Non-Lagrangian Models of Current Algebra*

KENNETH G. WILSON†
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
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An alternative is proposed to specific Lagrangian models of current algebra. In this alternative there are no explicit canonical fields, and operator products at the same point [say, $j_\mu(x)j^\mu(x)$] have no meaning. Instead, it is assumed that scale invariance is a broken symmetry of strong interactions, as proposed by Kastrup and Mack. Also, a generalization of equal-time commutators is assumed: Operator products at short distances have expansions involving local fields multiplying singular functions. It is assumed that the dominant fields are the $SU(3)\times SU(3)$ currents and the $SU(3)\times SU(3)$ multiplet containing the pion field. It is assumed that the pion field scales like a field of dimension Δ, where Δ is unspecified within the range $1\leq\Delta<4$; the value of Δ is a consequence of renormalization. These hypotheses imply several qualitative predictions: The second Weinberg sum rule does not hold for the difference of the K^* and axial-K^* propagators, even for exact $SU(2)\times SU(2)$; electromagnetic corrections require one subtraction proportional to the $I=1$, $I_z=0$ σ field; $\eta\to 3\pi$ and $\pi_0\to 2\gamma$ are allowed by current algebra. Octet dominance of nonleptonic weak processes can be understood, and a new form of superconvergence relation is deduced as a consequence. A generalization of the Bjorken limit is proposed.

I. INTRODUCTION

THERE are a number of problems in strong interactions which involve the short-distance behavior of the $SU(3)\times SU(3)$ currents but which cannot be solved by Gell-Mann's current algebra[1] alone. These problems include the convergence or divergence of Weinberg sum rules,[2] divergences in radiative corrections to strong interactions, the nature of the Bjorken limit,[3] etc. Various models have been proposed to handle these problems, such as the algebra of fields,[4] the quark model, or the σ model.[5] These models give conflicting answers to some of the problems mentioned. One therefore must consider what further alternatives to these models exist, and hence to get an idea of the range of answers possible to the problems listed.

This paper presents a framework in which one can discuss some alternatives to specific Lagrangian models. The present framework does not involve Lagrangians: There are no canonical fields in the formalism, and operator products at the same point, for example, the product $j_\mu(x)j^\mu(x)$ of two currents, have no meaning. To replace the Lagrangian methods of analyzing short-distance behavior, two hypotheses are proposed. The first is that the strong interactions become scale-invariant at short distances. This was proposed by Kastrup and Mac.[6] This means that scale invariance is a broken symmetry in the same sense as chiral $SU(3)\times SU(3)$. The other hypothesis is that there exist "operator-product expansions" for products of two (or more) local fields near the same point. For example, one can construct expansions for products such as $j_\mu(x)j_\nu(y)$ or $j_\mu(x)j_\nu(y)j_\pi(z)$ when y and z are near x. These expansions contain functions which are singular when $y=x$ or y is on the light cone through x. These expansions give a more detailed picture of the short-distance behavior of products than one gets if one only knows equal-time commutators. These expansions originated in detailed studies of renormalization in perturbation theory.[7] The importance of scale invariance for the analysis of short-distance behavior is apparent in the power-counting arguments of Dyson and in the relation between the renormalizability of an interaction and its dimension, pointed out by Umezawa *et al.*[8]

Scale invariance is sometimes thought of as a feature special to certain strictly Lagrangian theories. However, an analysis of the Thirring model[9] shows that scale invariance can persist in a theory where, for example, the canonical commutators have been destroyed by renormalization effects. (The Thirring model involves a spinor field in one space, one time dimension with a Fermi coupling.) While scale invariance persists, the scaling laws for particular fields change as the coupling constant changes. This will be assumed to hold for strong interactions also, so that the scaling laws for strongly interacting fields will be assumed to differ (because of renormalization effects) from free fields.

The hypotheses of this paper leave much to be determined; nevertheless, when combined in a simple way with current algebra, one can make a number of qualitative predictions. The applications considered in this

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[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, 63 (1964).
[2] S. Weinberg, Phys. Rev. Letters **18**, 507 (1967); S. Glashow, H. Schnitzer, and S. Weinberg, *ibid.* **19**, 139 (1967); T. Das, V. Mathur, and S. Okubo, *ibid.* **18**, 761 (1967).
[3] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).
[4] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967).
[5] M. Lévy, Nuovo Cimento **52A**, 23 (1967); M. Gell-Mann and M. Lévy, *ibid.* **16**, 705 (1960).
[6] G. Mack, Nucl. Phys. **B5**, 499 (1968), and references cited therein.
[7] J. Valatin, Proc. Roy. Soc. (London) **A222**, 93 (1954); **A222** 228 (1954); **A225**, 535 (1954); **A226**, 254 (1954); W. Zimmermann, Nuovo Cimento **10**, 597 (1958); K. Nishijima, Phys. Rev. **111**, 995 (1958); R. Haag, *ibid.* **112**, 668 (1958); R. Brandt, Ann. Phys. (N. Y.) **44**, 221 (1967); W. Zimmermann, Commun. Math. Phys. **6**, 161 (1967).
[8] S. Sakata, H. Umezawa, and S. Kamefuchi, Progr. Theoret. Phys. (Kyoto) **7**, 377 (1952); H. Umezawa, *ibid.* **7**, 551 (1952).
[9] See K. Johnson, Nuovo Cimento **20**, 773 (1961).

paper include the validity of Weinberg sum rules, divergences in radiative corrections to strong interactions, $\eta \to 3\pi$ and $\pi^0 \to 2\gamma$ decay, nonleptonic weak interactions, and the Bjorken limit.

This paper is only a summary of the ideas involved in the two hypotheses and a survey of their applications. Many details, some of considerable complexity, have been omitted.

Operator-product expansions are introduced in Sec. II. Scale invariance applied to operator-product expansions is explained in Sec. III. The scale-invariant part of strong interactions is discussed in Sec. IV. The effect of mass terms in the Lagrangian (the mass terms are to be treated as interaction Lagrangians in the sense of perturbation theory) is considered in Sec. V. A specific set of mass terms is proposed in Sec. VI. The applications are analyzed in Sec. VII. Section VIII contains final comments.

II. OPERATOR-PRODUCT EXPANSIONS

An equal-time commutator of two local fields $A(x)$ and $B(x)$ is expected to be of the general form

$$[A(x_0,\mathbf{x}),B(x_0,\mathbf{y})]=\sum_n D_n(\mathbf{x}-\mathbf{y})O_n(x)\,, \qquad (2.1)$$

where the $O_n(x)$ are a set of local fields at x [including the unit operator I which for the purposes of this paper is a local field $O_0(x)=I$; it is local because it commutes with the other local fields for spacelike separation]. The functions $D_n(\mathbf{x}-\mathbf{y})$ are δ functions or derivatives of δ functions.

The generalization proposed here is that an ordinary product $A(x)B(y)$ has an expansion when the four-vector y is near x, of the form

$$A(x)B(y)=\sum_n C_n(x-y)O_n(x)\,. \qquad (2.2)$$

Here the functions $C_n(x-y)$ depend on a four-vector, not a three-vector. Instead of being δ functions, they involve powers of $x-y$. They can have singularities on the light cone of the form $[(x-y)^2-i\epsilon(x_0-y_0)]^{-p}$, p being any real number (it need not be an integer). They also involve logarithms of $(x-y)^2$. The complete expansion in general involves an infinite number of local fields $O_n(x)$ but to any finite order in $x-y$ only a finite number of fields contribute. The expansion is valid in the weak sense: One must sandwich the product $A(x)B(y)$ between fixed final and initial states $\langle\alpha|$ and $|\beta\rangle$. The expansion is then valid for y sufficiently close to x.

These operator-product expansions exist for the free scalar and spinor field theories and for renormalized interacting fields to all orders in perturbation theory. In every case they are valid for any elementary or composite local fields: $A(x)$ and $B(y)$ can be elementary scalar or spinor fields or local currents or the stress-energy tensor or any local Wick product in a free-field theory. Similar expansions exist for T products or commutators at small distances or products of three or more fields close to the same point. (These statements will be demonstrated in a separate paper.) One can compute the equal-time commutator of A and B, given the operator-product expansion for $A(x)B(y)$ (see below); one can also compute equal-time commutators of any time derivatives of A and B.

There are several reasons for using operator-product expansions in place of equal-time commutators to describe the short-distance behavior of a field theory. One reason is that equal-time commutators can involve infinite constants, whereas the expansion coefficients $C_n(x-y)$ cannot. For example, the Schwinger term in the commutator of two quark currents contains a divergent constant.[10] In contrast, the functions $C_n(x-y)$ must be distributions in the four-vector $x-y$ since the operators $A(x)$ and $B(y)$ are, and a distribution cannot contain infinite constants.[11] A related result is the following: The Bjorken limit, formulated in terms of equal-time commutators, predicts that Fourier transforms of amplitudes, such as $\langle\alpha|TA(x)B(0)|\beta\rangle$, will behave as a power series in the transform variable q_0^{-1} when q_0 is large, $\mathbf{q}$ being held fixed.[3] With the more general operator-product expansion it is found that fractional powers of q_0 are also possible; they occur in the Thirring model in one space and one time dimension.[12] Other advantages of the operator-product expansions are: They are manifestly covariant, they exist for T products of operators, and one can give a simple discussion of symmetry-breaking effects to all orders.

One can relate the operator-product expansion of $A(x)B(y)$ to the equal-time commutator of A and B. Suppose $C_0(x-y)$ behaves as $[(x-y)^2-i\epsilon(x_0-y_0)]^{-p}$ for some power p. (If A and B are not Lorentz scalars, there would also be a polynomial in the components of $x-y$.) The $i\epsilon$ comes in because intermediate states have only positive energies and only intermediate states of very large energies (larger than the fixed initial- and final-state energies) contribute to the light-cone singularity. These high-energy states are exponentially damped if one gives x_0 a negative imaginary part. So $(x-y)^2$ becomes $(x_0-y_0-i\epsilon)^2-(\mathbf{x}-\mathbf{y})^2$. The commutator $[A(x),B(y)]$ for y near x (but *not* equal times) has an expansion

$$[A(x),B(y)]=\sum_n E_n(x-y)O_n(x)\,, \qquad (2.3)$$

where $E_0(x-y)$ is

$$E_0(z)=E[(-z^2+i\epsilon z_0)^{-p}-(-z^2-i\epsilon z_0)^{-p}] \qquad (2.4)$$

and E is a constant. The second term comes from the product $B(y)A(x)$: It makes $E_0(z)$ vanish for spacelike z. One can convert $E_0(z)$ into a sum of δ functions in $\mathbf{z}$, for any nonzero but small z_0. One uses the definition of a

[10] The calculation is similar to that of J. Schwinger, Phys. Rev. Letters **3**, 296 (1959).
[11] The Wightman axioms are assumed.
[12] This is a consequence of the analysis of Sec. VII plus known properties of the Thirring model; see K. Johnson, Nuovo Cimento **20**, 773 (1961).

