Relativistic Corrections to the Conductivity of a Collisional Plasma in a Magnetic Field

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The electrical conductivity tensor of a many component, relativistic plasma in a constant magnetic field, and near equilibrium is obtained. The collision term in the kinetic equation is left arbitrary, and the collisional part of the conductivity tensor is expressed as a momentum integral over the collision term directly. As an application, the conductivity due to electron motion is calculated to first order in both the relativistic and thermal corrections with the Beliaev-Budker collision integral. The thermal corrections are the same as in the non-relativistic theory. The leading contributions to the conductivity from the ion motion are also obtained to first order in the electron relativistic corrections.

I. INTRODUCTION

In the preceding paper[1] (hereafter referred to as I), a general method was developed for the direct calculation of the collisional contribution to the conductivity tensor of a nonrelativistic plasma in a constant magnetic field.

In the present paper we extend the discussion of I to the relativistic domain, and obtain the corresponding total conductivity (collision term arbitrary) for an s component, relativistic plasma. Whereas in I particle velocity variables were used, here it is convenient to do the calculations in terms of the particle momenta.

As an application of the general result, the conductivity is calculated to first order in the collision parameter from the Beliaev-Budker equation.[2] This equation is the relativistic analog of the Landau equation. Specifically, we consider the electron current, and in keeping with the neglect of radiation (see Krizan[3]), we restrict the discussion to order $(v/c)^2$. Thermal corrections are assumed small, and products of these by relativistic corrections are neglected. Thus, we need actually calculate only the $k=0$ part of the conductivity. The k^2 (thermal) corrections are identical to those given in the Landau equation calculation of I. The wavelength independent contributions to the conductivity tensor from ion motion are then obtained, and these include first-order relativistic corrections resulting from the electron motion in electron-ion and ion-electron collisions.

II. CONDUCTIVITY TENSOR FOR ARBITRARY COLLISION TERM

We consider a neutral, relativistic plasma composed of s species of charged particles. A uniform magnetic field is present, and the distribution function of the ith species is assumed to be perturbed slightly from its equilibrium value as in Eq. (2) of I. The equilibrium distribution $f_0^{(i)}$ is the relativistic Maxwellian at (common) temperature T, where

$$f_0^{(i)}=A^{(i)}\exp[-(\gamma^{(i)}-1)\xi^{(i)}],$$
$$\xi^{(i)}=m^{(i)}c^2/KT,$$
$$\gamma^{(i)}=[1+(p^{(i)}/m^{(i)}c)^2]^{\frac{1}{2}}, \tag{1}$$

and $A^{(i)}$ is the normalization constant. The species superscript will be suppressed whenever possible.

With the fixed coordinate system of I, introducing the cylindrical momentum coordinates

$$\vec{p}=u\cos\phi\hat{e}_x+u\sin\phi\hat{e}_y+p_z\hat{e}_z, \tag{2}$$

and using the relation

$$\vec{p}=\gamma m\vec{v}, \tag{3}$$

the linearized, relativistic kinetic equation for the perturbed distribution function $(f=f_1+f_c)$ becomes

$$\frac{\partial f}{\partial\phi}+i\gamma(a-b\cos\phi)f=\frac{\gamma q\vec{E}}{\Omega}\cdot\frac{\partial f_0}{\partial\vec{p}}-\frac{\gamma C(f)}{\Omega}. \tag{4}$$

$C(f)$ is the arbitrary, linearized, relativistic collision term. From Eq. (3), we see that Eq. (4) has the same explicit form in the ϕ variable as for the nonrelativistic case [see Eq. (4) of I]. Thus, the procedure of I applies (the calculations are done in terms of momenta here rather than velocity), and it is straightforward to show that the conductivity tensor σ_{kl} is obtained from

$$\sigma_{kl}E_l=\sum_{i=1}^{s}[q^{(i)}\int d^3p^{(i)}v_k^{(i)}f_1^{(i)}+\int d^3p^{(i)}\Psi_k^{(i)}C^{(i)}(f^{(i)})]. \tag{5}$$

In Eq. (5) $f_1=f_0\psi_lE_l/KT$,

$$(\psi_x, \psi_y, \psi_z) = iq\gamma \sum_{n=-\infty}^{\infty} \times \left(\frac{nwJ_n(\gamma b)}{\gamma b}, \frac{iwdJ_n(\gamma b)}{d(\gamma b)}, v_z J_n(\gamma b) \right) \times \exp[i(\gamma b \sin\phi - n\phi)]/(\gamma a - n)\Omega \ . \tag{6}$$

