Theory of Currents and the Nonstrong Interactions*

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General perturbations around Sugawara's theory of currents are considered. The usual forms for the semistrong, electromagnetic, and weak interactions are easily incorporated, and, using a modified interaction picture, a covariant Feynman-graphical approach to the perturbation is achieved. There is some degree of uniqueness in the allowed perturbation: If infinite polynomials are to be avoided in the theory, the perturbations must be of the form $V_\mu V^\mu$, or $V_\mu C^\mu$ (C^μ is a canonical vector field), or σ (the scalar density due to Bardakci, Frishman, and Halpern).

I. INTRODUCTION

WORK on Sugawara's theory of currents[1] has proceeded along several lines. (a) The limit procedure of Bardakci, Frishman, and Halpern,[2] relating the theory to the massive Yang-Mills theory, has led to the incorporation of electromagnetism, and at least two distinct ways[3] of introducing the semistrong interactions. (b) Fairly general canonical representations of the theory have been found,[4] corresponding roughly to the σ model of Gell-Mann and Lévy,[5] or a relativistic, second-quantized spinning top. Because of difficulties in perturbation expansions,[6] the particle content of the theory (even for the representations) remains an open question. (c) Certain new sum rules[7] and inequalities[8] based on the theory have been derived, and those that are testable seem to be in agreement with experiment.

Our purpose in this paper is to consider general perturbations around Sugawara's theory. This includes three questions and their answers: (1) Can all the usual semistrong, electromagnetic, and weak interactions be incorporated into the theory? (2) How does one actually perturb in such interactions? (3) Does the structure of the symmetric hadronic theory dictate,[9] to some degree, the forms of the perturbations possible? The answer to the first question is yes, and the details of the incorporation are found throughout the text. Our answer[10] to the second question is a modified-interaction picture, in which the operator time dependence is that of the symmetric theory. This results in a (covariant) Feynman-graphical expansion in the perturbations. The answer to the third question is somewhat striking: Within the framework of Sugawara's original assumption that the theory contain no infinite polynomials, the perturbations must be of the form $V_\mu V^\mu$ or $V_\mu C^\mu$, where C^μ is a canonical vector field, or σ, the scalar density of Ref. 2. The internal symmetry of these perturbations is, however, arbitrary.

The plan of the paper is as follows. In Sec. II, we work out the details of two representative types of perturbation, namely, (a) a "mass-mixing" model, applicable to semistrong interactions or current-current nonleptonic weak interactions, and (b) $SU(2)\otimes SU(2)$ nonleptonic weak interactions with W mesons. The Poincaré invariance of the interaction picture is checked explicitly. These are simple illustrations of the tech-

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[1] H. Sugawara, Phys. Rev. **170**, 1659 (1968); C. M. Sommerfield, *ibid.* **176**, 2019 (1968).

[2] K. Bardakci, Y. Frishman, and M. B. Halpern, Phys. Rev. **170**, 1353 (1968). It should be noted that, although the limit procedure often works (i.e., yields a consistent current theory), it is not guaranteed to do so. For example, the limit of a Yang-Mills model with symmetry breaking in the $F_{\mu\nu}$ terms ("vector" or "particle" mixing as opposed to the mass-mixing model of Ref. 3) is inconsistent. Such can usually be spotted quickly before the limit: The limit will not work if the divergences of the (Yang-Mills) fields contain $F_{\mu\nu}$.

[3] See Ref. 2 and H. Sugawara, Phys. Rev. Letters **21**, 772 (1968). The first method retains the usual algebra of currents, but at the expense of new scalar and pseudoscalar densities. One of the results of this paper is that this approach is the only way to introduce semistrong interactions while keeping the original algebra. The second method works entirely in terms of currents, but changes the algebra of currents. In the language of the spinning top, the first model corresponds to a sort of "potential" while the latter is an asymmetric top. It will be interesting to see which of these is the more realistic model.

[4] K. Bardakci and M. B. Halpern, Phys. Rev. **172**, 1542 (1968); H. Sugawara and M. Yoshimura, *ibid.* **173**, 1419 (1968). If the theory is taken as a classical field theory, these are the most general representations. As a quantum field theory, other solutions to the $(0,i)$ curl equations may exist.

[5] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

[6] If the obvious perturbation expansions are to be believed, then zero-mass (Goldstone) particles occur in the theory, and isospin invariance (including charge conservation) is spontaneously broken in the theory. Analogous expansions of the spinning top (in which angular momentum is not conserved) are easily found also, but these do not meet the boundary conditions. The hope is that "good" nonperturbative solutions exist for the current theory as well. To our knowledge, moreover, there is no reason why the theory, and, in particular, the given representations, cannot contain fermionic states.

[7] D. J. Gross, Phys. Rev. Letters **21**, 308 (1968); C. G. Callan and D. J. Gross, *ibid.* **21**, 311 (1968).

[8] S. Nussinov, University of California, Berkeley, Report (unpublished); D. J. Gross and S. Nussinov (unpublished).

[9] This is the spirit of R. F. Dashen and S. Frautchi, Phys. Rev. **145**, 1287 (1966).

[10] An alternate approach would be through Green's functions, staying in the Heisenberg picture.

niques necessary to treat two distinct cases, that is, perturbations with and without canonical fields. At the end of Sec. II, we mention the simpler case of perturbations in the scalar and pseudoscalar densities of Ref. 2.

Realistic models, involving combinations of the above cases, are given in Sec. III. In particular, we give results for the usual Cabibbo theory of weak interactions (with and without W mesons), with leptons and (mass-mixing) semistrong interactions. The useful perturbation here is in the weak interactions alone. Section IV is a collection of topics, beginning with the question of restrictions on the perturbations. The question of a "finite" current theory in the sense of Lee[11] and of Gell-Mann *et al.*[12] is also briefly discussed.

In essentially the usual manner,[13] the Appendix details a manifestly covariant (but frame-dependent) transition to the interaction picture.

II. SIMPLE MODELS

As mentioned in the Introduction, there are several distinct kinds of perturbations possible on the symmetric theory, characterized in general by the presence or absence of canonical fields. As the technique for calculating the perturbation differs somewhat from case to case, it is instructive to start with some simple examples of each. We begin with the case of no canonical fields.