δ function

$$\int_{\mathbf{z}} \delta^3(\mathbf{z})\rho(\mathbf{z}) = \rho(0)\,, \tag{2.5}$$

$$\int_{\mathbf{z}} [\nabla\delta^3(\mathbf{z})]\rho(\mathbf{z}) = -\nabla\rho(0)\,, \text{ etc.,} \tag{2.6}$$

where $\rho(\mathbf{z})$ is a differentiable function of $\mathbf{z}$, and $\int_{\mathbf{z}} \equiv \int d^3z$. One writes

$$\int_{\mathbf{z}} \rho(\mathbf{z})E_0(z) = \int_{\mathbf{z}} [\rho(0)+\mathbf{z}\cdot\nabla\rho(0)+\cdots]E_0(z)\,. \tag{2.7}$$

It makes sense to expand $\rho(\mathbf{z})$ in a Taylor series because $E_0(z)$ vanishes unless $|\mathbf{z}|\leq|z_0|$. One now sees that the function $E_0(z)$ is equivalent to a sum of δ functions:

$$E_0(z) = F_0(z_0)\delta^3(\mathbf{z}) + \mathbf{F}_1(z_0)\cdot\nabla\delta^3(\mathbf{z})+\cdots\,, \tag{2.8}$$

where

$$F_0(z_0) = \int_{\mathbf{z}} E_0(z_0,\mathbf{z})\,, \tag{2.9}$$

$$\mathbf{F}_1(z_0) = -\int_{\mathbf{z}} E_0(z_0,\mathbf{z})\mathbf{z}\,, \text{ etc.,} \tag{2.10}$$

The dependence of the function $F_0(z_0)$, etc., on z_0 is determined by dimensional analysis to be

$$F_0(z_0) = f_0 z_0^{-2p+3}\,, \tag{2.11}$$

$$\mathbf{F}_1(z_0) = \mathbf{f}_1 z_0^{-2p+4}\,, \text{ etc.,} \tag{2.12}$$

where f_0 and $\mathbf{f}_1$ are constants (proportional to E). Actually $\mathbf{f}_1$ vanishes because of rotational symmetry, but there will be tensor quantities f_{2ij}, etc., which do not vanish.

The equal-time commutator is obtained by letting $z_0 \to 0$. The coefficient of $\delta^3(\mathbf{x}-\mathbf{y})I$ in the commutator is $F_0(0)$, which is

$$\begin{aligned} &0\,, && \text{if } p<1.5\\ &f_0\,, && \text{if } p=1.5\\ &\infty\,, && \text{if } p>1.5\,. \end{aligned}$$

Similarly, the coefficient of $\nabla_i\nabla_j\delta^3(x-y)I$ is nonzero if $p\geqslant 2.5$ and infinite if $p>2.5$. This analysis is based on a particular definition of the equal-time commutator (the limit for $z_0\to 0$ of the unequal-time commutator). It may not apply to other definitions. It is clear that the operator-product expansions have greater flexibility in the form of the coefficient $C_n(x-y)$ than do the equal-time commutators unless one permits infinite coefficients in the equal-time commutators.

III. SCALE INVARIANCE

The nature of the singularities of the functions $C_n(x-y)$ is determined in known field theories (excluding finite-mass vector-meson theories[13]) by the exact and broken symmetries of the theory. The most crucial of these symmetries is broken scale invariance.[6,14] The free scalar and spinor field theories with zero mass are exactly scale-invariant. Mass terms and renormalizable interactions[15] break the symmetry but the ghost of scale invariance still governs the behavior of the singular functions.[16] Exact scale invariance means that the field theory is invariant to a one-parameter group of transformations $U(s)$. The local fields $O_n(x)$ transform as

$$U^\dagger(s)O_n(x)U(s) = s^{d(n)}O_n(sx)\,. \tag{3.1}$$

In free-field theories the constant $d(n)$ is the dimension of the field $O_n(x)$, that is, $O_n(x)$ has dimension $m^{d(n)}$ in mass units. For example, a free spinor field $\psi(x)$ is transformed to $s^{3/2}\psi(sx)$, the power of s being determined so that the canonical commutation rules are invariant. A free spinor field has dimension $m^{3/2}$, again because of the canonical commutation rules. It is important for the dimension of ψ that there is no dimensional constant in the canonical commutation rule. One can always change the dimension of ψ by multiplying it by a power of a mass m but this puts a dimensional constant into the commutation rule. Multiplying ψ by a constant does not change its transformation properties to $U(s)$.

In an exactly scale-invariant theory the behavior of the function $C_n(x-y)$ is determined except for a constant by scale invariance. Performing a scale transformation on Eq. (2.2), one has

$$s^{d_A+d_B}A(sx)B(sy) = \sum_n C_n(x-y)s^{d(n)}O_n(sx)\,. \tag{3.2}$$

Expanding the left-hand side,

$$s^{d_A+d_B}\sum_n C_n(sx-sy)O_n(sx) = \sum_n C_n(x-y)s^{d(n)}O_n(sx)\,. \tag{3.3}$$

If the fields $O_n(x)$ are linearly independent (this can always be arranged), one must have

$$C_n(sx-sy) = s^{-d_A-d_B+d(n)}C_n(x-y)\,. \tag{3.4}$$

This equation says that $C_n(x-y)$ must be homogeneous of order $-d_A-d_B+d(n)$ in $x-y$. The Lorentz transformation properties of $C_n(x-y)$ then determine the behavior of $C_n(x-y)$ completely except for one or more constants [one, if $C_n(x-y)$ must be a scalar]. In particular, the strength of the light-cone singularity is determined by the dimension $d_A+d_B-d(n)$. C_n can be singular only if $d_A+d_B\geqslant d(n)$ and becomes more singular the larger d_A+d_B is relative to $d(n)$.

In a free-field theory the fields $O_n(x)$ that occur in operator-product expansions are the free field itself, its

[13] The short-distance behavior of vector-meson theories is complicated by the longitudinal part of the vector-meson propagator. The analysis of this paper would have to be modified to take this into account; this the author has not done.

[14] Scale invariance in free-field theories is discussed (as part of the conformal group) by J. Wess, Nuovo Cimento **18**, 1086 (1960).

[15] Treated in perturbation theory.

[16] See Sec. V.

derivatives (of any order), and all possible local Wick products. For a free scalar field $\phi(x)$, examples of fields $O_n(x)$ are

$$\phi(x),\ \nabla_\mu\nabla_\nu\nabla_\pi\phi(x),\ :\phi^{29}(x):,\ \nabla_\mu:\phi^2(x)\nabla_\nu\phi(x)\nabla_\pi\phi(x):,\ \text{etc.}$$

It is usually desirable to use a linearly independent set of these fields, so that one would exclude $\nabla_\mu\nabla^\mu\phi(x)$ and $:\phi(x)\nabla_\mu\nabla^\mu\phi(x):$ and others which are fixed by the free-field equation. The set of fields $O_n(x)$ for a free-scalar-field theory can be ordered by dimension, starting with the unit operator (dimension zero) and the scalar field $\phi(x)$ itself (dimension 1). There are two fields of dimension 2 [$\nabla_\mu\phi(x)$ and $:\phi^2(x):$]. The number of fields multiplies rapidly as the dimension increases, but nevertheless there are only a finite number of linearly independent local fields of dimension D or less, for any finite bound D. This is true also of the free-spinor-field theory. The ordering by dimension is a useful concept because the functions $C_n(x-y)$ become less singular as the dimension $d(n)$ increases. $C_0(x-y)$ is the most singular function (if it does not vanish identically), and only a finite number of the $C_n(x-y)$ can be at all singular.

One can construct explicit operator-product expansions in a free-field theory. An illustration will suffice. Consider the product $:\phi^2(x):\ :\phi^2(y):$. It can be written by Wick's theorem:

$$:\phi^2(x):\ :\phi^2(y): = 2[D(x-y)]^2 I + 4D(x-y):\phi(x)\phi(y): + :\phi^2(x)\phi^2(y):, \quad (3.5)$$

where D is one of the free-field singular functions. It scales as $(x-y)^{-2}$. Expand the Wick products in a Taylor series in $y-x$, for example,

$$:\phi(x)\phi(y): = :\phi^2(x): + (y-x)_\mu :\phi(x)\nabla^\mu\phi(x): + \cdots, \quad (3.6)$$

and likewise for $:\phi^2(x)\phi^2(y):$. One then has an expansion of the desired form, except that the set of operators involved is not linearly independent [for example, $:\phi(x)\nabla_\mu\nabla^\mu\phi(x):$ is included]. The final step is to reduce the operators to a linearly independent set, which is formally straightforward but in practice rather complicated. The resulting expansion has all the properties discussed here.

In an exactly scale-invariant theory the singularities of the functions $C_n(x-y)$ are determined by pure symmetry considerations (scale invariance and Lorentz invariance), except for constants. If there are internal symmetries, some of these constants will be zero. The scale-invariance requirements override any other considerations; for example, one cannot demand that the equal-time commutators of all local fields be finite. If $A(x)$ and $B(y)$ are fields of high dimension, then their commutator will contain terrifyingly singular functions leading to an equal-time commutator with many derivatives of δ functions and many divergent constants.

The current commutators are a special case, where equal-time commutators must exist (apart from Schwinger terms). Let Q be the generator of an internal symmetry in an exactly scale-invariant theory. Then there will be fields $A(x)$ satisfying

$$[A(x),Q] = qA(x) \quad (3.7)$$

for some constant q. This equation is invariant to scale transformations only if Q is invariant:

$$U^\dagger(s)QU(s) = Q. \quad (3.8)$$

If Q is the space integral of a current $j_0(x)$, then $j_0(x)$ must transform as

$$U^\dagger(s)j_0(x)U(s) = s^3 j_0(sx). \quad (3.9)$$

Consider the unequal-time commutator of $j_\mu(y)$ with $A(x)$ which, for x near y, has an operator-product expansion

$$[A(x),j_\mu(y)] = \textstyle\sum_n K_{n\mu}(x-y)O_n(x). \quad (3.10)$$

Since j_μ is a conserved current, $K_{n\mu}(x-y)$ is conserved. This means that the integrals

$$k_n = \int_{\mathbf{z}} K_{n0}(z_0,\mathbf{z})$$

are independent of time. Hence they cannot diverge for $z_0 \to 0$. At equal times $(x_0 = y_0)$ one has

$$[A(x),j_0(y)] = \textstyle\sum_n k_n\delta^3(\mathbf{x}-\mathbf{y})O_n(x) + \text{ST}. \quad (3.11)$$

where the Schwinger terms (ST) involve derivatives of δ functions whose coefficients may be divergent. To be consistent with Eq. (3.7), one must have

$$qA(x) = \textstyle\sum_n k_n O_n(x). \quad (3.12)$$

Note that k_n can be a nonzero constant only if $K_{n0}(z)$ scales as z^{-3}, which means O_n has the same dimension as $A(x)$.

