}}(0)j_\nu^{\text{em}}(x))|0\rangle=(q^2\delta_{\mu\nu}-q_\mu q_\nu)\frac{1}{2\pi}\int\frac{\sigma^{(1)}(\mu^2)}{\mu^2+q^2-i\epsilon}d\mu^2 \tag{45}$$

is needed. The $q_\mu\to 0$ limit of Eq. (45) is zero. Also, none of the spin-1 integrals in the Weinberg sum rules would exist. Therefore, the results of this paper are valid only in the context of field-current identities independent of m_0. The spectral representation that we need then is

$$i\int d^4x e^{iq\cdot x}\langle 0|T(j_\mu^{\text{em}}(0)j_\nu^{\text{em}}(x))|0\rangle=\frac{\delta_{\mu\nu}}{2\pi}\int\frac{\rho^{(1)}(\mu^2)}{\mu^2+q^2-i\epsilon}d\mu^2+\frac{q_\mu q_\nu}{2\pi}\int\frac{\rho^{(1)}(\mu^2)}{\mu^2(\mu^2+q^2-i\epsilon)}d\mu^2. \tag{46}$$

Since the cross section for $e^+ + e^- \to$ hadrons involves $\sum_n \delta^4(q_n-p_+-p_-)|\langle 0|j_\mu^{\text{em}}(0)|n\rangle|^2$, it may be related to $\rho^{(1)}(q^2)$. The result is[7]

$$\sigma_T=8\pi^2(\alpha^2\rho^{(1)}(q^2)/q^4),\quad \alpha=e^2/4\pi. \tag{47}$$

Now, $\rho^{(1)}(q^2)$ consists of a contribution from the isovector current and the isoscalar current. The first Weinberg sum rule extended to the $SU(3)\times SU(3)$ current commutators relate spectral integrals of these contributions.

The relationship between the isovector and isoscalar spectral integrals can be derived from the equality of Schwinger terms in $[V_{30}(\mathbf{x},0),\ V_{3i}(\mathbf{y},0)]$ and $[V_{80}(\mathbf{x},0),\ V_{8i}(\mathbf{y},0)]$, and the following form for j_μ^{em}:

$$j_\mu^{\text{em}}(x)=V_{3\mu}(x)+\frac{1}{\sqrt{3}}V_{8\mu}(x). \tag{48}$$

It follows immediately that

$$\int\frac{\rho_S^{(1)}(\mu^2)}{\mu^2}d\mu^2=\frac{1}{3}\int\frac{\rho_V^{(1)}(\mu^2)}{\mu^2}d\mu. \tag{49}$$

However, Eqs. (43) and (49) imply that

$$\frac{1}{2\pi}\int\frac{\rho^{(1)}(\mu^2)}{\mu^2}d\mu^2=\tfrac{4}{3}f_\pi^2. \tag{50}$$

Therefore, using Eqs. (47) and (50), we obtain the following sum rule for $\sigma_T(E)$:

$$\frac{32}{3}\pi^3\alpha^2 f_\pi^2=\int_{2m_\pi}^{\infty}E^3\sigma_T(E)dE. \tag{51}$$

The c.m. energy of the electron-position pair is E.

[13] I wish to thank T. D. Lee for a discussion of this point.

Equation (51) provides a test of the sum rule, Eq. (30), and therefore, of the equality of the zero-mass limits for the vector-meson and axial-vector-meson decay amplitudes. To test Eq. (51), it is not necessary to have an accurate value of the ρ-meson parameters. In previously discussed methods of measuring g_ρ, the leptonic decays of the ρ meson were emphasized.[14] However, g_ρ as used in this paper is defined at zero-momentum transfer whereas the decays involve g_ρ on the mass shell. The difference is a factor of the ρ-meson renormalization constant,[15] $Z_3{}^\rho$. The results correspond only if $Z_3{}^\rho=1$, which is equivalent to pole dominance. What is needed to test Eq. (51) is an accurate plot of $\sigma_T(E)$ versus E over a range of energies from threshold to a value of E large enough to include most of the contributions to $\int E^3\sigma_T(E)dE$. If there is not a large number of high-mass resonances, ρ', ρ'', etc., with the quantum numbers of the ρ, then this cutoff energy is probably of the order of several ρ-meson masses. Saturation of Eq. (43) by a ρ and ρ' resonance yields

$$f_\pi{}^2=\frac{m_\rho{}^2}{g_{\rho\pi\pi}{}^2}+\frac{m_{\rho'}{}^2}{g_{\rho'\pi\pi}{}^2}. \tag{52}$$

Equation (52) has been derived previously by more conventional means by Moffat.[16] At present the data on the 2π contribution to σ_T are inconclusive as far as checking the validity of Eq. (51). Also, the existing data do not include contributions from a possible ρ'. Therefore, it is not possible to check Eq. (52). However, it will be important in the future to see whether more accurate data on $e^+ + e^- \to$ hardons verify Eq. (51).