Mass-Mixing Model

The most general mass-mixing Yang-Mills Lagrangian (see also Ref. 2) is[14]

$$\begin{aligned}\mathfrak{L}&=-\tfrac{1}{4}F_{a\mu\nu}F^{a\mu\nu}+\tfrac{1}{2}m_0^2\phi_{a\mu}D_{ab}\phi^{b\mu}\,,\\ F_{a\mu}&=\partial_\mu\phi_{a\nu}-\partial_\nu\phi_{a\mu}-\tfrac{1}{2}g_0C_{abc}[\phi_{b\mu},\phi_{c\nu}]_+\,,\end{aligned}\tag{2.1}$$

where D_{ab} is some real symmetric matrix. Going over to the Hamiltonian formalism, and taking the usual limit,[2] we obtain the most general mass-mixing current theory

$$\begin{aligned}\theta_{\mu\nu}&=(1/2C)(D^{-1})_{ab}\{[J_{a\mu},J_{b\nu}]_+\\&\qquad-g_{\mu\nu}J_{a\lambda}J^{b\lambda}\}\,,\\ [J_{a0}(\mathbf{x}),J_{b0}(\mathbf{y})]&=iC_{abc}J_{0c}(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\ [J_{a0}(\mathbf{x}),J_{bi}(\mathbf{y})]&=iD_{bc}C_{acd}(D^{-1})_{de}J_{ei}(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})\\&\qquad+iD_{ba}C\partial_i^{(x)}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\ [J_{ai}(\mathbf{x}),J_{bj}(\mathbf{y})]&=0\,,\end{aligned}\tag{2.2}$$

along with the associated dynamical equations

$$\begin{aligned}\partial^\mu J_{a\mu}&=-(1/2C)C_{abc}(D^{-1})_{cd}[J_{b\mu},J^{d\mu}]_+\,,\\ (D^{-1})_{ad}\{\partial_\mu J_{d\nu}-\partial_\nu J_{d\mu}\}&=(1/2C)C_{abc}(D^{-1})_{bd}\\&\qquad\times(D^{-1})_{cf}[J_{d\mu},J_{f\nu}]_+\,.\end{aligned}\tag{2.3}$$

In general, of course, D will be the unit matrix plus some perturbation. Our task is to find out how to calculate this perturbation. It will not do just to take matrix elements of the Hamiltonian because some of the interaction is in the algebra. This is a general problem for which we propose the following approach.

Interaction Picture

If we are to remain conventional in our approach, the first thing to do is get all the interaction in $\theta_{\mu\nu}$, i.e., prepare to get all the interaction out of the states. For simplicity, we discuss here the case of a diagonal mass mixing, $D_{ab}=\delta_{ab}\lambda_a$, but the generalization is straightforward. For this case, the algebra is swept free of interaction by the change of variable

$$\begin{aligned}\mathcal{J}_{a\mu}&\equiv(J_{a0},(1/\lambda_a)J_{ai})\,,\\ [\mathcal{J}_{a0}(\mathbf{x}),\mathcal{J}_{b0}(\mathbf{y})]&=iC_{abc}\mathcal{J}_{c0}(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\ [\mathcal{J}_{a0}(\mathbf{x}),\mathcal{J}_{bi}(\mathbf{y})]&=iC_{abc}\mathcal{J}_{ci}(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})\\&\qquad+iC\delta_{ab}\partial_i^{(x)}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\ [\mathcal{J}_{ai}(\mathbf{x}),\mathcal{J}_{bj}(\mathbf{y})]&=0\,.\end{aligned}\tag{2.4}$$

As detailed in the Appendix, $\mathcal{J}_{a\mu}$ is a frame-dependent vector, properly written as

$$\mathcal{J}_{a\mu}=(1/\lambda_a)J_{a\mu}+(1-1/\lambda_a)n_\mu n_\nu J^{a\nu}\,,\quad n_\mu n^\mu=1\,.\tag{2.5}$$

Throughout the text, we shall work in the (usual) frame $n^\mu=(1,0,0,0)$, contenting ourselves with the remark that it is common[15] to introduce such quantities when passing to the interaction picture in the presence of derivative coupling or spin-1 mesons.

In terms of $\mathcal{J}_{a\mu}$, we can rewrite $\theta_{\mu\nu}$:

$$\begin{aligned}\theta_{00}&=\theta_{00}^S(\mathcal{J})+\theta_{00}^I(\mathcal{J})\,,\\ \theta_{0i}&=\theta_{0i}^S(\mathcal{J})\,,\end{aligned}\tag{2.6}$$

and so on, where $\theta_{\mu\nu}^S(\mathcal{J})$ is the symmetric $\theta_{\mu\nu}$ written as a function now of $\mathcal{J}$, namely,

$$\theta_{\mu\nu}^S(\mathcal{J})=(1/2C)\{[\mathcal{J}_{a\mu},\mathcal{J}_{a\nu}]_+-g_{\mu\nu}\mathcal{J}_{a\sigma}\mathcal{J}^{a\sigma}\}\tag{2.7}$$

and $\theta_{00}^I(\mathcal{J})$, the interaction Hamiltonian density, is

$$\theta_{00}^I(\mathcal{J})=\frac{1-\lambda_a}{2C}\mathcal{J}_{a\mu}\mathcal{J}^{a\mu}+\frac{(1-\lambda_a)^2}{2C\lambda_a}\mathcal{J}_{a0}\mathcal{J}_{a0}\,.\tag{2.8}$$

Now we are ready for the interaction picture. Because the algebra of $\mathcal{J}_{a\mu}$ is that of the symmetric theory, we can hope to find a unitary transformation which leaves the operators with only the symmetric time dependence. Thus we seek a $U(t)$,

$$\mathcal{J}_{a\mu}(\mathbf{x},t)=U^{-1}(t)\mathcal{J}_{a\mu}{}^D(\mathbf{x},t)U(t)\,,\tag{2.9}$$

[11] T. D. Lee, Phys. Rev. **171**, 1731 (1968), and references therein.

[12] M. Gell-Mann, M. L. Goldberger, N. M. Kroll, and F. E. Low, this issue, Phys. Rev. **179**, 1518 (1969).

[13] See, e.g., P. T. Matthews, Phys. Rev. **76**, 1657 (1949).

[14] Where it overlaps, our notation is that of Ref. 2.

[15] See Ref. 13. For example, in a theory of pseudoscalar mesons derivative coupled to fermions $[\mathfrak{L}_I=-\lambda l_\mu\partial^\mu\varphi,\ l_\mu=\bar\psi\gamma_\mu\gamma_5\psi]$, the natural four-vector in the Heisenberg picture is $\pi^\mu=\partial^\mu\varphi-\lambda l^\mu$, but instead one introduces the frame-dependent vector $\tilde\pi^\mu=\partial^\mu\varphi-\lambda n^\mu n^\nu l_\nu$. In the usual frame ($n$ pure timelike), $\tilde\pi^\mu=(\partial^0\varphi-\lambda l^0,\partial^i\varphi)$. Then in the interaction picture $\tilde\pi^\mu$ becomes the frame-independent four-vector $\partial_\mu\varphi$. Similarly, our $\mathcal{J}_{a\mu}$ will be a four-vector in the interaction picture.

such that

$$\partial_\nu \mathcal{J}_{a\mu}{}^D(x) = i[P_\nu{}^D(\mathcal{J}^D), \mathcal{J}_{a\mu}{}^D(x)],$$
$$P_\nu{}^D(\mathcal{J}^D) = \int d^3x\, \theta_{0\nu}{}^S(\mathcal{J}^D). \tag{2.10}$$

This is easily accomplished in the usual manner, resulting in

$$U(t) = T \exp\left(-i\int_0^t dt \int d^3x\, \theta_{00}{}^I(\mathcal{J}^D)\right), \tag{2.11}$$

where T is the usual time ordering, along with the S matrix for the perturbation

$$S = T \exp\left(-i\int d^4x\, \theta_{00}{}^I(\mathcal{J}^D)\right). \tag{2.12}$$

To calculate the perturbation, this S matrix is to be taken, in the usual manner, between in and out states of the symmetric theory of currents.