A similar analysis of the translation generator P_μ shows that P_μ has dimension 1 and the local stress-energy tensor has dimension 4.

IV. HADRON SKELETON THEORY

Field theories with exact scale invariance are not physically interesting, since they cannot have finite-mass particles. But one can hypothesize that there exists a scale-invariant theory which becomes the theory of strong interactions when one adds mass terms to the Lagrangian. This leads to the idea of broken scale invariance proposed by Kastrup and Mack.[6] This idea will now be explained in detail. In this section the scale-invariant theory underlying strong interactions will be discussed, and in Secs. V and VI the effect of mass terms will be considered.

It is assumed that the strong interactions contain some arbitrary fundamental parameters just as the mass and charge of the electron are fundamental parameters in electrodynamics. However, the greater complication of strong interactions means that the parameters of strong interactions are not physical masses and coupling constants; they show up explicitly only in the short-distance behavior of strong interactions. Implicitly, they determine all of strong interactions, but to calculate physical masses and coupling constants one has to solve the strong interactions, which is not possible at present. In physics these parameters have particular values, but the theory of strong interactions is assumed to be self-consistent for any values of the parameters. In particular, if all the parameters are zero, it will be assumed that all partial symmetries become exact. The theory with all free parameters set equal to zero will be called the "skeleton theory." It is assumed that the skeleton theory is a theory with all physical masses equal to zero, which is exactly scale-invariant, and exactly invariant to P, C, PCT,[17] $SU(3)\times SU(3)$, and baryon number. The quark model suggests there should also be an "axial baryon number" but this will not be considered here for the sake of expediency. Many of the complications of the finite-mass theory should be absent from the zero-mass theory: All finite-mass thresholds are gone, replaced by a continuum starting from mass zero. If there are Regge families of particles, the whole family is telescoped into the zero-mass point. A two-point function $G(z)$, which could be extremely complicated in the finite-mass theory, is now a simple power of z because of scale invariance.

The skeleton theory has a set of local fields. This set divides into two linear spaces: the space of local Bose fields and the space of local Fermi fields. Each set can be defined as the set of fields which commute with the $SU(3)\times SU(3)$ currents for spacelike separations. A linear combination of such fields is also a local field, so that these sets define linear spaces. Each space will be assumed to have a countable linearly independent basis. The two bases will be lumped together and denoted by $\{O_n(x)\}$, How the basis is constructed is largely arbitrary, and there will be many equally valid choices of basis. However, it is convenient to define subsets of fields which belong to particular irreducible representations of the symmetries and choose the basis fields from these subsets. In practice, it is convenient to let a particular $O_n(x)$ be an individual field and not a multiplet; it might be a component of a vector field, for example.

The skeleton theory is especially elegant if the stress-energy tensor $\theta_{\mu\nu}$ of the skeleton is a traceless symmetric tensor, so that it belongs to an irreducible representation of the Lorentz group. This is the case for the quark model. It makes scale invariance and Lorentz invariance automatic, given translational invariance, because the generators of scale transformations and Lorentz transformations become

$$D=\int_{\mathbf{x}} x^{\mu}\theta_{\mu 0}(x)\,, \tag{4.1}$$

$$J_{\mu\nu}=\int_{\mathbf{x}} [x_{\mu}\theta_{\nu 0}(x)-x_{\nu}\theta_{\mu 0}(x)]\,. \tag{4.2}$$

Given that $\theta_{\mu\nu}$ is traceless, symmetric, and conserved, these generators are also conserved.

The skeleton theory is presumed to have operator-product expansions like Eq. (2.2). The coefficients $C_n(x-y)$ are determined except for constants by scale invariance and other symmetries. No proposal will be made here for determining these constants. However, it will be assumed that these constants are all unique and dimensionless.[18] Then dimensional analysis of the operator-product expansion shows that the dimension of a local field $A(x)$ is the same as its scale-invariance quantum number d_A. There are also operator-product expansions for products of three local fields $A_1(x)A_2(y)A_3(z)$, or even more. In this case, the expansion functions $C_n(x-y,\, x-z)$ can depend in an arbitrarily complicated way on the ratio $(x-y)^2/(x-z)^2$ without violating any invariance. No procedure will be offered for determining the dependence on such dimensionless arguments.

It will be assumed that the fields $O_n(x)$ of the basis are ordered by dimension. For any given dimension there will be one or more multiplets of fields labeled by their Lorentz representation, baryon number, $SU(3)\times SU(3)$ representation, and P and C properties. Of particular importance are the fields of low dimension, since these fields have the most singular coefficients in operator-product expansions. As a result, they will determine singularities in radiative corrections, convergence of Weinberg sum rules, etc. In practice, it is the fields of dimension 4 or less that are important.[19] There are several fields that *must* have dimension 4 or less. First, there are the $SU(3)\times SU(3)$ currents and the baryon current. These will be assumed to satisfy Gell-Mann's current algebra in the skeleton theory as well as the physical theory. So the currents have dimension 3. The stress-energy tensor has dimension 4. There must also be an $SU(3)\times SU(3)$ multiplet including the pion field, with a dimension Δ less than 4, and greater than or equal to 1. The dimensional restriction is necessary to make partial conservation of axial-vector current (PCAC) work when $SU(3)\times SU(3)$ is broken; this will be explained later.[20] This multiplet will be assumed to be an irredubible multiplet transforming as the $(3,\bar{3})\oplus(\bar{3},3)$ representation of $SU(3)\times SU(3)$.[21] All these properties are true of the $SU(3)\times SU(3)$ σ model in which the pion-field multiplet has dimension 1, and the quark model in which the pion field has dimension

[17] Since PCT is automatic in a local-field theory, it can and will be ignored.

[18] For each field $O_n(x)$ there is one arbitrary normalization factor, which one chooses to be a dimensionless constant.

[19] An exception is in the problem of nonleptonic weak interactions where the crucial question is the dimension of the first local field in the basis with the quantum numbers assumed for the nonleptonic weak Hamiltonian. See Sec. VII E.

[20] See the end of Sec. VI.

[21] This was proposed by Gell-Mann (Ref. 1).

3.[22] Here no assumption will be made about the dimension Δ beyond the restriction $1 \leqslant \Delta < 4$.

Are there other fields besides the ones mentioned above with dimension 4 or less? It will become evident that this is a vital question. The author has no way to answer it conclusively. As an *ad hoc* approach, it will be assumed, to start with, that there are no other fields of dimension 4 or less. An extra field [an $SU(3)\times SU(3)$ singlet scalar field] will be proposed later because it is needed as a mass term in the Lagrangian.

In free-field models [the $SU(3)\times SU(3)$ σ model or the free-quark model] the basis set of fields $\{O_n(x)\}$ can be constructed from Wick products and derivatives of the elementary fields. Thus, in the quark model, the fields of dimension 4 or less and zero baryon number are fields of the form $:\bar{\psi}\alpha\psi:$ or $:\nabla_\mu\bar{\psi}\alpha\psi:$ or $:\bar{\psi}\alpha\nabla_\mu\psi:$, where α is any spin and $SU(3)$ matrix and ψ is the quark field. These fields all have dimension 3 or 4.[23] A study[24] of the Thirring model has convinced the author that the dimensions of fields in free-field models have little or no bearing on the dimensions of fields in strong interactions. In the Thirring model there is a single dimensionless coupling constant λ and the theory is scale-invariant for all values of λ. In the free-field limit of the Thirring model the spinor field ψ has dimension $\frac{1}{2}$ as expected from the canonical commutation rules in one space dimension. However, the dimension varies continuously with λ and approaches ∞ as λ approaches 2π. This is a consequence of renormalization; after renormalization the field does not satisfy canonical commutation rules and hence does not have to have dimension $\frac{1}{2}$. However, the Thirring model has charge and axial-charge conservation and the equal-time commutators of the charges with the spinor field force the corresponding currents to have dimension 1, which they do. This is the same as the dimension of the product $\bar{\psi}\gamma_\mu\psi$, only in the free-field limit. So in thinking about the dimensions of operators in the strong interactions we do not assume that there are elementary fields satisfying canonical commutation rules and do not assume that the other local fields have the same dimensions as products of elementary fields. It is also not assumed that fields exist with the same dimension as the product of two pion fields or the product of two currents. (The latter assumption affects the theory of nonleptonic weak interactions.[19]) This leaves one with enormous flexibility in choosing dimensions of fields.

V. MASS TERMS: GENERALITIES

Now consider the problem of interactions, that is, departures from the skeleton theory. If the skeleton theory were a free-field theory, there would be four types of interactions: mass terms quadratic in the free field, super-renormalizable interactions (which require subtractions only in low orders of perturbation theory), renormalizable interactions, and nonrenormalizable interactions. Each interaction corresponds to a local Lagrangian density scalar to Lorentz transformations and invariant to exact internal symmetries. Clearly any Lagrangian density must be a linear combination of the basis of local fields $\{O_n(x)\}$, so that it is logical to use the subset of fields from the basis which are scalars and exact to internal symmetries as the specific interactions. One then has a basis $\{\mathfrak{L}_i(z)\}$ of possibe interactions, culled from the complete basis $\{O_n(x)\}$. It was shown by Umezawa *et al.*[8] that a renormalizable interaction must have dimension 4 or less. If it has dimension less than 4, it is a super-renormalizable interaction (e.g., the $:\phi^3:$ interaction of a scalar field). Free-field mass terms (for scalar or spinor fields) have dimension less than 4.

As long as interactions on a free-field theory are treated in perturbation theory, one finds that operator-product expansions at small distances [like Eq. (2.2)] exist in the presence of interaction, but that the different types of interaction have profoundly different effects on the expansion coefficients $C_n(z)$. If one has only mass terms and super-renormalizable interactions, then the dominant term in $C_n(z)$, for z small, is the skeleton term; terms depending on the free-field mass or the coupling constant are smaller by a power of z.[25] If one has a renormalizable interaction, the interaction generates terms logarithmically more singular than the skeleton term. Finally, a nonrenormalizable interaction treated in lowest order generates a term more singular by a power of z than the skeleton term. For every type of interaction or mass term, the functions $C_n(x-y)$ can be written as a power series in the coupling constant or mass (only to first order for nonrenormalizable interactions). The nth-order term in the series scales as $z^{n(4-d_i)}$ relative to the skeleton term, where d_i is the dimension of the interaction. There can also be logarithms of $(z)^2/m^2$, where m is the free-field mass.

Similar conclusions apply to interactions on the hadron skeleton. One defines the basis of possible interactions as the subset of fields $\{\mathfrak{L}_i(x)\}$ from the basis $\{O_n\}$ which are Lorentz scalars, even to P and C, and have $I=Y=0$. (One excludes the unit operator and all derivatives of fields from this list, since these are not meaningful interactions.) There are three types of interactions, depending on their dimensions: Generalized mass terms have dimension less than 4,[26] renormalizable interactions, if any, have dimension 4 exactly, and nonrenormalizable interactions have dimension greater than 4. A general perturbation formula can be set up to describe any of these interactions. To avoid innumerable complications of perturbation theory to all orders one

[22] The difference in dimension affects the behavior of the pion field, in particular its operator products at short distances. This was noted by M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Ref. 20.