V. INVESTIGATION OF THE ϕKK SYSTEM

Applying the techniques described in Sec. III to the amplitude for $\phi\to K^+ + K^-$, we may derive the analog of Eqs. (30) and (36) for the strangeness-changing axial-vector-current. Since this has been done in a previous paper, we just state the result

$$-\frac{3}{4}\frac{m_\phi{}^2}{g_Y{}^2}\frac{\cos\theta_Y\cos(\theta_Y-\theta_N)}{\cos\theta_N}=f_K{}^2\left[1+\frac{m_K{}^4}{2\pi}\int\frac{\sigma_K'(\mu^2)}{\mu^4}d\mu^2\right]. \tag{53}$$

In Eq. (53), g_Y is the isoscalar coupling constant, θ_Y and θ_N are the two ω-ϕ mixing angles, f_K is the kaon decay constant, and $\sigma_K'(\mu^2)$ is the continuum part of the kaon propagator spectral function. Application of the techniques in Sec. II to the $K_A\rho K$ system, where K_A is the chiral partner of the K^* meson, yields the following sum rule:

$$\int\frac{\rho_K{}^{(1)}(\mu^2)}{\mu^2}d\mu^2=\int\rho_K{}^{(0)}(\mu^2)d\mu^2, \tag{54}$$

where the spectral functions are those that occur in the K-type axial-vector current. Equation (54) is the extension of Eq. (30) to the strangeness-changing axial-vector current. Equation (53) is the extension of Eq. (36) to the kaon spectral integral. In the next section, we discuss the equivalence between Eqs. (54) and (53) and discuss the application of all these sum rules when they are used in conjunction with the first Weinberg sum rules.

VI. APPLICATIONS OF THE SPECTRAL SUM RULES

A. Equivalence Eqs. (53) and (54)

To prove the equivalence of Eqs. (53) and (54), we must first examine the derivation of Eq. (53) carefully. Following Kroll, Lee, and Zumino we treat ω-ϕ mixing with a matrix formalism.[15] Applying the techniques discussed in Sec. III to the ϕKK system,[10] we obtain the following result for the PCAC extrapolation of $f_{\phi KK}(-m_\phi{}^2,-m_K{}^2,-m_K{}^2)$ which is defined analogously to $f_{\rho\pi\pi}(-m_\rho{}^2,-m_\pi{}^2,-m_\pi{}^2)$ [see Eq. (34)]:

$$f_{\phi KK}(0,0,0)=\frac{3m_\phi{}^2}{4f_K{}^2}\left[g^{-1}M^2\int\frac{d\mu^2}{\mu^2}\sigma_{\phi\omega}(\mu^2)\right]_{11}. \tag{55}$$

The relevant matrices in Eq. (55) are defined below[15]:

$$g=T^{-1}g_D, \tag{56}$$

$$T=\begin{pmatrix}\cos\theta_Y & -\sin\theta_Y\\ \sin\theta_N & \cos\theta_N\end{pmatrix}, \tag{57}$$

$$g_D=\begin{pmatrix}g_D & 0\\ 0 & g_N\end{pmatrix}, \tag{58}$$

$$M^2=\begin{pmatrix}m_\phi{}^2 & 0\\ 0 & m_\omega{}^2\end{pmatrix}. \tag{59}$$

In Eq. (58), g_N is the coupling of the baryon current. The spectral density $\sigma_{\phi\omega}(\mu^2)$ in Eq. (55) is a matrix which occurs in the Källén-Lehmann representation of the time-ordered product of the renormalized ϕ and ω fields.