Notice that, in the interaction picture, the generators of the Lorentz group $M_{\mu\nu}{}^D$ must be constructed out of $\theta_{\mu\nu}{}^S(\mathcal{J}^D)$ in the usual way

$$M_{\mu\nu}{}^D = \int d^3x [x_\mu \theta_{0\nu}{}^S(\mathcal{J}^D) - x_\nu \theta_{0\mu}(\mathcal{J}^D)]. \tag{2.13}$$

Thus, as promised in Ref. 15, $\mathcal{J}_{a\mu}{}^D(x)$ is a (frame-independent) four-vector[16]

$$[M_{\mu\nu}{}^D, \mathcal{J}_{a\rho}{}^D] = -i(x_\mu\partial_\nu - x_\nu\partial_\mu)\mathcal{J}_{a\rho}{}^D + i(g_{\nu\rho}\mathcal{J}_{a\mu}{}^D - g_{\mu\rho}\mathcal{J}_{a\nu}{}^D). \tag{2.14}$$

This completes the identification of $\mathcal{J}_{a\mu}{}^D(x)$ with the currents in the symmetric Sugawara theory. Thus the S matrix, Eq. (2.12), is a power series in the symmetric currents. The remaining problem is the Poincaré invariance of the S matrix.

During the rest of the discussion of this model, we will remain in the interaction picture. Thus we suppress the label D. The form of Eq. (2.8) is typical of derivative coupling theories, that is, $\theta_{00}{}^I(\mathcal{J})$ is not just a scalar, but the (0,0) component of a tensor. In fact, the extra noncovariant piece is just what is needed to make S a scalar, that is, to have Schwinger terms cancel seagulls.[17] Let us see how this goes in a perturbation expansion. Suppose $\lambda_a = 1 + \epsilon\Delta_a$, where $\epsilon \ll 1$; then to second order in ϵ,

$$\theta_{00}{}^I \cong -\frac{\epsilon\Delta_a}{2C}\mathcal{J}_{a\mu}\mathcal{J}^{a\mu} + \frac{\epsilon^2\Delta_a{}^2}{2C}\mathcal{J}_{a0}\mathcal{J}_{a0}. \tag{2.15}$$

Orders ϵ^0 and ϵ^1 are obviously covariant, while the second-order perturbation is

$$\frac{-\epsilon^2\Delta_a\Delta_b}{8C^2}\int d^4x \int d^4y\, T\{\mathcal{J}_{a\mu}(x)\mathcal{J}^{a\mu}(x), \mathcal{J}_{b\sigma}(y)\mathcal{J}^{b\sigma}(y)\} \frac{-i\epsilon^2\Delta_a{}^2}{2C}\int d^4x\, \mathcal{J}_{a0}(x)\mathcal{J}_{a0}(x). \tag{2.16}$$

The reader is invited to verify that this expression commutes with all the generators of the Poincaré group. (The only nontrivial commutators are, of course, with M_{0i}.) That is, the noncovariant part of the time-ordered product (Schwinger terms) exactly cancels the noncovariant part of the second term (seagull). Similarly, one can check covariance to all orders. In fact, however, this model satisfies a general sufficient condition for covariance of the S matrix, namely, Schwinger's[18] condition for both $\theta_{\mu\nu}{}^S$ and $\theta_{\mu\nu}$, together with

$$\int d^3x\, \theta_{0i}{}^I(x) = \int d^3x\, x_j\theta_{0i}{}^I(x) = 0, \quad j \neq i.$$

The proof of this sufficient condition will be given elsewhere.[19]

We end our discussion of this model with the comment that, evidently, our analysis suffices to give a Feynman-graphical interpretation to semistrong interactions (in the mass-mixing model), and to current-current nonleptonic weak interactions (with a nondiagonal D_{ab}). The details of the weak theory will be given in Sec. III.

Perturbation by Canonical Field

As a simple example of perturbation via a canonical field, we shall discuss the case of $SU(2)\otimes SU(2)$ nonleptonic weak interactions with W mesons. We begin with the Lagrangian for W's coupled to the massive Yang-Mills field

$$\begin{aligned}
\mathcal{L} &= -\tfrac{1}{4}F_{a\mu\nu}(\tilde{\varphi})F^{a\mu\nu}(\tilde{\varphi}) + \tfrac{1}{2}m_0{}^2\varphi_{a\mu}\varphi^{a\mu} + \mathcal{L}_W, \\
\tilde{\varphi}_{a\mu} &= \varphi_{a\mu} + (\lambda/g_0)\xi_{ab}W_{b\mu}, \\
\xi_{ab} &= \delta_{a1}\delta_{b1} + \delta_{a2}\delta_{b2}, \\
\mathcal{L}_W &= -\tfrac{1}{4}G_{b\mu\nu}G^{b\mu\nu} + \tfrac{1}{2}\mu_0{}^2W_{b\mu}W^{b\mu}, \\
G_{b\mu\nu} &= \partial_\mu W_{b\nu} - \partial_\nu W_{b\mu}.
\end{aligned} \tag{2.17}$$

The internal index on φ_μ runs from 1 to 3, while the internal index on W_μ runs over 1 and 2; μ_0 is the (bare) W mass, and λ is the weak-coupling constant. As usual, a trivial chiral index (say, in ξ_{ab}) is implied which guarantees that W couples only to the left-handed combination of $\varphi_{a\mu}$.

This system is structurally analogous to electromagnetism[2]; taking the usual limit, we obtain the

[16] What has happened is what always happens in derivative-coupling theories: The frame dependence of the variables, introduced by quantizing along a special surface, has been cancelled by the frame dependence of $U(t)$. This is, of course, what interaction pictures are designed to do.

[17] L. S. Brown, Phys. Rev. **150**, 1338 (1966); S. G. Brown, *ibid.* **158**, 1444 (1967).

[18] J. Schwinger, Phys. Rev. **130**, 406 (1963); **130**, 800 (1963).