[23] There is also the unit operator.

[24] The results stated below can be gleaned from Ref. 12.

[25] This was pointed out by J. Estberg, Ph.D. thesis, Cornell University (unpublished).

[26] One cannot distinguish between mass terms and super-renormalizable interactions in a strongly interacting theory because one does not write local fields as products of elementary fields.

writes only a first-order formula giving the change in any local (Heisenberg) field $O_n(x)$ when any coupling constant is changed. That is, if $\{\lambda_i\}$ are the set of coupling constants associated with the interactions $\{\mathfrak{L}_i\}$, one obtains a formula for $\partial O_n(x)/\partial\lambda_i$. The usual (unrenormalized) formula is

$$\frac{\partial O_n(x)}{\partial\lambda_i}=i\int_y [O_n(x),\mathfrak{L}_i(y)]_{\rm ret}\,, \qquad (5.1)$$

where $[\ ]_{\rm ret}$ means the retarded commutator $(y_0<x_0)$. This formula has to be corrected both for nonadiabatic effects (when physical particle masses vary with λ_i) and for ultraviolet singularities at $x=y$. The nonadiabatic effects are easily accounted for and will not be considerd here. The ultraviolet singularities can be analyzed using the operator-product expansion for the commutator $[O_n(x),\mathfrak{L}_i(y)]$ and the singular terms can then be removed by subtraction. One can then show[27] that operator-product expansions continue to hold in the presence of the perturbation and obtain formulas for derivatives of expansion functions such as $\partial C_n(z)/\partial\lambda_i$. These formulas will not be quoted here. These formulas can be used to show that the expansion functions have power series in all interactions. One finds as in the free-field case that the nth-order term scales as $z^{n(4-d_i)}$ relative to the skeleton term, where d_i is the dimension of the interaction. There may also be logarithms of $x-y$. One can avoid logarithms of physical masses (which depend in a very complicated way on the coupling constants λ_i) by introducing an arbitrary subtraction constant a, so that logarithms have the form $\ln[(x-y)^2/a^2]$, not $\ln[(x-y)^2/m^2]$, in the expansion functions.[28] If this is done, it is no longer possible to choose the subtractions for Eq. (5.1), so that the vacuum expectation value $\langle\Omega|O_n(x)|\Omega\rangle$ vanishes.

The generalized mass terms, with $d_i<4$, give corrections to expansion functions smaller by a power of z than the skeleton term. In this case, the divergences in Eq. (5.1) are primarily due to the skeleton theory. Also, the equal-time commutators of symmetry generators are to some extent unaffected by the presence of generalized mass terms [more precisely, the equal-time commutator of a current $j_\mu(y)$ with a local field $A(x)$ is changed only by fields of dimension less than A itself]. So generalized mass terms are the logical choice of interaction when one wants a symmetry of the skeleton to be a broken symmetry of the theory with interaction.

The renormalizable interactions produce corrections to expansion functions which are logarithmically more singular than the skeleton terms. Hence, if an interaction is not invariant to a symmetry, it can destroy the equal-time commutators associated with the symmetry.[29] However, the number of subtractions needed in Eq. (5.1) is unchanged by the presence of interaction, so that one has a renormalizable interaction in the conventional sense. Nonrenormalizable interactions produce expansion functions more singular by a power of $x-y$ than the skeleton terms and hence force one to make extra subtractions due to the interaction in Eq. (5.1), so that they are nonrenormalizable in the conventional sense.

It will now be assumed that the interactions on the hadron skeleton theory are all generalized mass terms. This means that scale invariance, as well as $SU(3)\times SU(3)$, is a broken symmetry. This is an *ad hoc* assumption motivated in part by the success of broken $SU(3)\times SU(3)$ and in part because it is hard to use the ideas of this paper if renormalizable or unrenormalizable interactions are permitted (this is not true if the renormalizable or nonrenormalizable interactions are present but small; thus electrodynamic and weak corrections to strong interactions do not create difficulties).

In order that all terms in a given expansion function $C_n(x-y)$ have the same dimensions, one must assign each coupling constant λ_i the dimension m^{4-d_i}, where d_i is the dimension of $\mathfrak{L}_i$. So all generalized mass terms have coupling constants which carry dimensions; hence, it is possible for physical masses to be generated by any generalized mass term.

VI. HADRON MASS TERMS

Consider now the possible generalized mass terms in strong interactions. Given the list of fields of dimension less than 4 assumed earlier, the only possibilities are two fields from the pion-field multiplet. These are σ_8 and σ_0, where σ_8 is the $I=Y=0$, $SU(3)$-octet scalar field, and σ_0 is the $SU(3)$-singlet scalar field. Assume that there is an $SU(3)\times SU(3)$ singlet field, $w(x)$, also with dimension less than 4. The need for this will be seen shortly. Then the interaction Lagrangian can be written[30]

$$\mathfrak{L}_I(x)=\lambda_0\sigma_0(x)+\lambda_8\sigma_8(x)+\lambda w(x)\,. \qquad (6.1)$$

One would like an order-of-magnitude estimate for each term in $\mathfrak{L}_I$. The term $\lambda_8\sigma_8(x)$ is the $SU(3)$-breaking term; this is known to cause energy separations within a multiplet of 450 MeV (the N^* decuplet) or less. As a mean energy, say $\lambda_8\sigma_8\sim 300$ MeV. This is a relatively small energy, since it is now known that one should measure energies relative to m_ρ or maybe m_N, and not m_π. One can get an estimate on the $\lambda_0\sigma_0$ term but by a more indirect argument. One knows that in the limit

[27] This paragraph summarizes a very complex analysis.

[28] This possibility was noted by M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954). It is essential to the subsequent analysis that logarithms do not involve the couplings λ_i. This is because such logarithms would destroy the spurion analysis of $SU(3)\times SU(3)$-symmetry breaking used in Sec. VII, for which a pure power series in λ_i is essential.

[29] This is related to observations of K. Johnson and F. E. Low, Progr. Theoret. Phys. (Kyoto) Suppl. **37–38**, 74 (1966).

[30] The $SU(3)\times SU(3)$-symmetry-breaking terms are as proposed by M. Gell-Mann, Phys. Rev. **125**, 1067 (1962), Eq. (5.21). For further discussion of this choice of symmetry breaking, see S. L. Glashow and S. Weinberg, Phys. Rev. Letters **20**, 224 (1968), and M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).

of exact $SU(2)\times SU(2)$ symmetry the pion mass must be zero.[31] So the $SU(2)\times SU(2)$-breaking term must raise the pion mass from 0 to 140 MeV; in other words, the $SU(2)\times SU(2)$-breaking term is smaller than even the $SU(3)$-breaking term. To discuss $SU(2)\times SU(2)$ breaking, one must express both σ_0 and σ_8 in terms of an $SU(2)+SU(2)$-violating operator σ_9 [belonging to the (2,2) representation] and $SU(2)\times SU(2)$-conserving field σ_{10}:

$$\sigma_0=-(\sqrt{\tfrac{2}{3}})\sigma_9+(\sqrt{\tfrac{1}{3}})\sigma_{10}, \tag{6.2}$$

$$\sigma_8=(\sqrt{\tfrac{1}{3}})\sigma_9+(\sqrt{\tfrac{2}{3}})\sigma_{10}. \tag{6.3}$$

Since σ_0 and σ_8 are part of a single irreducible representation of $SU(3)\times SU(3)$, this decomposition of σ_0 and σ_8 by $SU(2)\times SU(2)$ representation is unique. Now one has

$$\lambda_0\sigma_0+\lambda_8\sigma_8=[-(\sqrt{\tfrac{2}{3}})\lambda_0+(\sqrt{\tfrac{1}{3}})\lambda_8]\sigma_9 +[(\sqrt{\tfrac{1}{3}})\lambda_0+(\sqrt{\tfrac{2}{3}})\lambda_8]\sigma_{10}. \tag{6.4}$$

If m_π were zero, the coefficient of σ_9 would be zero. With m_π only 140 MeV, the coefficient of σ_9 should be roughly zero, that is,

$$\lambda_0\simeq(\sqrt{\tfrac{1}{2}})\lambda_8. \tag{6.5}$$

The error turns out to be of order $m_\pi{}^2$, which even compared to $(300\text{ MeV})^2$ is small. This means the $SU(3)$-symmetric term $\lambda_0\sigma_0$ represents an energy of, perhaps, 200 MeV. This is why another mass term is needed to change the ρ, nucleon, and other heavier particles from mass zero (in the skeleton theory) to their observed masses. This requires energies of order 1 BeV, so that λw should be of order 1 BeV.

Given the interaction Lagrangian, one can derive the PCAC formula. It follows from Eq. (5.1) (even in renormalized form), that if $j_\mu(x)$ is a conserved current in the skeleton theory and $\mathfrak{L}_I$ contains only generalized mass terms, then

$$\nabla^\mu j_\mu(x)=i[Q(x_0),\mathfrak{L}_I(x)], \tag{6.6}$$

where $Q(x_0)$ is the charge associated with j_μ. In the case of the strangeness-conserving isovector axial-vector current A_μ, one obtains

$$\nabla^\mu A_\mu(x)=[-\tfrac{1}{3}\sqrt{3}\lambda_8+(\sqrt{2}/\sqrt{3})\lambda_0]\phi(x), \tag{6.7}$$

where ϕ is the pion field.

One can now see why the pion field must have dimension Δ less than 4. The reason is that the pion field must be in the same $SU(3)\times SU(3)$ representation as the $SU(3)\times SU(3)$-breaking terms in $\mathfrak{L}_I$, due to Eq. (6.6). But the Lagrangian contains only generalized mass terms in order not to jeopardize the current commutation relations. The lower bound $\Delta\geqslant 1$ is an elementary consequence of the Källén-Lehmann representation.

[31] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961). See also S. L. Adler and R. F. Dashen, *Current Algebras* (W. A. Benjamin, Inc., New York, 1968).