For the pole-dominance extrapolation, we obtain

$$f_{\phi KK}'(0,0,0)=m_\phi{}^2(M^{-2}g)_{11}\frac{m_K{}^4}{2\pi}\int\frac{\sigma_K(\mu^2)}{\mu^4}d\mu^2 \tag{60}$$

Equation (6) is equivalent to Eq. (16) in Ref. 10. However, Eq. (55) reduces to its equivalent in Ref. 10 [see Eqs. (15) and (17) in Ref. 10] only when the pole

[14] J. J. Sakurai, Phys. Rev. Letters **17**, 1021 (1966).
[15] N. M. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. **157**, 1376 (1967).
[16] J. W. Moffat, Phys. Rev. Letters **20**, 620 (1968); **20**, 977 (E) (1968).

approximation is used for the propagator in the resonance region $-m_\phi^2 \leq q^2 \leq -m_\omega^2$. Therefore, Eq. (53) is an approximate equation. The exact equation is obtained by equating $f_{\phi KK}(0,0,0)$ and $f_{\phi KK}'(0,0,0)$, which occur in Eqs. (55) and (60), respectively. Before proceeding, however, we must introduce the Weinberg sum rules for the ω-ϕ system. Using the field-current identity for the ω-ϕ system, which reads[15]

$$[j_\mu^{\mathrm{em}}(x)]_{I=0} = -\tfrac{1}{2}(g^{-1}M^2\psi_\mu)_1, \tag{61}$$

$$\psi_\mu = \begin{pmatrix} \phi_\mu \\ \omega_\mu \end{pmatrix}, \tag{62}$$

we obtain the following form for the relevant Weinberg sum rule:

$$\frac{4}{3}\frac{m_\rho^2}{g_\rho^2} = \left[g^{-1}M^2 \int \frac{\sigma_{\phi\omega}(\mu^2)}{\mu^2} d\mu^2 M^2 g_T^{-1} \right]_{11}, \tag{63}$$

where T denotes transpose.

Introducing a similar field-current identity for the baryon current,[15] and noting that the Weinberg sum rules are diagonal in the SU_3 quantum numbers, we find

$$\left[g^{-1}M^2 \int \frac{\sigma_{\phi\omega}(\mu^2)}{\mu^2} d\mu^2 M^2 g_T^{-1} \right]_{12} = \left[g^{-1}M^2 \int \frac{\sigma_{\phi\omega}(\mu^2)}{\mu^2} d\mu^2 M^2 g_T^{-1} \right]_{21} = 0. \tag{64}$$

Therefore, the matrix

$$g^{-1}M^2 \int \frac{\sigma_{\phi\omega}(\mu^2)}{\mu^2} d\mu^2 M^2 g_T^{-1}$$

is diagonal.

If we have two 2×2 matrices, A and B, such that

$$AB^{-1} = \Lambda, \tag{65}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}; \tag{66}$$

then it follows that

$$A_{11}/B_{11} = \lambda_1. \tag{67}$$

We now let

$$A = g^{-1}M^2 \int \frac{d\mu^2}{\mu^2} \sigma_{\phi\omega}(\mu^2), \tag{68}$$

$$B = g_T M^{-2}, \tag{69}$$

and note that since M^2 is diagonal,

$$B_{11} = (g_T M^{-2})_{11} = (M^{-2}g)_{11}. \tag{70}$$

The result of equating $f_{\phi KK}(0,0,0)$ and $f_{\phi KK}'(0,0,0)$ is

$$\frac{m_K^4 f_K^2}{2\pi} \int \frac{\sigma_K(\mu^2)}{\mu^4} d\mu^2 = \tfrac{3}{4} A_{11}/B_{11} = \frac{m_\rho^2}{g_\rho^2}. \tag{71}$$

Using the equivalent of Eq. (13) for K-type currents and PCAC for kaons, we see that Eq. (71) is equivalent to Eq. (54) and again the consistency of our initial dynamical assumption is demonstrated. It is interesting to note that if $\sigma_{\phi\omega}$ is dominated by ϕ and ω poles in the resonance region (local pole dominance), Eq. (64) implies

$$\tan\theta_Y = (m_\phi^2/m_\omega^2) \tan\theta_N. \tag{72}$$

This result, which corresponds to the current-mixing model,[15] has been derived previously from the Weinberg sum rules under the more restricted assumption of pole dominance of $\sigma_{\phi\omega}(\mu^2)$ for $0 \leq \mu^2 \leq \infty$ (global pole dominance).[17]

B. Vector-Meson and Axial-Vector-Meson Coupling Constants and f_K/f_π

We discussed an experimental test of the new sum rule for π type currents in Sec. IV. We now proceed to examine the experimental consequences of Eq. (54). If we are interested in approximate relations among vector-meson and axial-vector-meson coupling constants, we may use the assumption of global pole dominance though it is possible that this may not in general be a good approximation.