$\vec{\psi}$ is the vector obtained from Eq. (10) of I by replacing a, b by γa, γb and then multiplying by γ, and

$$\tilde{\psi}_k(\vec{p}, \Omega) = \psi_k(\vec{p}, -\Omega) . \tag{7}$$

Equation (5) is exact in the sense that it is valid to all orders in the collision parameter. From here on, however, we limit the discussion to first order in the collision parameter (collision effects assumed small) by replacing $C(f)$ by $C(f_1)$. Using Eq. (15) of I, we obtain to this order

$$\sigma_{kl} = (KT)^{-1} \sum_{i=1}^{S} \times [q^{(i)} \int d^3p^{(i)} v_k^{(i)} f_0^{(i)} \psi_l^{(i)} + \sum_{j=1}^{S} \times \int d^3p^{(i)} \tilde{\psi}_k^{(i)} C^{(ij)}(\psi_l^{(i)} f_0^{(i)}; \psi_l^{(j)} f_0^{(j)})] . \tag{8}$$

It is worth emphasizing that the relativistic corrections do not alter the explicit form in the ϕ variable of the linearized kinetic equation. For this reason, the discussion of I is extended naturally to yield a conductivity which is formally the same as in the nonrelativistic case.

III. CONDUCTIVITY TENSOR FROM THE BELIAEV-BUDKER EQUATION

A. General Results

In this section we calculate the conductivity tensor using the Beliaev-Budker[2] collision integral. Substituting the linearized Beliaev-Budker collision term into Eq. (8), and integrating by parts in the p_m variable, we obtain

$$\sigma_{kl}^{(c)} = -\frac{2\pi}{KT} \sum_{i,j=1}^{S} (q^{(i)} q^{(j)})^2 L^{(ij)} \int d^3p^{(i)} \int d^3p^{(j)} \frac{\partial \tilde{\psi}_k^{(i)}}{\partial p_m^{(i)}} \frac{W_{mn}^{(ij)}}{\gamma^{(i)} \gamma^{(j)}} \left(\frac{\partial \psi_l^{(i)}}{\partial p_n^{(i)}} - \frac{\partial \psi_l^{(j)}}{\partial p_n^{(j)}} \right) f_0^{(i)} f_0^{(j)} . \tag{9}$$

$L^{(ji)} = L^{(ij)}$ is the Coulomb logarithm, and, with $\vec{P} = \vec{p}/mc\gamma$,

$$W_{nm}^{(ij)} = W_{mn}^{(ji)} = W_{mn}^{(ij)} = \frac{(\gamma^{(i)}\gamma^{(j)})^2 (1 - P_\sigma^{(i)} P_\sigma^{(j)})^2}{c[(\gamma^{(i)}\gamma^{(j)})^2 (1 - P_\sigma^{(i)} P_\sigma^{(j)})^2 - 1]^{\frac{3}{2}}}$$

$$\times \{ [(\gamma^{(i)}\gamma^{(j)})^2 (1 - P_\sigma^{(i)} P_\sigma^{(j)})^2 - 1] \delta_{mn} - (\gamma^{(i)})^2 P_m^{(i)} P_n^{(i)} - (\gamma^{(j)})^2 P_m^{(j)} P_n^{(j)}$$

$$+ (\gamma^{(i)}\gamma^{(j)})^2 (1 - P_\sigma^{(i)} P_\sigma^{(j)}) (P_m^{(i)} P_n^{(j)} + P_m^{(j)} P_n^{(i)}) \} . \tag{10}$$