VII. APPLICATIONS

A. Weinberg Sum Rules

To illustrate the applications of the operator-product expansions, consider first the question of convergence of Weinberg sum rules.[2] Let $\Delta_{\mu\nu}(x)$ be an arbitrary linear combination of vector and axial-vector meson propagators, and $G_{\mu\nu}(p)$ be its Fourier transform. The first Weinberg sum rule holds for $G_{\mu\nu}(p)$ if $\Delta_{\mu\nu}(x)$ is less singular than x^{-4} as $x\to 0$.[32] The second sum rule holds if it is less singular than x^{-2}. More precisely, one can get the second sum rule if either the $g_{\mu\nu}$ term or the $x_\mu x_\nu$ term in $\Delta_{\mu\nu}(x)$ is less singular than x^{-2}. So one needs the short-distance behavior of the product $TV_{\mu\alpha}(x)\times V_{\nu\beta}(0)$ and $TA_{\mu\alpha}(x)A_{\nu\beta}(0)$ for x near 0; α and β are $SU(3)$ indices. The behavior is needed to order x^{-2}. Since $V_{\mu\alpha}$ and $A_{\mu\alpha}$ have dimension 3, one will need to expand in terms of operators O_n of dimension 4 or less. Since one wants the vacuum expectation value of the T product, one needs only operators O_n with a nonzero vacuum expectation value. The only possibilities are I, σ_0, σ_8, and w. Terms involving w will always be less singular than terms involving I, so that w will be ignored. Let σ_0 and σ_8 be the vacuum expectation values $\langle\Omega|\sigma_0(x)|\Omega\rangle$ and $\langle\Omega|\sigma_8(x)|\Omega\rangle$. The linear combination $\Delta_{\mu\nu}(x)$ has an expansion

$$\Delta_{\mu\nu}(x)\simeq H_{\mu\nu}(x)+H_{1\mu\nu}(x)\sigma_0+H_{2\mu\nu}(x)\sigma_8 \tag{7.1}$$

for x small, with unknown functions H, H_1, and H_2.

The quantity $\Delta_{\mu\nu}(x)$ is a vacuum expectation value of a linear combination of T products. The linear combination of T products can be chosen to belong to a particular irreducible representation of $SU(3)\times SU(3)$. The currents themselves belong to $(8,1)\oplus(1,8)$,[1] so the product of two currents can belong to (1,1), $(8,1)\oplus(1,8)$, $(10,1)\oplus(1,10)\oplus(\overline{10},1)\oplus(1,\overline{10})$, $(27,1)\oplus(1,27)$, or (8,8). When one takes the vacuum expectation value, the decuplet representation disappears. The propagator combinations which correspond to the other representations are listed in Table I. To determine the singularity of $H(x)$, $H_1(x)$, and $H_2(x)$ for each linear combination, one can do a spurion analysis. If $\Delta_{\mu\nu}(x)$ belongs to the same representation of $SU(3)\times SU(3)$ as σ_0, then $H_1(x)$ will have a skeleton term. If not, one must find out how many powers of the symmetry-breaking parameters λ_0 and λ_8 are needed for $H_1(x)$ not to be zero. This is determined by a spurion analysis with the spurion representing the symmetry-breaking interaction. The spurion belongs to $(3,\bar{3})\oplus(\bar{3},3)$. Combining one spurion with σ_0 or σ_8, which also belong to $(3,\bar{3})\oplus(\bar{3},3)$, one can produce all representations except $(27,1)\oplus(1,27)$: To produce the latter requires three spurions at least. The skeleton term in $H_1(x)$ would have behaved as $x^{-6+\Delta}$ (6 for two currents, Δ for the dimension of the σ field). With one spurion, one has one power of λ_0 or λ_8, and correspond-

[32] This would mean $\Delta_{\mu\nu}(x)$ being less singular than $g_{\mu\nu}/x^4$ and $x_\mu x_\nu/x^6$.

TABLE I. Combinations of vector and axial-vector meson propagators, associated $SU(3)\times SU(3)$ and $SU(2)\times SU(2)$ representations, and maximum singularity of H, H_1, and H_2.[a]

$\Delta_{\mu\nu}(x)$	$SU(3)\times SU(3)$	$SU(2)\times SU(2)$	H	H_1, H_2
$\rho-A_1$	(8,8)	(3,3)	$2-2\Delta$	-2
$K^*-K_A^*$	(8,8)	(2,2)	$2-2\Delta$	-2
$\omega-D$	(8,8)	(1,1)	$2-2\Delta$	-2
x	$(8,1)\oplus(1,8)$	(1,1)	$2-2\Delta$	-2
y	$(27,1)\oplus(1,27)$	(1,1)	$10-4\Delta$	$6-2\Delta$
z	(1,1)	(1,1)	-6	-2
x: $3(\rho+A_1)-2(K^*+K_A^*)-(\omega+D)$				
y: $(\rho+A_1)-4(K^*+K_A^*)+3(\omega+D)$				
z: $\rho+A_1+K^*+K_A^*+\omega+D$				

[a] Shorthand: ρ is the ρ propagator, D is the axial propagator counterpart to ω propagator, the column labeled H gives most singular power of x in $H(x)$, and ω is the propagator made from $I=Y=0$ $SU(3)$-octet current, not baryon current.

ingly a factor $x^{4-\Delta}$ (apart from logarithms). Hence $H_1(x)$ scales as x^{-2} apart from logarithms, except for the linear combination of propagators associated with the $(27,1)\oplus(1,27)$ representation. The same is true of $H_2(x)$. The complete results are shown in Table I. One sees from Table I that the second Weinberg sum rule can hold only for the $(27,1)\oplus(1,27)$ combination. However, as pointed out by Weinberg,[2] it should be a good approximation to go to the exact $SU(2)\times SU(2)$ limit ($\lambda_8=\sqrt{2}\lambda_0$). Then the operator-product expansion of Eq. (7.1) must obey $SU(2)\times SU(2)$. But the fields σ_0 and σ_8 belong to the (1,1) and (2,2) representations of $SU(2)\times SU(2)$, so that they cannot appear in the expansion of the $\rho-A_1$ linear combination. So the original second Weinberg sum rule converges in the exact $SU(2)\times SU(2)$ limit. This is not true of any of the $SU(3)\times SU(3)$ generalizations [except the $(27,1)\oplus(1,27)$ term, if it converges without the $SU(2)\times SU(2)$ limit].[33] All the $SU(3)\times SU(3)$ sum rules converge in the exact $SU(3)\times SU(3)$ limit; how good an approximation this would be to physics, I do not know. If Δ is 3 or larger, then even the first Weinberg sum rule does not converge, except for the $(27,1)\oplus(1,27)$ combination. This whole analysis is modified if there are scalar fields of dimension 4 or less belonging to the (8,8), $(8,1)\oplus(1,8)$, or $(27,1)\oplus(1,27)$ representations of $SU(3)\times SU(3)$. Such fields would destroy the second Weinberg sum rule for the relevant linear combinations of propagators, whether or not $SU(3)\times SU(3)$ is exact.

[33] This contradicts the assumptions of S. L. Glashow and S. Weinberg, Phys. Rev. Letters **20**, 224 (1968). In particular, Glashow and Weinberg assume the second Weinberg sum rule for the difference of the K^* and K_A^* propagators, which contradicts the analysis of this paper. Note, also, that the symmetry-breaking mechanism of this paper is more complicated than that of T. Das, V. S. Mathur, and S. Okubo, *ibid.* **19**, 470 (1967); Das *et al.* propose that the second Weinberg sum rule should be violated in first order of $SU(3)$ breaking at least for differences of vector-meson propagators, but hold in second-order $SU(3)$ breaking. In the present paper, the violation of the second Weinberg sum rule is caused by a combination of first-order $SU(3)\times SU(3)$ breaking in the Lagrangian and violation of $SU(3)\times SU(3)$ in the vacuum expectation values of the σ fields, which means one can have violation of the Weinberg sum rule even in second order of $SU(3)$ breaking.

B. Divergence in Radiative Corrections

As a second example, consider the question of divergences in the radiative corrections to strong interactions. Consider only the second-order radiative corrections; these are described by an effective interaction Lagrangian

$$\mathcal{L}_I(0)=e^2\int_x Tj_\mu(x)j_\nu(0)D^{\mu\nu}(x)\,, \tag{7.2}$$

where $j_\mu(x)$ is the hadron electromagnetic current and $D^{\mu\nu}(x)$ is the photon propagator. The divergences in radiative corrections come from divergences in the x integral at $x=0$; they occur if the T product is as singular as x^{-2}. One is concerned only with the isospin-1 and -2 parts of $\mathcal{L}_I(0)$. So one needs only fields in the expansion of the T product which carry isospin. To cause a singularity x^{-2} or greater, these fields must have dimension 4 or less. So one writes

$$Tj_\mu(x)j_\nu(0)=\sum_n C_{n\mu\nu}(x)O_n(0)\,, \tag{7.3}$$

with the sum over $O_n(0)$ so restricted. Finally, the integrals of $C_{n\mu\nu}(x)D^{\mu\nu}(x)$ over x will vanish because of Lorentz invariance unless $O_n(0)$ is a scalar field. So one s restricted to the $I=1$, $I_z=0$ scalar field σ_3 in the pion-field multiplet. So the only possible divergent term in $\mathcal{L}_I(0)$ has the form

$$e^2\left(\int_x C_{\mu\nu}(x)D^{\mu\nu}(x)\right)\sigma_3(0)\,. \tag{7.4}$$

The behavior of $C_{\mu\nu}(x)$ is found by a spurion analysis to be x^{-2}, except in the case of exact $SU(2)\times SU(2)$ symmetry, when it is identically zero. This means that, for finite pion mass, $C_{\mu\nu}(x)$ has a factor m_π^2. The standard procedure for obtaining a finite effective Lagrangian is to subtract the offending term and add an arbitrary constant times σ_3:

$$\mathcal{L}_I(0)=e^2\int_x [Tj_\mu(x)j_\nu(0)-C_{\mu\nu}(x)\sigma_3(0)]D^{\mu\nu}(x)+f\sigma_3(0)\,, \tag{7.5}$$

where f is an arbitrary constant. The $\sigma_3(0)$ term is the tadpole term proposed by Coleman and Glashow.[34] If there are other isospin-1 or isospin-2 scalar fields with dimension 4 or less, they too can cause divergences in radiative corrections.

C. $\eta\to\pi^+\pi^0\pi^-$ Decay

Two specific problems involving electromagnetic corrections will be considered here: the $\eta\to 3\pi$ decay and the $\pi^0\to 2\gamma$ decay. First, consider the $\eta\to 3\pi$ decay. Experimentally the amplitude for $\eta\to 3\pi$ is well fitted

[34] S. Coleman and S. L. Glashow, Phys. Rev. **134**, B671 (1964).

by the formula[35]

$$A[\eta \to \pi^+ + \pi^0 + \pi^-] = \bar{A}_\eta [1 - m_\pi^{-2} a (S_0 - \bar{S}_\eta)], \quad (7.6)$$

where

$$S_0 = [q(\eta) - q(\pi^0)]^2, \quad (7.7)$$

$$\bar{S}_\eta = m_\pi^2 + \tfrac{1}{3} m_\eta^2, \quad (7.8)$$

$$a = -0.2 \pm 0.015, \quad (7.9)$$

and $q(\eta)$ and $q(\pi^0)$ are the η and π^0 four-momenta. It was shown by Sutherland[36] that the amplitude A should vanish when any pion four-momentum goes to zero. If the π^+ four-momentum goes to zero, then $q(\eta) - q(\pi^0)$ must equal $q(\pi^-)$, and S_0 is m_π^2. In this case, Eq. (7.6) gives an amplitude less than $0.1\bar{A}_\eta$, in good agreement with Sutherland's prediction. But when the π^0 four-momentum goes to zero, S_0 becomes M_η^2 and the $\eta \to 3\pi$ amplitude is about $3\bar{A}_\eta$.