The relevant coupling for the pole-dominance approximation are defined as follows:

$$\langle 0 | V_{3\mu} | \rho^0 \rangle = [2q_0(2\pi)^3]^{-1/2} \epsilon_\mu^\rho (m_\rho^2/g_\rho), \tag{73}$$

$$\langle 0 | (A_\mu)_{S=0} | A_1 \rangle = (2q_0(2\pi)^3)^{-1/2} \epsilon_\mu^A (m_A^2/g_A), \tag{74}$$

$$\langle 0 | (V_\mu)_{S=1} | K^* \rangle = (2q_0(2\pi)^3)^{-1/2} \epsilon_\mu^{K*} (m_{K^*}^2/g_{K^*}), \tag{75}$$

$$\langle 0 | (A_\mu)_{S=1} | K_A \rangle = (2q_0(2\pi)^3)^{-1/2} \epsilon_\mu^{KA} (m_{K_A}^2/g_{K_A}). \tag{76}$$

These couplings may all be related using the following field-current identities in conjunction with Eqs. (6) and (14)[8]:

$$(V_\mu)_{S=1} = -\sqrt{2} m_\rho^2/g_\rho K_\mu^*, \tag{77}$$

$$(A_\mu)_{S=1} = -\sqrt{2} m_\rho^2/g_\rho K_{A\mu}. \tag{78}$$

We now apply global pole dominance to the Weinberg sum rules and ignore the scalar contribution to $(V_\mu)_{S=1}$

$$\frac{m_A^2}{g_A^2} + f_\pi^2 = \frac{2m_\rho^2}{g_\rho^2}, \tag{79}$$

$$\frac{m_{K_A}^2}{g_{K_A}^2} + f_K^2 = \frac{2m_\rho^2}{g_\rho^2}, \tag{80}$$

$$\frac{m_{K^*}^2}{g_{K^*}^2} = \frac{2m_\rho^2}{g_\rho^2}. \tag{81}$$

[17] R. J. Oakes and J. J. Sakurai, Phys. Rev. 19, 1266 (1967).

Equations (71) and (36) imply, neglecting the continuum, that[18]

$$f_\pi = f_K. \tag{82}$$

Actually, Eq. (82) provides a test of our sum rule, Eq. (54), if according to our estimates the continuum contributions to these pseudoscalar spectral integrals are negligible. The neglect of the 0^+ contributions to Eq. (81) imposes an additional limitation on the applicability of our sum rules. However, the derivation of Eq. (82) does not involve this assumption but depends only on the neglect of continuum contributions to (36) and (71). Therefore, Eq. (81) should hold, if our equal-limit hypothesis is valid, independently of any significant contribution from 0^+ states. At present, the experimental situation on f_K/f_π is not clear because of the existence of two Cabibbo angles θ_V and θ_A which do not necessarily have to be equal.[9]

The field-current identities imply that

$$\sqrt{2}(m_\rho{}^2/g_\rho) = m_A{}^2/g_A\,, \tag{83}$$

$$\sqrt{2}(m_\rho{}^2/g_\rho) = m_{K^*}{}^2/g_{K^*}\,, \tag{84}$$

$$\sqrt{2}(m_\rho{}^2/g_\rho) = m_{K_A}{}^2/g_{K_A}. \tag{85}$$

Combining Eqs. (83)–(85) with Eqs. (79)–(82), we obtain the following mass relationships:

$$m_{K_A} = m_A = \sqrt{2}m_\rho\,, \tag{86}$$

$$m_{K^*} = m_\rho\,, \tag{87}$$

and the following coupling-constant relationships:

$$g_A = g_{K_A} = \sqrt{2}g_\rho = \sqrt{2}(m_\rho/f_\pi)\,, \tag{88}$$

$$g_{K^*} = (1/\sqrt{2})g_\rho = (1/\sqrt{2})(m_\rho/f_\pi)\,. \tag{89}$$

The mass relationships, Eqs. (86) and (87), which are most easily accessible to experiment, are valid within about 10–20%. This can be considered reasonable agreement considering the approximate nature of global pole dominance.