From Eqs. (9) and (10) it is easily verified that, as in the nonrelativistic case,

$$\sigma_{kl}^{(c)}(\Omega) = \sigma_{lk}^{(c)}(-\Omega) . \tag{11}$$

B. Electron Current

We now consider specifically the contribution to the conductivity from the electron current. The relativistic corrections to the electron motion are assumed small, and we retain only $(v/c)^2$ corrections to the conductivity. Ion motion is neglected altogether so that we ignore the electron-ion mass ratio compared to unity. We assume that "thermal" corrections to "cold plasma" theory are small by using Eqs. (24) and (25) of I, and do not consider the phenomenon of resonant damping (Landau damping, cyclotron damping). As discussed by Krizan,[3] a particle Hamiltonian is well defined to order $(v/c)^2$, and the use of the Beliaev-Budker equation to this order seems justified (at least for $\Omega \ll \omega_p$, $\omega < \omega_p$). Neglecting products of thermal by relativistic corrections, and remembering that the Beliaev-Budker equation is the relativistic generalization of the Landau equation, we see that we need calculate only the wavelength independent $(k=0)$ part of the conductivity denoted by $\sigma_{kl}(k=0)$. The k^2 corrections will be identical to those given in I. With the above assumptions, it is straightforward to show that, to lowest order in the relativistic correction, electron-electron collisions contribute nothing to $\sigma_{kl}(k=0)$, and that

$$\sigma_{kl}(k=0)=\frac{-e}{KT}\int d^3p v_k f_0 \psi_l(k=0)-\frac{2\pi Ze^4 L^{\mathrm{ei}} n_e m}{KT}\int d^3p\gamma\,\frac{\partial\psi_k}{\partial p_m}\left(\frac{\delta_{mn}}{p}-\frac{p_m p_n}{p^3}\right)\frac{\partial\psi_l}{\partial p_n}f_0\,, \tag{12}$$

where the second term is due to electron-ion collisions. In Eq. (12), the "average" ion charge Ze is defined by

$$Zn_e e^2=\sum_{j=1}^{s-1} n^{(j)}(q^{(j)})^2\,,$$

and to order $(v/c)^2$,

$$f_0=n_e(\xi/2\pi m^2c^2)^{3/2}(1-15/8\xi)(1+\xi p^4/8m^4c^4)\exp(-\xi p^2/2m^2c^2)\,. \tag{13}$$

Also, the vector $\vec{\psi}(k=0)$ is

$$[\psi_x(k=0),\ \psi_y(k=0),\ \psi_z(k=0)]=-\frac{ie}{m}\left(\frac{(p_x\omega\gamma-ip_y\Omega)}{(\omega^2\gamma^2-\Omega^2)}\,,\ \frac{i(p_x\Omega-ip_y\omega\gamma)}{(\omega^2\gamma^2-\Omega^2)}\,,\ \frac{p_z}{\omega\gamma}\right)\,, \tag{14}$$

where Ω is negative for the electrons. Using Eqs. (7), (13), and (14), the integrals are readily performed in Eq. (12) with the result that

$$\sigma_{kl}(k=0)=i\omega_p{}^2T_{kl}(k=0)/4\pi\omega+(2/\pi)^{1/2}\omega_p{}^3LZK_{kl}(k=0)/4\pi\omega^2\Lambda\,. \tag{15}$$

To the desired order, the only nonvanishing components of $T_{kl}(k=0)$, and $K_{kl}(k=0)$ are

$$\begin{aligned}
&T_{xx}=T_{yy}=[\omega^2/(\omega^2-\Omega^2)][1-(5/2\xi)(\omega^2+\Omega^2)/(\omega^2-\Omega^2)]\,,\\
&T_{xy}=-T_{yx}=[i\Omega\omega/(\omega^2-\Omega^2)][1-5\omega^2/\xi(\omega^2-\Omega^2)]\,,\quad T_{zz}=1-5/2\xi\,,\\
&K_{xx}=K_{yy}=[\omega^2(\omega^2+\Omega^2)/(\omega^2-\Omega^2)^2]\{1-[15/8\xi(\omega^4-\Omega^4)][\omega^2(\omega^2+3\Omega^2)+(\Omega^2/15)(3\omega^2+\Omega^2)]\}\,,\\
&K_{xy}=-K_{yx}=[2i\omega^3\Omega/(\omega^2-\Omega^2)^2]\{1-[15/16\xi(\omega^2-\Omega^2)][(3\omega^2+\Omega^2)+\tfrac{1}{15}(\omega^2+3\Omega^2)]\}\,,\\
&K_{zz}=1-15/8\xi\,.
\end{aligned} \tag{16}$$

Since $\sigma_{kl}(k=0)$ is linear in the normalization of f_0, a factor of

$$n_e(\xi/2\pi m^2c^2)^{3/2}(1-15/8\xi)$$

is contributed to each of the above components from that normalization.