Sutherland's calculation assumed the unsubtracted form (7.2) for the effective electrodynamic Lagrangian. With the subtracted form (7.5) the calculation is different.[37] Sutherland's argument applies to the first term in Eq. (7.5) but not to the term $f\sigma_3(0)$. The field σ_3 commutes with the charged axial-vector currents but not with the neutral axial-vector current. [The relevant $SU(3) \times SU(3)$ commutators are given by Gell-Mann.[1]] As a result, one still predicts that the $\eta \to 3\pi$ amplitude vanishes for zero π^+ momentum but not for zero π^0 momentum. This agrees with experiment. When $q(\pi^0) \to 0$, the $\eta \to 3\pi$ amplitude will be proportional to the subtraction constant f. Since f is an unknown parameter, it can be chosen to fit the $\eta \to 3\pi$ decay amplitude. There remains a problem in $\eta \to 3\pi$ decay, namely, its relation to the K and π mass differences obtained by first taking two pions to zero momentum in the $\eta \to 3\pi$ amplitude, which reduces it to the η-π electromagnetic mixing amplitude, then relating the mixing amplitude via $SU(3)$ to the K and π mass differences.[37] This analysis does not work very well, but it involves more dubious approximations than the simple current-algebra calculation.[38]

It has been proposed that the tadpole contributions to electromagnetic corrections be obtained dynamically—in terms, say, of the A_2 Regge-pole contribution to the virtual forward Compton amplitude.[39] It remains to be seen whether such a term has a different $SU(2) \times SU(2)$ behavior from the unsubtracted electromagnetic Lagrangian, which is necessary if it is to account for the $\eta \to 3\pi$ decay.

D. $\pi^0 \to \gamma\gamma$ Problem

The decay $\pi^0 \to 2\gamma$ has been shown by Veltman and Sutherland[40] to be forbidden by current algebra and gauge invariance. The invariant amplitude for π^0 decay can be written

$$T_{\mu\nu}(p,k) = \epsilon_{\mu\nu\alpha\beta} p^\alpha k^\beta T(k^2), \quad (7.10)$$

where p is one of the photon momenta and k is the π^0 momentum. The photon momenta will be kept on the mass shell $[p^2 = (k-p)^2 = 0]$. The argument of Veltman and Sutherland predicts that $T(0)$ will be zero. Bell and Jackiw[41] have recently pointed out that there exists a specific calculation of $T(0)$ in the σ model, and that $T(0)$ does not vanish. The calculation is an old calculation of Steinberger.[42]

The resolution of this contradiction is in a breakdown of the usual arguments that one uses in writing Ward identities. Namely, one is used to taking a matrix element such as $\langle A | T j_\mu(x) \nabla^\pi j_\pi(y) | B \rangle$ and rewriting it

$$\nabla_y^\pi \langle A | T j_\mu(x) j_\pi(y) | B \rangle + \delta(x_0 - y_0) \langle A | [j_\mu(x), j_0(y)] | B \rangle.$$

But in the case of the $\pi^0 \to 2\gamma$ amplitude, which involves the matrix element $\langle \Omega | T j_\mu(x) j_\nu(y) \nabla^\pi A_\pi(z) | \Omega \rangle$, extra terms arise when ∇^π is brought outside the matrix element. The extra terms involve the singularity at $x = y = 0$ and involve complicated integrals. The author has not evaluated these integrals even in the σ-model example, let alone strong interactions; however, one can argue that they do not obviously vanish in either case.

A brief account of the analysis will now be given. The $\pi^0 \to 2\gamma$ amplitude can be written

$$T_{\mu\nu}(p,k) = (k^2 - m_\pi^2)(F_\pi m_\pi^2)^{-1} \int_x \int_z e^{ip\cdot x} e^{ik\cdot z} \times \langle \Omega | T j_\mu(x) j_\nu(0) \nabla^\pi A_\pi(z) | \Omega \rangle, \quad (7.11)$$

where F_π is the PCAC constant,

$$\nabla^\pi A_\pi(x) = F_\pi m_\pi^2 \phi(x), \quad (7.12)$$

ϕ is the π^0 field, and A_π is the neutral axial-vector field. The quantity $T(0)$ can be extracted from the formula

$$\nabla_\alpha^p \nabla_\beta^k T_{\mu\nu}(p,k)|_{p=k=0} = \epsilon_{\mu\nu\alpha\beta} T(0). \quad (7.13)$$

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[35] The value for a is taken from M. Gormley, E. Hyman, W. Lee, T. Nash, J. Peoples, C. Schulz, and S. Stein, Phys. Rev. Letters **21**, 402 (1968).

[36] D. G. Sutherland, Phys. Letters **23**, 384 (1966).

[37] S. K. Bose and A. M. Zimmerman, Nuovo Cimento **43A**, 1165 (1966); R. Ramachandran, *ibid.* **47A**, 669 (1967); R. H. Graham, L. O'Raifeartaigh, and S. Paksava, *ibid.* **48A**, 830 (1967); Y. T. Chiu, J. Schechter, and Y. Ueda, Phys. Rev. **161**, 1612 (1967); D. G. Sutherland, Nucl. Phys. **B2**, 433 (1967). For a review of other theories of $\eta \to 3\pi$ decay, see J. S. Bell and D. G. Sutherland, *ibid.* **B4**, 315 (1968).

[38] See S. L. Adler and R. L. Dashen (Ref. 31), p. 137.

[39] S. Okubo, Phys. Rev. Letters **18**, 256 (1967); D. J. Gross and H. Pagels, Phys. Rev. **172**, 1381 (1968).

[40] M. Veltman, Proc. Roy. Soc. (London) **A301**, 107 (1967); D. G. Sutherland, Nucl. Phys. **B2**, 433 (1967). For related work on the $\omega\rho\pi$ system, see R. Perrin, Phys. Rev. **170**, 1367 (1968); S. G. Brown and G. B. West, *ibid.* **174**, 1777 (1968); R. Arnowitt, M. H. Friedman, and P. Nath, Northeastern University (unpublished).

[41] J. S. Bell and R. Jackiw, Nuovo Cimento (to be published). I wish to thank Dr. Jackiw for very helpful discussions on the $\pi^0 \to 2\gamma$ problem. See also S. L. Adler, Phys. Rev. **177**, 2426 (1969); his conclusions are equivalent to ours.

[42] J. Steinberger, Phys. Rev. **76**, 1180 (1949).

So

$$\epsilon_{\mu\nu\alpha\beta}T(0)=F_\pi^{-1}\int_x\int_z x_\alpha z_\beta\langle\Omega|\,Tj_\mu(x)j_\nu(0)\nabla^\pi A_\pi(z)\,|\Omega\rangle. \tag{7.14}$$

In the σ-model calculation in lowest-order perturbation theory, this matrix element involves the currents of a free charged Dirac field ψ:

$$j_\mu(x)=:\bar\psi(x)\gamma_\mu\psi(x):, \tag{7.15}$$

$$A_\pi(x)=\tfrac{1}{2}i:\bar\psi(x)\gamma_\pi\gamma_5\psi(x):. \tag{7.16}$$

These currents commute for equal times (except for c-number Schwinger terms which cannot affect the three-current matrix element), just as the currents j and A of strong interactions do. Also, the free-field current $j_\mu(x)$ is conserved and $A_\pi(x)$ is partially conserved. Furthermore, $T(0)$ is defined by a convergent integral; in fact, if one rotates the x_0 and z_0 integrals to the imaginary axis (thus going over to a Euclidean metric), the integral for $T(0)$ is absolutely convergent. (This can be verified explicitly for the free-field currents, and is a consequence of the hypotheses of this paper for the currents of strong interactions.) Consider the effect of the conventional rules for interchanging a T product with a gradient. One can simply bring ∇^π outside the T product because all equal-time commutators are zero. Define

$$F_{\mu\nu\pi}(x,z)=\langle\Omega|\,Tj_\mu(x)j_\nu(0)A_\pi(z)\,|\Omega\rangle, \tag{7.17}$$

and consider the following identity:

$$\begin{aligned}x_\alpha z_\beta\nabla_z{}^\pi F_{\mu\nu\pi}(x,z)=\nabla_z{}^\pi[x_\alpha z_\beta F_{\mu\nu\pi}(x,z)]-(\nabla_x{}^\pi+\nabla_z{}^\pi)\\ \times[x_\alpha z_\nu F_{\mu\pi\beta}]+\nabla_x{}^\pi[x_\mu z_\nu F_{\pi\alpha\beta}]+x_\alpha z_\nu(\nabla_x{}^\pi+\nabla_z{}^\pi)F_{\mu\pi\beta}\\ -x_\mu z_\nu\nabla_x{}^\pi F_{\pi\alpha\beta}.\end{aligned} \tag{7.18}$$

The first three terms are total derivatives; the last two terms vanish according to the conventional analysis because of the conservation of j_μ. The integrals of the first three terms are zero; hence $T(0)$ should vanish.

This type of argument treats cavalierly the intrinsic singularities of $F_{\mu\nu\pi}(x,z)$ when two or more operators are evaluated at the same point. To set up a more careful treatment, the first step is to define the integral (7.14) such that the singular points do not occur in the range of integration. This is easy to do: One calculates the integral excluding the regions $|x_0|<\epsilon$, $|z_0|<\epsilon'$, and $|x_0-z_0|<\epsilon''$, for small but finite ϵ, ϵ', and ϵ''. Then one takes the limits $\epsilon\to0$, $\epsilon'\to0$, and $\epsilon''\to0$. The integral, being absolutely convergent, is independent of the order that one takes these limits.

One now calculates the integral for finite ϵ, ϵ', and ϵ'' using the identity (7.18). Because the singular points of $F_{\mu\nu\pi}(x,z)$ are excluded from the range of integration, the identity holds without question, and the last two terms are zero. One is left with

$$\epsilon_{\mu\nu\alpha\beta}T(0)=F_\pi^{-1}(I_1+I_2+I_3), \tag{7.19}$$

where

$$I_1=\int_x\int_z\nabla_z{}^\pi[x_\alpha z_\beta F_{\mu\nu\pi}(x,z)], \tag{7.20}$$

and I_2 and I_3 are analogous. The z integral can be written in terms of surface integrals on the surfaces $z_0=\pm\epsilon'$ and $z_0=x_0\pm\epsilon''$. Consider, for example, the surface integrals at $z_0=x_0\pm\epsilon''$: They are

$$\int_x\int_z[\delta(z_0-x_0+\epsilon'')-\delta(z_0-x_0-\epsilon'')]x_\alpha z_\beta F_{\mu\nu\pi}(x,z). \tag{7.21}$$

The x integral must be broken into two parts. In one part, x is restricted to be far away from the origin, compared to ϵ''. With this restriction one can let $\epsilon''\to0$ inside the integral without making much error[43] and the integrand reduces to an equal-time commutator which vanishes. This is no longer true if x is of order ϵ'' or smaller, for then $F_{\mu\nu\pi}(x,z)$ has a nontrivial dependence on the variable z as well as $x-z$[44] (this is evident in the case of free-field currents), and one can no longer replace ϵ'' by zero. Actual evaluation of the integral is now nontrivial.