[19] D. J. Gross and M. B. Halpern (unpublished).

current theory with W mesons.

$$\theta_{\mu\nu}=\theta_{\mu\nu}{}^S(J)+\theta_{\mu\nu}{}^W, \tag{2.18}$$

where $\theta_{\mu\nu}{}^S(J)$ is the symmetric $SU(2)\otimes SU(2)$ Sugawara tensor, and

$$\theta_{\mu\nu}{}^W=\tfrac{1}{2}[G_{b\mu\lambda},G_b{}^\lambda{}_\nu]_+ +\tfrac{1}{2}\mu_0{}^2[W_{\mu b},W_{\nu b}]_+ -g_{\mu\nu}(-\tfrac{1}{4}G_{b\lambda\sigma}G^{b\lambda\sigma}+\tfrac{1}{2}\mu_0{}^2W_{b\lambda}W^{b\lambda}) \tag{2.19}$$

would be the free-W-meson stress-energy tensor, except that

$$W_{b0}=(1/\mu_0{}^2)(\lambda\xi_{ab}J_{a0}-\partial_iG_{b0i}). \tag{2.20}$$

This is to be taken with the algebra

$$\begin{aligned}
[J_{a0}(\mathbf{x}),J_{b0}(\mathbf{y})]&=iC_{abc}J_{c0}(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\
[J_{a0}(\mathbf{x}),J_{bi}(\mathbf{y})]&=iC_{abc}\tilde{J}_{ci}(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})+iC\delta_{ab}\partial_i{}^{(x)}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\
[G_{b0i}(\mathbf{x}),W_{aj}(\mathbf{y})]&=-i\delta_{ab}\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\
[J_{ai}(\mathbf{x}),G_{b0j}(\mathbf{y})]&=-i\lambda C\xi_{ab}\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\
\tilde{J}_{a\mu}&=J_{a\mu}+\lambda C\xi_{ab}W_{b\mu},
\end{aligned} \tag{2.21}$$

all other commutators vanishing. Taken together, $\theta_{\mu\nu}$ and the algebra imply the dynamical equations

$$\begin{aligned}
\partial_\mu\tilde{J}_{a\nu}-\partial_\nu\tilde{J}_{a\mu}&=(1/2C)C_{abc}[\tilde{J}_{b\mu},\tilde{J}_{c\nu}]_+,\\
\partial^\mu J_{a\mu}&=-\tfrac{1}{2}\lambda C_{abc}\xi_{cd}[J_{b\mu},W^{d\mu}]_+,\\
\partial^\mu G_{b\mu\nu}+\mu_0{}^2W_{b\nu}&=\lambda\xi_{ab}J_{a\nu}.
\end{aligned} \tag{2.22}$$

This completes the model. The task again is to calculate the perturbation.

As in the mass-mixing model, the first step is to eliminate the interaction from the algebra. This is accomplished by rewriting the theory in terms of the frame-dependent four-vector[20]

$$\mathcal{J}_{a\mu}\equiv(J_{a0},\tilde{J}_{ai}). \tag{2.23}$$

The algebra becomes

$$\begin{aligned}
[\mathcal{J}_{a0}(\mathbf{x}),\mathcal{J}_{b0}(\mathbf{y})]&=iC_{abc}\mathcal{J}_{c0}(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\
[\mathcal{J}_{a0}(\mathbf{x}),\mathcal{J}_{bi}(\mathbf{y})]&=iC_{abc}\mathcal{J}_{ci}(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})+iC\delta_{ab}\partial_i{}^{(x)}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\
[G_{b0i}(\mathbf{x}),W_{aj}(y)]&=-i\delta_{ab}\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}),
\end{aligned} \tag{2.24}$$

all other commutators vanishing. This is the algebra then of the symmetric theory in the presence of free W mesons. The stress-energy tensor goes over to

$$\begin{aligned}
\theta_{00}&=\theta_{00}{}^S(\mathcal{J})+\theta_{00}{}^{FW}+\theta_{00}{}^I,\\
\theta_{0i}&=\theta_{0i}{}^S(\mathcal{J})+\theta_{0i}{}^{FW},
\end{aligned} \tag{2.25}$$

and so on, where $\theta_{\mu\nu}{}^S(\mathcal{J})$ is the symmetric tensor, given in (2.7), $\theta_{\mu\nu}{}^{FW}$ has the form of a free-stress-energy tensor for W's,

$$\begin{aligned}
\theta_{00}{}^{FW}&=\tfrac{1}{2}G_{b0i}G_{b0i}+(1/2\mu_0{}^2)\partial_iG_{b0i}\partial_jG_{b0j}+\tfrac{1}{2}\mu_0{}^2W_{bi}W_{bi}+\tfrac{1}{4}G_{bij}G_{bij},
\end{aligned} \tag{2.26}$$

$$\theta_{0i}{}^{FW}=\tfrac{1}{2}[G_{b0j},G_b{}^j{}_i]_+-\tfrac{1}{2}[\partial_jG_{b0j},W_{ib}]_+,$$

and the interaction term is

$$\theta_{00}{}^I=\lambda\mathcal{J}_{b\mu}W^{b\mu}+(\lambda^2/2\mu_0{}^2)\mathcal{J}_{b0}\mathcal{J}_{b0}+\tfrac{1}{2}\lambda^2CW_{bi}W_{bi}. \tag{2.27}$$

In $\theta_{00}{}^I$, and from now on in this section, we mean $W_{b0}=-(1/\mu_0{}^2)\partial_iG_{b0i}$; that is, we have used (2.20) to eliminate the interaction from W_{b0}. Moreover, all isospin summations in $\theta_{00}{}^I$ run only over 1 and 2.

To go to the interaction picture, we seek again a $U(t)$ which leaves $\mathcal{J}_{a\mu}$ with the symmetric space-time dependence, and $W_{b\mu}$ a free field, i.e., the space-time dependence of the first two terms in θ_{00} and θ_{0i}. The result for the S matrix is

$$\begin{aligned}
S&=T\exp\left(-i\int d^4x\,\theta_{00}{}^I(\mathcal{J}^D,W^D)\right),\\
\theta_{00}{}^I(\mathcal{J}^D,W^D)&=\lambda\mathcal{J}_{b\mu}{}^DW_D{}^{b\mu}+(\lambda^2/2\mu_0{}^2)\mathcal{J}_{b0}{}^D\mathcal{J}_{b0}{}^D+\tfrac{1}{2}\lambda^2CW_{bi}{}^DW_{bi}{}^D,
\end{aligned} \tag{2.28}$$

where now, as before, $\mathcal{J}_{a\mu}{}^D$ and $W_{b\mu}{}^D$ are four-vectors (with respect to the interaction-picture Poincaré generators).

S is evidently covariant to zeroth and first orders in λ. The second-order perturbation is (suppressing D's)

$$-\tfrac{1}{2}\lambda^2\int d^4x\int d^4y\,T\{\mathcal{J}_{b\mu}(x)W^{b\mu}(x),\mathcal{J}_{c\nu}(y)W^{c\nu}(y)\} -\frac{i\lambda^2}{2\mu_0{}^2}\int d^4x\,\mathcal{J}_{b0}(x)\mathcal{J}_{b0}(x) -\tfrac{1}{2}i\lambda^2C\int d^4x\,W_{bi}(x)W_{bi}(x). \tag{2.29}$$

As in the mass-mixing model, one can check explicitly that this expression commutes with the Poincaré generators, so that Schwinger terms cancel seagulls. With intermediate boson theories, it is customary to be more explicit about covariance, exhibiting the results in terms of W propagators and covariant time-ordered products. Consider (2.29) between arbitrary (symmetric) states. The fact that $[\mathcal{J},W]=0$ allows us to write the first term as one time-ordered product for the $\mathcal{J}$'s, times another for the W's. Because the symmetric states contain no W's, we can drop the normal-ordered part of the W time-ordered product. Thus (2.29) is proportional to

$$T\{\mathcal{J}_{a\mu}(x),\mathcal{J}_{b\nu}(y)\}\langle 0|T\{W^{a\mu}(x),W^{b\nu}(y)\}|0\rangle +(i/\mu_0{}^2)\mathcal{J}_{b0}(x)\mathcal{J}_{b0}(x)\delta^{(4)}(x-y) +iCW_{bi}(x)W_{bi}(x)\delta^{(4)}(x-y). \tag{2.30}$$