C. Effects of Ion Motion

As discussed in I, ion motion cannot be ignored at low frequencies ($\omega \lesssim \Omega_i$). For completeness then, we now discuss briefly the effects of ion motion, and give the corresponding wavelength free contributions to the conductivity tensor. There are two contributions: those due to ion motion in the electron current, and those from the ion currents. To keep the discussion of reasonable length, we consider here only one species of ion with charge Ze, and density n_e/Z. Specifically, we are interested only in wavelength independent ($k=0$) results to first order in electron relativistic corrections, and we treat the ion motion nonrelativistically. Consequently, the noncollisional contributions will be those discussed in I, and ion-ion collisions will not contribute (the k^2 contributions from ion-ion collisions are those discussed in I). However, the effects of ion motion on the electron-ion collisional contribution to the electron current, and the ion-electron collisional contribution to the ion current will contain electron relativistic corrections. These terms are not difficult to calculate from Eq. (9), and we merely state the resulting additions to Eq. (16) denoted by K_{kl}':

$$K_{xx}'=K_{yy}'=Z^2\,\frac{m_e^2}{m_i^2}\,\frac{\omega^2(\omega^2+\Omega_i^2)}{(\omega^2-\Omega_i^2)^2}\left(1+\frac{1}{8\xi}\right)+\frac{2m_e Z\omega^2(\omega^2+\Omega_e\Omega_i)}{m_i(\omega^2-\Omega_i^2)(\omega^2-\Omega_e^2)}\left[1+\frac{1}{\xi}\left(\frac{\omega^2}{\omega^2+\Omega_e\Omega_i}-\frac{15(\omega^2+\Omega_e^2/15)}{8(\omega^2-\Omega_e^2)}\right)\right],$$

$$K_{xy}'=-K_{yx}'=2iZ^2\,\frac{m_e^2}{m_i^2}\,\frac{\omega^3\Omega_i}{(\omega^2-\Omega_i^2)^2}\left(1+\frac{1}{8\xi}\right)+2iZ\,\frac{m_e}{m_i}\,\frac{\omega^3\Omega_e}{(\omega^2-\Omega_i^2)(\omega^2-\Omega_e^2)}\left(1-\frac{15}{8\xi}\,\frac{(\omega^2+\Omega_e^2/15)}{(\omega^2-\Omega_e^2)}\right). \tag{17}$$

The K_{zz}' term of $(Zm_e/m_i)^2(1+1/8\xi)+2Z(m_e/m_i)(1-7/8\xi)$ is negligible at *all* frequencies.

IV. DISCUSSION

We have generalized the discussion of I to obtain an expression for the total conductivity of a collisional, relativistic plasma in a constant magnetic field [Eq. (5)]. Important simplifications result from being able to write $\overleftrightarrow{\sigma}$ as an integral over the collision term directly rather than over f_c. The conductivity, as it stands in Eq. (5), is in a convenient form for iteration. Iterating once, we obtained the conductivity tensor for a weakly collisional plasma to first order in the collision parameter [Eq. (8)]. This conductivity is formally identical to that in the nonrelativistic case. The derivation was easily extended to the relativistic case since relativistic corrections to the linearized kinetic equation do not alter the explicit ϕ nature of that equation.

As an application of the general result, the $k=0$ part of the conductivity due to the electron motion was calculated to lowest order in the relativistic correction from the Beliaev-Budker collision integral. The actual computation of $\overleftrightarrow{\sigma}(k=0)$ is surprisingly little complicated by the inclusion of relativistic effects to order $(v/c)^2$. The k^2 corrections to "cold plasma" theory are identical to the Landau equation results of Pytte[1] (products of relativistic by thermal corrections were ignored). The results are consistent with the dispersion relations for electron waves given by McBride[4] for propagation parallel, and perpendicular to the magnetic field. With the results of this paper and the preceding one (I), we have in effect derived the dispersion relation for electron wave propagation at an arbitrary angle with respect to the magnetic field to first order in the relativistic and thermal corrections. In particular, we are now in a position to calculate the collisional damping for *any* electron wave mode ($\omega \gg \Omega_i$) to first order in the relativistic and thermal corrections.

Contributions to the conductivity from the ion motion, which become important at low frequencies ($\omega \lesssim \Omega_i$), were calculated for $k=0$, and for a two-component plasma to first order in electron relativistic corrections, and the collision parameter.

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