The difficulty caused by x being of order ϵ'' is avoided if one requires that ϵ be much larger than ϵ''; then x is always much larger than ϵ''. In this event, the surface terms at $z_0=x_0\pm\epsilon''$ approximately cancel. Likewise, if $\epsilon\gg\epsilon'$, the surface terms at $z_0=\pm\epsilon'$ approximately cancel. To be precise,

$$\lim(\epsilon'\to0)\lim(\epsilon''\to0)I_1=0 \tag{7.22}$$

for fixed nonzero ϵ. (If the currents j_μ and A_π had a nonzero equal-time commutator, I_1 would have reduced to an equal-time commutator term.) So one can make I_1 vanish by letting $\epsilon\to0$ be the last limit that one takes. Similarly, I_2 vanishes if $\epsilon''\to0$ last, and I_3 vanishes if $\epsilon'\to0$ last. Unfortunately, these three requirements are incompatible. If one lets $\epsilon''\to0$ last, eliminating I_2, then I_1 and I_3 need not be zero. In this case, I_1 reduces to the expression (7.21) with x unrestricted (so that I_1 now depends only on ϵ''). Only small x and z (of order ϵ'') are important in the integral: If either x or z is large compared to ϵ'', the term at $z_0=x_0+\epsilon''$ cancels the term at $z_0=x_0-\epsilon''$. For x and z of order ϵ'', $F_{\mu\nu\pi}(x,z)$ scales as $(\epsilon'')^{-9}$ because of scale invariance at short distances.[45] With this scaling law for $F_{\mu\nu\pi}$, the integral (7.21) is *independent of* ϵ''.

[43] This is a consequence of the fact that $F_{\mu\nu\pi}(x,z)$ is not more singular than $(x-z)^{-3}$ for $z\to x$, $x\neq0$. The details of the analysis use the operator-product expansion for $Tj_\mu(x)A_\pi(z)$ and the techniques of Sec. II.

[44] In the free-field case $F_{\mu\nu\pi}(x,z)$ includes a propagator $S^F(z)$. When x is large and z is near x, one can approximate $S^F(z)$ by $S^F(x)$; this approximation is essential in relating (7.21) to an equal-time commutator. But when x and $z-x$ are both $\sim\epsilon''$, this approximation is no good.

[45] The $(\epsilon'')^{-9}$ scaling law for $F_{\mu\nu\pi}(x,z)$ can be seen explicitly in the free-field example. For strong interactions the hadronic skeleton contribution to $F_{\mu\nu\pi}(x,z)$ scales as $(\epsilon'')^{-9}$ by dimensional analysis; finite mass corrections are less singular and can be neglected.

The conclusion of this analysis is that, if one takes the limits $\epsilon \to 0$, $\epsilon' \to 0$, and then $\epsilon'' \to 0$ last, I_2 is zero and I_1 and I_3 approach constants, but there is no reason to suppose that I_1 and I_3 are zero: On the contrary, in the case of free-field currents, $T(0)$ is not zero, so that one of I_1 or I_3 must be nonzero.[46]

For $T(0)$ not to be zero, it is crucial that $F_{\mu\nu\pi}(x,z)$ scales as $(\epsilon'')^{-9}$ when x and z are of order ϵ''. If $F_{\mu\nu\pi}(x,z)$ were less singular than this for x and $z \to 0$, then I_1 and I_3 would vanish in the limit $\epsilon'' \to 0$ because of the small range of integration (x, $z \sim \epsilon''$). In this case, $T(0)$ would have to be zero. The author suspects that $F_{\mu\nu\pi}(x,z)$ will be less singular than ϵ''^{-9} in the algebra of fields.[47] Then the Sutherland argument would apply to $\pi^0 \to \gamma\gamma$ decay in the algebra of fields. This would make the algebra of fields in bad contradiction with the observed $\pi^0 \to \gamma\gamma$ decay rate, which agrees well with the Steinberger formula.

There is no proof that a theory of strong interactions satisfying scale invariance at short distances will give the numerical $\pi^0 \to \gamma\gamma$ rate correctly, but at least it will not be in error by a factor of m_π^4 (m_π^2 in the amplitude and m_π^4 in the decay rate).

E. Nonleptonic Weak Interactions

The effective Lagrangian for weak interactions can be analyzed qualitatively, using the hypotheses of this paper. Octet dominance in nonleptonic decays can be understood without introducing neutral currents and some new "four-dimensional" superconvergence relations are proposed. Violation of CP is ignored.

Consider the $\Delta Y = 1$ component of the effective weak-interaction Hamiltonian. It will be assumed to have the form

$$\mathfrak{L}_W(0) = \tfrac{1}{2}\sqrt{2}G \cos\theta \sin\theta \int_x Tj_{\mu W}(x) j_{\nu S}(0) W^{\mu\nu}(x,M), \tag{7.23}$$

where G is the Fermi constant, θ is the Cabibbo angle, $j_{\mu W}$ is the nonstrange weak current with $\Delta Q = -1$, and $j_{\nu S}$ is the $\Delta Y = 1$, $\Delta Q = 1$ strangeness-changing current. The function $W^{\mu\nu}(x,M)$ might be a W-meson propagator (apart from a scale factor), or it might be a more complicated function. M is the effective cutoff mass of $W^{\mu\nu}(x,M)$, i.e., $W^{\mu\nu}(x,M)$ falls off exponentially when xM is large and spacelike; it is assumed that dimensionally $W^{\mu\nu}(x,M)$ is of the form $M^4 W^{\mu\nu}(xM)$. The function $W^{\mu\nu}(x,M)$ is assumed to satisfy the normalization condition

$$\int_x W^{\mu\nu}(x,M) = g^{\mu\nu}. \tag{7.24}$$

This condition arises if the same cutoff function occurs in semileptonic weak decays. The cutoff mass M will be assumed to be large compared to typical strong-interaction masses, i.e., $M \gg 1$ GeV.

Because $W^{\mu\nu}(x,M)$ has a short range in position space, the integral in (7.23) involves only small x, and in this region one can write an operator-product expansion for $Tj_{\mu W}(x) j_{\nu S}(0)$. This will have the form

$$Tj_{\mu W}(x) j_{\nu S}(0) = \textstyle\sum_n C_{n\mu\nu}{}^W(x) O_n(0), \tag{7.25}$$

which, substituted in Eq. (7.23), gives

$$\mathfrak{L}_W(0) = \tfrac{1}{2}\sqrt{2}G \cos\theta \sin\theta \textstyle\sum_n C_n(M) O_n(0), \tag{7.26}$$

with

$$C_n(M) = \int_x C_{n\mu\nu}{}^W(x) W^{\mu\nu}(x,M). \tag{7.27}$$

Because of exact symmetry requirements the fields $O_n(0)$ that contribute to Eq. (7.26) must be scalar or pseudoscalar fields with $\Delta Y = 1$, $\Delta I_z = -\frac{1}{2}$, and $\Delta I = \frac{1}{2}$ or $\frac{3}{2}$.

The leading term in $C_{n\mu\nu}{}^W(x)$, for any given n and small x, scales as $x^{\alpha(n)}$, where $\alpha(n)$ depends on the dimension of $O_n(d_n)$ and the number of symmetry-breaking spurions needed to make $C_{n\mu\nu}{}^W(x)$ nonzero. If no spurions are needed, then

$$\alpha(n) = d_n - 6. \tag{7.28}$$

By dimensional analysis, $C_n(M)$ behaves as

$$C_n(M) \propto M^{-\alpha(n)}. \tag{7.29}$$

In free-field models of current algebra there are an infinite number of $SU(3) \times SU(3)$ multiplets of scalar and pseudoscalar fields which can contribute to the expansion (7.26) for $\mathfrak{L}_W(0)$. But if M is large, the coefficients $C_n(M)$ will be very different for different multiplets because different multiplets usually have different values for $\alpha(n)$. So only one or a few multiplets will dominate in Eq. (7.26); apart from symmetry-breaking considerations, the multiplet of lowest dimension will have the largest coefficient. The ϕ-σ multiplet is the multiplet of lowest dimension, by assumption. The relevant fields for Eq. (7.26) are the $\Delta Y = 1$ σ and ϕ fields; however, these fields are both divergences (of the strangeness-changing vector and axial-vector currents), so they will not contribute to any decay process.[48] Hence one must go beyond the list of fields proposed in Sec. IV.

In the free-quark model, the next scalar and pseudoscalar multiplets with appropriate quantum numbers have dimension 6 (Wick products of two currents).

[46] The ambiguity in I_1 associated with different limiting procedures is analogous to the ambiguity in the Fourier transform of $F_{\mu\nu\pi}(x,z)$ already pointed out by Bell and Jackiw (Ref. 41) and Adler (Ref. 41).

[47] Field algebraists claim that the vacuum expectation value of two currents $\langle\Omega|Tj_\mu(x)j_\nu(0)|\Omega\rangle$ is less singular at $x=0$ in the algebra of fields than it is in an approximately scale-invariant theory such as the quark model. See J. Dooher, Phys. Rev. Letters **19**, 600 (1967); M. B. Halpern and G. Segrè, *ibid.* **19**, 611 (1967).

[48] See S. L. Adler and R. F. Dashen (Ref. 31), p. 133; C. Bouchiat, J. Iliopoulos, and J. Prentki, CERN Report No. Th.908 (unpublished); S. Nussinov and G. Preparata, Phys. Rev. **175**, 2180 (1968).

There are several irreducible multiplets with the same dimension—in fact, all multiplets contained in the product of two currents. In strong interactions, the author considers such degeneracy in dimension unlikely: Renormalization effects should be different for each different irreducible $SU(3)\times SU(3)$ multiplet. This means that one can hope that a single $SU(3)\times SU(3)$ multiplet will dominate Eq. (7.26). The experimental fact of octet dominance indicates that the dominant multiplet should contain only $SU(3)$ octets and singlets; this means that the dominant multiplet must be either an $(8,1)\oplus(1,8)$ or a $(3,\bar{3})\oplus(\bar{3},3)$ multiplet. The author knows of no way to distinguish these two possibilities, but will guess that the dominant multiplet is $(8,1)\oplus(1,8)$. This mutliplet will be denoted σ'-ϕ'. Its dimension will be called Δ'. Neglecting other multiplets in Eq. (7.26), and denoting the $\Delta Y=1$ $I_z=-\frac{1}{2}$ fields by σ_5' and ϕ_5', the effective weak $\Delta Y=1$ Lagrangian is

$$\mathcal{L}_W(0)=\tfrac{1}{2}\sqrt{2}G\cos\theta\sin\theta\, aM^{6-\Delta'}[\sigma_5'(0)-\phi_5'(0)], \quad (7.30)$$

where a is an unknown constant depending only on strong interactions. The combination σ_5'-ϕ_5' results from assuming pure chiral weak currents (equal vector and axial-vector Cabibbo angles).