Now define

$$\begin{aligned}
&\langle 0|T\{W_{a\mu}(x),W_{b\nu}(y)\}|0\rangle\\
&\quad\equiv\delta_{ab}[\Delta^{\mu\nu}(x-y)-(i/\mu_0{}^2)\delta_{\mu0}\delta_{\nu0}\delta^{(4)}(x-y)],\\
&T^*\{\mathcal{J}_{a\mu}(x),\mathcal{J}_{b\nu}(y)\}=T\{\mathcal{J}_{a\mu}(x),\mathcal{J}_{b\nu}(y)\}\\
&\qquad-i\delta_{ab}C(g_{\mu\nu}-\delta_{\mu0}\delta_{\nu0})\delta^{(4)}(x-y).
\end{aligned} \tag{2.31}$$

[20] In the frame-dependent language,

$$\mathcal{J}_{a\mu}=J_{a\mu}+\lambda C\xi_{ab}(g_{\mu\nu}-n_\mu n_\nu)W^\nu.$$

Then $\Delta^{\mu\nu}$ is the usual covariant W propagator;

$$\Delta^{\mu\nu}=(g^{\mu\nu}+\partial^\mu\partial^\nu/\mu_0^2)\tfrac{1}{2}\Delta_F(x-y)\,,\qquad (\Box^2+\mu_0^2)\Delta_F(x)=2i\delta^{(4)}(x)\,, \tag{2.32}$$

and T^* satisfies

$$\partial_{(x)}{}^\mu T^*\{\mathcal{J}_{a\mu}(x),\mathcal{J}_{b\nu}(y)\}=iC_{abc}J_{c\nu}(x)\delta^{(4)}(\mathbf{x}-\mathbf{y}) \tag{2.33}$$

as well as, in an obvious notation,

$$\begin{aligned}[M_{\mu\nu},T_{\lambda\rho}{}^*(x,0)]=&-i(x_\mu\partial_\nu-x_\nu\partial_\mu)T_{\lambda\rho}{}^*(x,0)\\ &+i(g_{\nu\lambda}T_{\mu\rho}{}^*-g_{\mu\lambda}T_{\nu\rho}{}^*)\\ &+i(g_{\nu\rho}T_{\lambda\mu}{}^*-g_{\mu\rho}T_{\lambda\nu}{}^*)\,.\end{aligned} \tag{2.34}$$

Thus T^* is the covariant time-ordered current product. Not surprisingly then, Eq. (2.30) is precisely equal to

$$T^*\{\mathcal{J}_{b\mu}(x),\mathcal{J}_{b\nu}(y)\}\Delta^{\mu\nu}(x-y)\,. \tag{2.35}$$

A similar analysis can be carried out to all orders. In fact, however, the sufficient conditions[19] mentioned above are again satisfied for this model. The S matrix is covariant to all orders.

An entirely analogous discussion can be given for electromagnetism.[2] Again all the noncovariant terms (including now the peculiarities of the Coulomb gauge) cancel. Thus, e.g., the Cottingham formula follows in the current theory.

Scalar and Pseudoscalar Densities

Our last model is the simplest. A model of semistrong interactions has been proposed[2] in which the algebra of currents stays the same, but scalar and pseudoscalar densities are introduced. As an example, we give the perturbation theory for the broken $SU(2)\otimes SU(2)$ model. The theory is defined by $\Delta\theta_{\mu\nu}(x)=-g_{\mu\nu}f_\pi\mu^2\sigma(x)$, and the algebra

$$\begin{aligned}&[A_{a0}(\mathbf{x}),\sigma(\mathbf{y})]=-i\varphi_a(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\ &[A_{a0}(\mathbf{x}),\varphi_b(\mathbf{y})]=i\sigma(\mathbf{x})\delta_{ab}\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\\ &[V_{a0}(\mathbf{x}),\varphi_b(\mathbf{y})]=i\epsilon_{abc}\varphi_c(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})\,,\end{aligned} \tag{2.36}$$

together with the symmetric algebra among $A_{a\mu}$, $V_{a\mu}$ (axial-vector and vector currents). All other commutators vanish. Because the algebra is already free of the interaction, going to the interaction picture is a simple matter. The S matrix is

$$S=T\exp\left(-if_\pi\mu^2\int d^4x\,\sigma^D(x)\right), \tag{2.37}$$

to be taken as usual between states of the symmetric theory. The space-time dependence of σ^D is, of course, that of $\theta_{\mu\nu}{}^S(V^D,A^D)$; that is,

$$\begin{aligned}\partial_\mu\sigma^D&=(1/2C)[A_{a\mu}{}^D,\varphi_a{}^D]_+\,,\\ \partial_\mu\varphi_a{}^D&=(1/2C)\{\epsilon_{abc}[V_{b\mu}{}^D,\varphi_c{}^D]_+-[A_{a\mu}{}^D,\sigma^D]_+\}\,,\end{aligned} \tag{2.38}$$

where $V_{a\mu}{}^D$ and $A_{a\mu}{}^D$ are the currents of the symmetric theory. σ^D is a scalar, which commutes with itself, so that S is trivially a scalar.

III. REALISTIC MODELS: WEAK AND SEMISTRONG INTERACTIONS

In Sec. II, we outlined the general methods for performing perturbation theory around Sugawara's theory of currents. We shall now present in some detail realistic models which include the weak interactions of hadrons and leptons (with and without W mesons), in the presence of semistrong interactions; in particular, we shall incorporate the Cabibbo theory[21] together with the mass-mixing model[3] for semistrong interactions. We shall omit the details of attaching the weak interactions to the symmetry-breaking scheme using scalar and pseudoscalar densities,[2] since this is relatively straightforward. We shall also omit electromagnetism, which was introduced into the theory in Ref. 2, and for which the passage to the interaction picture is similar to the case of W mesons.

Current-Current Cabibbo Theory

The theory and its perturbation expansion can be derived in two ways. As in Sec. II, we can start with a massive Yang-Mills theory, including nondiagonal mass-mixing and leptonic weak interactions, take the limit, and go to the interaction picture. Alternatively, we could take the symmetric energy-momentum tensor to be the free tensor in the interaction picture, and add to it a suitable interaction tensor in a way that preserves Poincaré invariance of the total theory. The second approach will be described in Sec. IV. Here we shall follow the usual method.