VII. DISCUSSION OF RESULTS

We have shown that a sum rule, equivalent to the KSRF relation, can be derived under the assumption that a unique zero-mass limit exists for the $A_1\rho\pi$, $\rho\pi\pi$, ϕKK, and $K_A\rho K$ amplitudes. It is logical to ask for a motivation for this postulate. Indeed, it is logical to ask for a motivation of the assumption that certain extrapolations to zero mass are slowly varying. Since different methods of using the PCAC partial-integration procedure in multipion processes determine different extrapolations, it is not clear which extrapolation is the most slowly varying.

Of course, those extrapolations which yield interesting results are generally assumed to be slowly varying. However, other extrapolations may be slowly varying also. For example, one method of extrapolating the $\rho\pi\pi$ amplitude consists in first reducing the amplitude to the Fourier transform of a three-point function, applying the PCAC partial integration, and the current algebra, and then taking the zero-mass limit.[11] The result is equivalent to the Ward identity approach and the assumption of slow variation yields the equation

$$g_{\rho\pi\pi} = g_\rho \tag{90}$$

which is just global pole dominance of the pion electromagnetic form factor. Equation (90) is not considered to be very interesting at present. However, the KSRF relation, Eq. (1), which is the result of a different PCAC extrapolation, is considered to be a very interesting relation. Phenomenological theories are constructed to yield Eq. (1).[19] It may turn out that Eq. (1) is not a good approximation. In that case, the most slowly varying extrapolation would be that leading to Eq. (90), and therefore, Eq. (90) would be considered to be the PCAC and current-algebra result. We have taken the easy way out of this dilemma by replacing the assumption of slow variation of a particular well-chosen extrapolation by the assumption that the zero-mass limit is independent of the particular method of extrapolation used. We have shown that such an assumption is consistent when applied to several vector-meson and axial-vector-meson decay amplitudes. It remains to be seen if it is useful to generalize the equal-limit hypothesis to other processes.

It is not possible to derive such a sum rule for amplitudes like $K_AK^*\pi$ which do not involve a conserved current. This is because of the presence of scalar excitations in the strangeness-changing vector current which lead to additional terms when the Ward identity approach is applied. However, this new sum rule is not inconsistent with the Weinberg sum rules with scalar excitations included. Indeed, several calculations using the Weinberg sum rules ignore the scalar excitation of the strangeness-changing current and these calculations have yielded successful results.[17]

This procedure of defining a unique limit is really not new. For example, Sakurai's derivation of KSRF assumes that the zero-momentum limit for the πN elastic scattering amplitude obtained by current algebra is the same as that gotten by using perturbation theory and keeping only those diagrams which contribute to ρ meson exchange.[20] In a sense, we have just extended this concept to decay amplitudes such as $\rho\pi\pi$. Sakurai's limit using ρ exchange is now replaced by a limit which is taken using either perturbation theory for the three-point function or the Ward identity since they must yield equivalent results.

Of course, we may relax our restriction of equality of limits and assume only that the limits are approximately

[18] This result has also been derived using local pole dominance by Acharya and Aly. See R. Acharya and H. H. Aly, Phys. Rev. Letters **27B**, 166 (1968).

[19] Ken Kawarabayashi and Shinsaku Kitakado (to be published).

[20] J. J. Sakurai, Phys. Rev. Letters **17**, 522 (1966).

equal (assumption of slow variation). If we then assume that SU_3 is valid for zero-mass extrapolations of $\rho\pi\pi$, ϕKK, $K_A\rho K$, and $A_1\rho\pi$, we obtain the result

$$\int \frac{\rho_K^{(1)}(\mu^2)}{\mu^2}d\mu^2 = \int \frac{\rho_\pi^{(1)}(\mu^2)}{\mu^2}d\mu^2 = c\int \rho_\pi^{(0)}(\mu^2)d\mu^2 = c\int \rho_K^{(0)}(\mu^2)d\mu^2, \quad (91)$$

where c is close to 1. This result has been obtained by Oakes using a special model[9] for SU_3 breaking. The constant c may then be determined by the colliding-beam experiments by comparing the left and right sides of Eq. (51). Finally, we may say that it is clear that the principle of equality of zero-mass limits removes some ambiguity in the application of PCAC and current algebra and leads to a new sum rule with dynamical content. If experiments bear out the implications of this sum rule, particularly Eqs. (51) and (82), then the equality of zero-mass limits must be seriously considered in the construction of any Lagrangian theory.