Formula (7.30) is not expected to be exact, but the error will depend inversely as some power of the weak cutoff M. The ratio of K^+ to K_{0S} lifetimes should depend on the weak cutoff M, since the K^+ decay will be determined by a multiplet containing $\Delta I=\frac{3}{2}$ fields which will appear as one of the corrections to Eq. (7.30).[49] Equation (7.30) predicts octet dominance in all strangeness-changing weak processes. It must be made clear that octet dominance is not a prediction of the hypotheses of this paper. The above analysis simply allows one to incorporate the experimentally observed octet dominance into the theory without introducing neutral currents into the current-current Lagrangian.

The factor $M^{6-\Delta'}$ in Eq. (7.30) means that nonleptonic decays can be enhanced or suppressed compared to semileptonic processes, depending on the value of Δ'. However, recent calculations[50] using a local current-current weak Lagrangian suggest that neither enhancement nor suppression occurs. The relation of these calculations to the operator-product analysis will now be discussed.[51]

The calculations of Ref. 50 involve (for baryon decays) eliminating the pion by current algebra, after which one has a matrix element of the form $\langle A|j_{\mu N}(0)j_S^{\mu}(0)|B\rangle$ (denoted M_{AB}), where $|A\rangle$ and $|B\rangle$ are one-baryon states (for example, $|A\rangle$ might be a nucleon and $|B\rangle$ a Σ state); $j_{\mu N}(0)$ is a nonstrange current and $j_S^{\mu}(0)$ a strangeness-changing current. It is assumed that the product of currents at the same point exists. It is calculated by putting in a complete set of intermediate states; in practice only a few single-particle or resonance states are kept. Define $M_{AB}(q)$ to be the sum over those intermediate states with momentum q relative to the initial state; then one computes first $M_{AB}(q)$, and one has

$$M_{AB}=\int_q M_{AB}(q)\,. \quad (7.31)$$

To compare with the operator-product analysis, some further assumptions will be made. Assume that Δ' is either equal to 6 or slightly smaller. This means that for the $\sigma'-\phi'$ multiplet the constant $\alpha(n)$ is almost zero. Assume that for all other scalar-pseudoscalar multiplets (except σ-ϕ), $\alpha(n)$ is appreciably above zero. Then one can write an expansion for the ordinary product $j_{\mu N}(x)j_S^{\mu}(0)$:

$$j_{\mu N}(x)j_S^{\mu}(0)=B_1(x)\sigma_m(0)+B_2(x)\phi_m(0) +B'(x)[\sigma_m'(0)-\phi_m'(0)], \quad (7.32)$$

where m is 4 or 5. The functions $B_1(x)$ and $B_2(x)$ scale as x^{-2} for small x (see Sec. VII A), $B'(x)$ as $x^{-6+\Delta'}$. The σ-ϕ multiplet must be considered because the baryon states $|A\rangle$ and $|B\rangle$ do not have the same energy. The error in Eq. (7.32) goes to zero as $x\to 0$. As a result, one can write the following momentum-space formula:

$$\int_q [M_{AB}(q)-B_1(q)\sigma_{AB}-B_2(q)\phi_{AB} -B'(q)(\sigma_{AB}'-\phi_{AB}')]=0\,, \quad (7.33)$$

where $B_1(q)$, etc., are the Fourier transforms of $B_1(x)$, etc., and

$$\sigma_{AB}=\langle A|\sigma_m(0)|B\rangle\,,\ \text{etc.} \quad (7.34)$$

Because the integral converges, one can hope that only moderate values of q are the most important values and cut if off at a not too large $q_{\max}$. Then one has

$$\int_q^{q_{\max}} M_{AB}(q)=B_{I1}(q_{\max})\sigma_{AB}+B_{I2}(q_{\max})\phi_{AB} +B_I'(q_{\max})(\sigma_{AB}'-\phi_{AB}')\,, \quad (7.35)$$

where

$$B_{I1}(q_{\max})=\int_q^{q_{\max}} B_1(q)\,,\ \text{etc.} \quad (7.36)$$

Now $B_{I1}(q_{\max})$ and $B_{I2}(q_{\max})$ behave as $q_{\max}^2$ for large $q_{\max}$, owing to the behavior x^{-2} of $B_1(x)$ and $B_2(x)$. For very large $q_{\max}$, $B_I'(q_{\max})$ behaves as $q_{\max}^{(6-\Delta')}$, but for not too large $q_{\max}$, $B_I'(q_{\max})$ is approximately 1 due to $(x)^{\Delta'-6}$ being $\simeq 1$ except at very small x.

What one needs for calculating weak-interaction matrix elements are the matrix elements σ_{AB}' and ϕ_{AB}'.

[49] V. S. Mathur and P. Olesen, Phys. Rev. Letters **20**, 1527 (1968). [The specific mechanism of these authors for making the violation of $\Delta I=\frac{1}{2}$ be cutoff-dependent does not work (Ref. 48).]

[50] Y. Hara, Progr. Theoret. Phys. (Kyoto) **37**, 710 (1967); Y. T. Chiu and J. Schechter, Phys. Rev. Letters **16**, 1022 (1966); Y. T. Chiu, J. Schechter, and Y. Ueda, Phys. Rev. **150**, 1201 (1966); **157**, 1317 (1967); S. Biswas, A. Kumar, and R. Saxena, Phys. Rev. Letters **17**, 268 (1966); S. Nussinov and G. Preparata (Ref. 48).

[51] I wish to thank Professor J. Schechter for raising this question.

So one must somehow eliminate the matrix elements σ_{AB} and ϕ_{AB} in Eq. (7.35). One can do this by going to the exact $SU(3)\times SU(3)$ limit. The matrix elements σ_{AB} and ϕ_{AB} do not vanish in this limit but the functions $B_1(x)$ and $B_2(x)$ do [because jj and σ-ϕ belong to orthogonal $SU(3)\times SU(3)$ multiplets]. If $B_I'(q_{\max})$ is approximately 1, then Eq. (7.35) gives the desired matrix elements. If Δ' is somewhat different from 6, the matrix element σ_{AB}' would differ from the matrix elements needed for the weak-decay calculations only by a universal factor depending on $q_{\max}$ and M, but not the states A and B. Even if one cannot eliminate the σ_{AB} matrix elements, one can derive sum rules where σ_{AB}, etc., all vanish, namely, by taking the linear combination of Σ to nucleon matrix elements which projects out the $\Delta I=\frac{3}{2}$ part of the current-current product. Call this linear combination $M_{\Sigma N\frac{3}{2}}(q)$; then we predict

$$\int_q M_{\Sigma N\frac{3}{2}}(q)=0. \tag{7.37}$$

One might call this a four-dimensional superconvergence relation. The author is not prepared to discuss the validity in practice of either Eq. (7.35) or (7.37).

F. Bjorken Limit

As a final application, consider the Bjorken limit.[3] The problem is to determine the behavior of an amplitude

$$T_{\alpha\beta}(q)=\langle\alpha|\int_x e^{iq\cdot x}TA(x)B(0)|\beta\rangle, \tag{7.38}$$

when $q_0\rightarrow\infty$ with $\mathbf{q}$ fixed. With the operator-product expansion, there is no need to keep $\mathbf{q}$ fixed; one only has to have q^2 large. Then the integral is dominated by the behavior of the T product for small x; if $C_n(x)$ are expansion functions for the T product, then

$$T_{\alpha\beta}(q)\simeq\sum_n\langle\alpha|O_n(0)|\beta\rangle\int_x e^{iq\cdot x}C_n(x), \tag{7.39}$$

which can be written

$$T_{\alpha\beta}(q)\simeq\sum_n R_n(q)\langle\alpha|O_n(0)|\beta\rangle. \tag{7.40}$$

By dimensional analysis, $R_n(q)$ contains a skeleton term scaling as

$$R_n(q)\sim q^{d_A+d_B-d(n)-4}$$

for large q. There are also finite mass corrections which are smaller. Because of Lorentz invariance, each term in $R_n(q)$ will be a power of q^2, not necessarily an integral power, multiplying a polynomial in q. There may also be logarithms of q^2. To obtain the Bjorken limit specifically, one can let $q_0\rightarrow\infty$ with $\mathbf{q}$ held fixed, which means that q^2 is replaced by $q_0{}^2$ to a first approximation, and one gets an expansion in terms of $q_0{}^{-1}$, but one may get fractional powers of $q_0{}^{-1}$ in the expansion. The fractional powers are seldom of importance when A and B are two currents. They will be important if A and B are both from the ϕ-σ multiplet and $O_n(0)$ is a current; then a fractional power dominates $R_n(q)$ unless Δ is an integer or half-integer.

VIII. FINAL COMMENTS

What is proposed here is a new language for describing the short-distance behavior of fields in strong interactions. One talks about operator-product expansions for products of operators near the same point, instead of equal-time commutators. One discusses the dimension of an operator instead of how it is formed from products of canonical fields. Analyses of divergences in radiative corrections, etc., are carried out in position space rather than momentum space. Furthermore, one has qualitative rules for the strength of $SU(3)\times SU(3)$-symmetry-violating corrections at short distances. To the extent that one can analyze problems at short distances using only the $SU(3)\times SU(3)$ currents and the σ-ϕ multiplet, the hypotheses of this paper have the elegance of simplicity, once one is used to the language. Even in the nonleptonic weak interactions, where a new multiplet σ'-ϕ' is introduced, one can easily obtain a rapport between theory and the experimentally observed octet dominance. The results of the hypotheses are all qualitative, but with them one can resolve some of the qualitative difficulties with previous current-algebra calculations of $\eta\rightarrow 3\pi$ and $\pi^0\rightarrow 2\gamma$ decay.

There are formidable obstacles to be overcome before the hypotheses of this paper can be made quantitative. This is best seen by returning to the $\pi^0\rightarrow 2\gamma$ problem. To calculate the $\pi^0\rightarrow 2\gamma$ rate, one needs to know the vertex function $\langle\Omega|Tj_\mu(x)j_\nu(0)A_\pi(z)|\Omega\rangle$ when x and z are small. One needs this matrix element for the hadron skeleton theory. However, if one knows this matrix element for small x and z, one knows it for *all* x and z in the hadron skeleton theory due to scale invariance. It is hard to imagine that one could have a complete formula for this vertex function without having a complete solution of the hardon skeleton theory. The prospects for obtaining such a solution seem dim at present.

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[52] See, e.g., L. Kadanoff, W. Götze, D. Hamblen, R. Hecht, E. A. S. Lewis, V. V. Palciauskas, M. Rayl, J. Swift, D. Aspnes, and J. Kane, Rev. Mod. Phys. 39, 395 (1967).