We begin with the Lagrangian

$$\begin{aligned}\mathcal{L}=&-\tfrac{1}{4}F_{a\mu\nu}F^{a\mu\nu}+\tfrac{1}{2}m_0^2D_{ab}\varphi_{a\mu}\varphi^{b\mu}+\bar\psi(i\partial-M)\psi\\ &-g_0G\sqrt{2}C\varphi^{a\mu}\eta_a[\xi_a{}^+l_\mu{}^-+\xi_a{}^-l_\mu{}^+]-\tfrac{1}{2}\sqrt{2}Gl_{\alpha\mu}l^{\alpha\mu}\,.\end{aligned} \tag{3.1}$$

Our notation is as follows. The $SU(3)\otimes SU(3)$ Yang-Mills fields are $\varphi_{a\mu}$, where Roman letters a, b, c go from 1 to 16, 1 to 8 being vector labels and 9 to 16 being axial vector. The lepton currents are $l_{\alpha\mu}=\frac{1}{2}\sqrt{2}\bar\psi\gamma_\mu\times(1-\gamma_5)\tau^\alpha\psi$ where greek letters α, β, $\gamma=1$, 2; ψ is a 16-component spinor:

$$\psi=\begin{pmatrix}\psi_e\\ \psi_{\nu_e}\\ \psi_\mu\\ \psi_{\nu_\mu}\end{pmatrix},\quad \tau^\alpha=\begin{pmatrix}\sigma^\alpha & 0\\ 0 & \sigma^\alpha\end{pmatrix}, \tag{3.2}$$

where σ^α are the Pauli matrices. M is the usual (diagonal) lepton mass matrix, and the weak leptonic currents are, of course, $l_\mu{}^\pm=\frac{1}{2}\sqrt{2}(l_\mu{}^1\pm il_\mu{}^2)$. The symmetric matrix D_{ab} is given by

$$D_{ab}=\eta_a\delta_{ab}-GC\sqrt{2}\eta_a(\xi_a{}^+\xi_b{}^-+\xi_a{}^-\xi_b{}^+)\eta_b\,, \tag{3.3}$$

[21] N. Cabibbo, Phys. Rev. Letters 10, 531 (1963).

in which the diagonal part $\eta_a\delta_{ab}$ describes the semistrong interactions,[3] while the second term describes the current-current weak interactions. G is the Fermi coupling constant, and $\xi_a^{\pm}$ is defined by the Cabibbo form for the hadronic fields (θ_C is the Cabibbo angle),

$$\xi_a^{\pm}\varphi_{a\mu}=[(\delta_{a1}-\delta_{a9}\pm i\delta_{a2}\mp i\delta_{a10})\cos\theta_C+(\delta_{a4}-\delta_{a12}\pm i\delta_{a5}\mp\delta_{a13})\sin\theta_C]\varphi_{a\mu}. \quad (3.4)$$

Taking the limit in the usual manner[22] leads to the total stress-energy-momentum tensor for the current theory:

$$\begin{aligned}\theta_{\mu\nu}=(1/2C)(D^{-1})_{ab}\{&[\tilde{J}_{a\mu},\tilde{J}_{b\nu}]_+-g_{\mu\nu}\tilde{J}_{a\lambda}\tilde{J}^{b\lambda}\\&-\tfrac{1}{2}\sqrt{2}GC\eta_a[\xi_a^+l_\mu^-+\xi_a^-l_\mu^+,\tilde{J}_{a\nu}]_+\\&-\tfrac{1}{2}\sqrt{2}GC\eta_a[\xi_a^+l_\nu^-+\xi_a^-l_\nu^+,\tilde{J}_{a\mu}]_+\}\\&+\tfrac{1}{4}i[\bar{\psi}(\gamma_\mu\partial_\nu+\gamma_\nu\partial_\mu)\psi-\partial_\mu\bar{\psi}\gamma_\nu\psi-\partial_\nu\bar{\psi}\gamma_\mu\psi]\\&-\tfrac{1}{2}\sqrt{2}Gg_{\mu\nu}l_{\alpha\lambda}l^{\alpha\lambda}. \quad (3.5)\end{aligned}$$

Here $J_{a\mu}$ are the $SU(3)\times SU(3)$ currents, but the more convenient variable at this stage is

$$\tilde{J}_{a\mu}\equiv J_{a\mu}+\sqrt{2}CG\eta_a(\xi_a^+l_\mu^-+\xi_a^-l_\mu^+). \quad (3.6)$$

Note that the hadronic weak current is simply

$$\begin{aligned}J_{W\mu}^{\pm}=[(\delta_{a1}-\delta_{a9}\pm i\delta_{a2}\mp i\delta_{a10})\cos\theta_C&+(\delta_{a4}-\delta_{a12}\pm i\delta_{a5}\mp i\delta_{a13})\sin\theta_C]J_{a\mu}\\&=\xi_a^{\pm}J_{a\mu}. \quad (3.7)\end{aligned}$$

This is to be taken with the algebra

$$\begin{aligned}&[J_{a0}(\mathbf{x}),J_{b0}(\mathbf{y})]=iC_{abc}J_{c0}(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\&[J_{a0}(\mathbf{x}),J_{bi}(\mathbf{y})]=iD_{bd}(D^{-1})_{ec}C_{ade}\tilde{J}_{ci}(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})+iD_{ab}C\partial_i^{(x)}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\&[\tilde{J}_{ai}(\mathbf{x}),\tilde{J}_{bj}(\mathbf{y})]=[\tilde{J}_{ai}(\mathbf{x}),l_{\beta\mu}(\mathbf{y})]=[J_{a0}(\mathbf{x}),l_{\beta\mu}(\mathbf{y})]=0,\\&[l_{\alpha0}(\mathbf{x}),l_{\beta\mu}(\mathbf{y})]=i\epsilon_{\alpha\beta\gamma}l_{\gamma\mu}(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}),\end{aligned} \quad (3.8)$$

plus the usual anticommutator for the lepton field with itself. In addition, the time derivatives of the lepton fields are to be evaluated (in the usual manner) via the lepton equation of motion

$$\begin{aligned}(i\gamma\cdot\partial-M)\psi=&\tfrac{1}{2}\sqrt{2}G\gamma_\mu(1+\gamma_5)\tau^\alpha\psi l^{\alpha\mu}\\&+\tfrac{1}{2}\sqrt{2}G(D^{-1})_{ab}\tilde{J}_{b\mu}\eta_a(\xi_a^+\tau^-+\xi_a^-\tau^+)\times\gamma^\mu(1+\gamma_5)\psi. \quad (3.9)\end{aligned}$$

It is easy to see that when $D_{ab}=\delta_{ab}$ and $G=0$, the energy-momentum tensor becomes $\theta_{\mu\nu}{}^S(\mathcal{J})+\theta_{\mu\nu}$(free leptons), and the algebra is the symmetric algebra of fields. It is much harder, but possible, to show that the system is Poincaré-invariant (and that $J_{a\mu},l_{\alpha\mu}$ transform as four-vectors).

The resulting equations of motion are

$$\begin{aligned}\partial^\mu J_{a\mu}&=(1/2C)C_{abc}(D^{-1})_{bd}[J_{d\mu},\tilde{J}^{c\mu}]_+,\\\partial^\mu\tilde{J}_{a\mu}&=(1/2C)C_{abc}(D^{-1})_{bd}[\tilde{J}_{d\mu},\tilde{J}^{c\mu}]_+,\\\partial_\mu\tilde{J}_{a\nu}-\partial_\nu\tilde{J}_{a\mu}&=(1/2C)C_{def}D_{ad}(D^{-1})_{eg}(D^{-1})_{fh}\times[\tilde{J}_{g\mu},\tilde{J}_{h\nu}]_+.\end{aligned} \quad (3.10)$$

The current-current interaction is rather well hidden in this system, but the perturbation theory will find it.