APPENDIX: SPECTRAL REPRESENTATION FOR

$$i\int d^4x\, e^{iq\cdot x}\langle 0|T(A_\mu^+(0),\partial_\alpha A_\alpha^-(x))|0\rangle$$

We first integrate by parts using the following identity:

$$\frac{\partial}{\partial x_\alpha}[e^{iq\cdot x}\langle 0|T(A_\mu^+(0),A_\alpha^-(x))|0\rangle] = iq_\alpha e^{iq\cdot x}\langle 0|T(A_\mu^+(0),A_\alpha^-(x))|0\rangle + e^{iq\cdot x}\langle 0|T(A_\mu^+(0),\partial_\alpha A_\alpha^-(x))|0\rangle - (2im_\rho^2/g_\rho^2)e^{iq\cdot x}\delta_{n\mu}[\partial\delta^4(x)/\partial x_n]. \quad (A1)$$

In Eq. (A1), we have used the following commutation relations with the Schwinger term exhibited:

$$\delta(x_0)[A_0^-(x),A_\mu^+(0)] = -2\delta^4(x)V_{3\mu}(x) - \frac{2im_\rho^2}{g_\rho^2}\delta_{n\mu}\frac{\partial\delta^4(x)}{\partial x_n}. \quad (A2)$$

Using Eq. (A1), we find

$$i\int d^4x\, e^{iq\cdot x}\langle 0|T(A_\mu^+(0),\partial_\alpha A_\alpha^-(x))|0\rangle = -iq_\alpha\Delta_{\mu\alpha}{}^A(q) + \frac{2im_\rho^2}{g_\rho^2}q_n\delta_{n\mu}. \quad (A3)$$

To correctly evaluate the first term on the right side of Eq. (A3), we must include the Schwinger term

$$\Delta_{\mu\alpha}{}^A(q) = \frac{\delta_{\mu\alpha}}{2\pi}\int d\mu^2\frac{\rho_\pi^{(1)}(\mu^2)}{\mu^2+q^2-i\epsilon} + \frac{q_\mu q_\alpha}{2\pi}\int d\mu^2\frac{\rho_\pi^{(1)}(\mu^2)}{\mu^2(\mu^2+q^2-i\epsilon)} + \frac{q_\mu q_\alpha}{2\pi}\int d\mu^2\frac{\rho_\pi^{(0)}(\mu^2)}{\mu^2+q^2-i\epsilon} - 2\delta_{\mu 4}\delta_{\alpha 4}\frac{m_\rho^2}{g_\rho^2}. \quad (A4)$$

Using Eq. (A4), the result is

$$i\int d^4x e^{iq\cdot x}\langle 0|T(A_\mu^+(0),\partial_\alpha A_\alpha^-(x))|0\rangle = -\frac{iq_\mu}{2\pi}\int d\mu^2\frac{\rho_\pi^{(1)}(\mu^2)}{\mu^2+q^2-i\epsilon} - iq_\mu q^2\int d\mu^2\frac{\rho_\pi^{(1)}(\mu^2)}{\mu^2(\mu^2+q^2-i\epsilon)} - iq_\mu\frac{q^2}{2\pi}\int d\mu^2\frac{\rho_\pi^{(0)}(\mu^2)}{\mu^2+q^2-i\epsilon} + \frac{2im_\rho^2}{g_\rho^2}q_\mu. \quad (A5)$$

We may now combine the first two terms on the right side of Eq. (A5). We then replace the last term by the sum of spectral integrals in Eq. (13).

The final result is

$$i\int d^4x\, e^{iq\cdot x}\langle 0|T(A_\mu^+(0),\partial_\alpha A_\alpha^-(x))|0\rangle = \frac{iq_\mu}{2\pi}\int d\mu^2\frac{\mu^2\rho_\pi^{(0)}(\mu^2)}{\mu^2+q^2-i\epsilon}. \quad (A6)$$

This is the desired spectral representation.