We can think of perturbing in the semistrong interactions, the weak interactions, or both. All cases can be done, but the semistrong perturbation was described in Sec. II, and the most interesting perturbation is presumably only in G anyway, so we shall confine ourselves to this case. To pass to the interaction picture, we define the frame-dependent four-vectors

$$\begin{aligned}\mathcal{J}_{a0}&\equiv J_{a0},\\\mathcal{J}_{ai}&\equiv\eta_a(D^{-1})_{ab}\tilde{J}_{bi}.\end{aligned} \quad (3.11)$$

As above, these structures are defined so as to remove the weak interactions from the algebra. That is, the algebra of $\mathcal{J}_{a\mu}$ is the mass-mixing algebra, and hadron variables commute with lepton variables. In terms of $\mathcal{J}_{a\mu}$, the stress-energy-momentum tensor may be rewritten as $\theta_{\mu\nu}=\theta_{\mu\nu}{}^F+\theta_{\mu\nu}{}^I$, where

$$\theta_{\mu\nu}{}^F=\frac{1}{2C}\frac{1}{\eta_a}\{[\mathcal{J}_{a\mu},\mathcal{J}_{a\nu}]_+-g_{\mu\nu}\mathcal{J}_{a\lambda}\mathcal{J}^{a\lambda}\}+\theta_{\mu\nu}{}^L, \quad (3.12)$$

$$\theta_{\mu\nu}{}^L=\tfrac{1}{4}i[\bar{\psi}(\gamma_\mu\partial_\nu+\gamma_\nu\partial_\mu)\psi-\partial_\mu\bar{\psi}\gamma_\nu\psi-\partial_\nu\bar{\psi}\gamma_\mu\psi], \quad (3.13)$$

where now the lepton time derivatives are to be evaluated via the free Dirac equation. $\theta_{\mu\nu}{}^F$ has the algebraic structure of the strong and semistrong interactions in the presence of free leptons. The interaction is described by

$$\begin{aligned}\theta_{0i}{}^I&=0,\\\theta_{00}{}^I&=\tfrac{1}{2}\sqrt{2}G[\mathcal{J}_{W\mu}{}^+,\mathcal{J}_{W}{}^{\mu-}]_++G^2C(D^{-1})_{ab}\eta_a\eta_b\times(\xi_a^+\mathcal{J}_{W0}{}^-+\xi_a^-\mathcal{J}_{W0}{}^+)\times(\xi_b^+\mathcal{J}_{W0}{}^-+\xi_b^-\mathcal{J}_{W0}{}^+),\end{aligned} \quad (3.14)$$

where

$$\mathcal{J}_{W\mu}{}^{\pm}=\xi_a^{\pm}\mathcal{J}_{a\mu}+l_\mu{}^{\pm} \quad (3.15)$$

will be the total weak-interaction current in the interaction picture. As above, the transition to the interaction picture is achieved by the unitary operator

$$U(t,0)=T\left\{\exp\left[-i\int_0^t d^4x\,\theta_{00}{}^I(\mathcal{J}^D)\right]\right\}$$

and the S matrix for the perturbation is $U(\infty,-\infty)$. In the interaction picture $\mathcal{J}_\mu{}^D$ and $l_\mu{}^D$ are four-vectors. We see immediately that, to first order in G, the theory describes the current-current Cabibbo interaction.[23]

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[22] During the limit, we define time components of the hadronic currents as the usual form $J_{a0}\equiv-(g_0)^{-1}\partial_i\pi_{ai}+C_{abc}\varphi_{bi}\pi_{ci}$; that is, the charges are "purely hadronic" in that they will not rotate the leptons. If desired, one can think of adding $l_{\alpha\mu}$ to $J_{a\mu}$ (with appropriate internal indices), and constructing the charges from these. As mentioned in Ref. 23, this is purely a matter of definition and does not change the theory.

[23] If we had followed the suggestion of Ref. 22, the Heisenberg theory would have looked different, but we would have insisted that $\theta_{\mu\nu}{}^F$ remain $\theta_{\mu\nu}$(Sugawara)$+\theta_{\mu\nu}$(free leptons), leading to the same perturbation. The point of Ref. 22 is still good: One might want to think of the interaction-picture charges as being constructed out of structures like (3.15).

Again, the S matrix may be (formally) shown to be Poincaré-invariant to all orders; that is, the non-covariant second term in $\theta_{00}{}^I$ (seagull) always cancels the noncovariant Schwinger terms arising from commutators of $\theta_{00}{}^I$ with itself.

Cabibbo Theory with W's

The mechanics of such models were outlined in Sec. II for the case of $SU(2)\otimes SU(2)$ with no semistrong interaction. In the case of $SU(3)\otimes SU(3)$ with mass-mixing symmetry breaking, we begin with the Lagrangian

$$\begin{aligned}\mathcal{L}&=-\tfrac{1}{4}\tilde{F}_{a\mu\nu}\tilde{F}^{a\mu\nu}+\tfrac{1}{2}m_0{}^2\eta_a\varphi_{a\mu}\varphi^{a\mu}-\tfrac{1}{4}G_{\alpha\mu\nu}G^{\alpha\mu\nu}\\&\quad+\tfrac{1}{2}\mu_0{}^2W_{\alpha\mu}W^{\alpha\mu}+\bar{\psi}(i\gamma\cdot\partial-M)\psi-gW_{\alpha\mu}l^{\alpha\mu},\\ \tilde{F}_{a\mu\nu}&=\partial_\mu\tilde{\varphi}_{a\nu}-\partial_\nu\tilde{\varphi}_{a\mu}-\tfrac{1}{2}g_0C_{abc}[\tilde{\varphi}_{b\mu},\tilde{\varphi}_{c\nu}]_+,\\ \tilde{\varphi}_{a\mu}&=\varphi_{a\mu}+(g/g_0)(\xi_a{}^+W_\mu{}^-+\xi_a{}^-W_\mu{}^+),\end{aligned}\tag{3.16}$$

where

$$g^2/\mu_0{}^2=\tfrac{1}{2}\sqrt{2}G.$$

In the usual limit, we obtain

$$\theta_{\mu\nu}=(1/2C)\eta_a{}^{-1}\{[J_{a\mu},J_{a\nu}]_+-g_{\mu\nu}(J_{a\lambda}J^{a\lambda})\}+\theta_{\mu\nu}{}^W+\theta_{\mu\nu}{}^L,\tag{3.17}$$

where $\theta_{\mu\nu}{}^W$ is given by (2.19), and would be the free-W-meson energy-momentum tensor except that

$$W_0{}^\pm=(1/\mu_0{}^2)[\partial_iG_{i0}{}^\pm+g(J_{W0}{}^\pm+l_0{}^\pm)].\tag{3.18}$$

The form of $\theta_{\mu\nu}{}^L$ is given in (3.13), except that now

$$i\gamma_0\partial_0\psi=(i\gamma_i\partial_i+M)\psi+\tfrac{1}{2}\sqrt{2}gW^{\alpha\mu}\gamma_\mu\tau^\alpha(1-\gamma_5)\psi.\tag{3.19}$$

In addition to the usual algebra among the leptons, the nonvanishing commutators are

$$\begin{aligned}&[J_{a0}(\mathbf{x}),J_{b0}(\mathbf{y})]=iC_{abc}J_{c0}(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\&[J_{a0}(\mathbf{x}),J_{bi}(\mathbf{y})]=i\eta_b\eta_c{}^{-1}C_{abc}\tilde{J}_{ci}(x)\delta^{(3)}(\mathbf{x}-\mathbf{y})\\&\qquad+i\eta_a\delta_{ab}C\partial_i{}^{(x)}\delta^{(3)}(x-y),\\&[G_{0i}{}^\beta(\mathbf{x}),W_j{}^\alpha(\mathbf{y})]=-i\delta_{\alpha\beta}\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\&[G_{0i}{}^\pm(\mathbf{x}),J_{bj}(\mathbf{y})]=igC\eta_b\xi_b{}^\pm\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\&\tilde{J}_{a\mu}=J_{a\mu}+\eta_aCg(\xi_a{}^+W_\mu{}^-+\xi_a{}^-W_\mu{}^+).\end{aligned}\tag{3.20}$$

The equations of motion are

$$\begin{aligned}&\partial^\mu J_{a\mu}=-(1/2C)C_{abc}\eta_c{}^{-1}[J_{b\mu},\tilde{J}^{c\mu}]_+,\\&\partial^\mu G_{\mu\nu}{}^\pm+\mu_0{}^2W_\nu{}^\pm=g(J_{W\nu}{}^\pm+l_\nu{}^\pm),\\&\partial_\mu\tilde{J}_{a\nu}-\partial_\nu\tilde{J}_{a\mu}=(1/2C)C_{abc}\eta^a\eta_b{}^{-1}\eta_c{}^{-1}[\tilde{J}_{b\mu},\tilde{J}_{d\nu}]_+.\end{aligned}\tag{3.21}$$

In order to perform perturbation theory in the weak interactions, we define the usual frame-dependent vectors;

$$\mathcal{J}_{a\mu}\equiv(J_{a0},\tilde{J}_{ai}).\tag{3.22}$$

The algebra of $\mathcal{J}_{a\mu}$ is free of weak interactions (in particular, it commutes with all W-meson variables). In terms of $\mathcal{J}_{a\mu}$, the stress-energy tensor $\theta_{\mu\nu}=\theta_{\mu\nu}{}^F+\theta_{\mu\nu}{}^I$ has the form

$$\begin{aligned}\theta_{\mu\nu}{}^F&=(1/2C)\eta_a\{[\mathcal{J}_{a\mu},\mathcal{J}_{a\nu}]_+-g_{\mu\nu}(\mathcal{J}_{a\lambda}\mathcal{J}^{a\lambda})\}\\&\quad+\theta_{\mu\nu}{}^W(\text{free})+\theta_{\mu\nu}{}^L(\text{free}),\\ \theta_{00}{}^I&=g[W_\mu{}^+\mathcal{J}_W{}^{\mu-}+W_\mu{}^-\mathcal{J}_W{}^{\mu+}]+(g^2/2\mu_0{}^2)\\&\quad\times[\mathcal{J}_{W0}{}^+,\mathcal{J}_{W0}{}^-]_++\tfrac{1}{2}g^2C\eta_a\\&\qquad\times(\xi_a{}^+W_i{}^-+\xi_a{}^-W_i{}^+)^2,\quad\theta_{0i}{}^I=0.\end{aligned}\tag{3.23}$$

$\theta_{\mu\nu}{}^W$ and $\theta_{\mu\nu}{}^L$ have the forms of Eqs. (2.19) and (3.13), respectively, but are free in the sense that now

$$W_0{}^\pm=(1/\mu_0{}^2)\partial_iG_{i0}{}^\pm,\quad i\gamma_0\partial_0\psi=(i\gamma_i\partial_i+M)\psi.\tag{3.24}$$

This is to be understood in $\theta_{00}{}^I$ as well. The unitary transformation

$$U(t,0)=T\exp\left(-i\int_0^t\theta_{00}{}^I(\mathcal{J}^D)d^4x\right)$$

leads to the usual interaction picture in which $\mathcal{J}_{a\mu}{}^D$, $W_\mu{}^D$, and $l_\mu{}^D$ are all four-vectors. The interaction is evidently the usual Cabibbo form, and the S matrix $U(\infty,-\infty)$ is formally Poincaré-invariant to all orders. As in Sec. II, the Feynman series can be rewritten in terms of W propagators.

IV. GENERAL PERTURBATION

In the previous sections we have shown how to incorporate semistrong, electromagnetic, and weak interactions into Sugawara's theory, and in particular how to pass to the interaction picture and construct a covariant perturbation theory. We now address ourselves to the problem of the most general perturbation.

Sugawara[1] has shown that if one demands that (a) the currents satisfy the algebra of fields, (b) the energy-momentum tensor $\theta_{\mu\nu}$ is an $SU(3)\otimes SU(3)$ scalar, and (c) $\theta_{\mu\nu}$ is a finite polynomial in the currents, then the proposed quadratic form is essentially unique. The restriction to this form follows from Lorentz invariance, which requires $\theta_{\mu\nu}$ to satisfy the Schwinger relation

$$[\theta_{00}(\mathbf{x}),\theta_{00}(\mathbf{y})]=i[\theta_{0i}(\mathbf{x})+\theta_{0i}(\mathbf{y})]\partial_i{}^{(x)}\delta^{(3)}(\mathbf{x}-\mathbf{y}).\tag{4.1}$$

It is easy to see that, given conditions (a)–(c), $\theta_{\mu\nu}$ must be quadratic in the currents to be consistent with (4.1). However, if one adds symmetry-breaking interactions, then, at least from our work above, conditions (a) and (b) are not satisfied. What, then, are the most general perturbations? By working primarily in the interaction picture, we shall be able to carry over some parts of Sugawara's reasoning.

Let us abstract what we know about the interaction picture. In general, the total energy-momentum tensor has the form

$$\theta_{\mu\nu}{}^T=\theta_{\mu\nu}{}^S(\mathcal{J}^D)+\theta_{\mu\nu}{}^F(\varphi_r{}^D)+\theta_{\mu\nu}{}^I(\mathcal{J}^D,\varphi_r{}^D),\tag{4.2}$$

where $\mathcal{J}_\mu{}^D$ is the interaction-picture current and $\varphi_r{}^D$ is some set of additional (canonical) fields in the interaction picture. $\theta_{\mu\nu}{}^S$ has the functional form of the symmetric Sugawara tensor, and $\theta_{\mu\nu}{}^F(\varphi_r{}^D)$ is the free tensor for the φ fields. The algebra of $\mathcal{J}^D$ is the symmetric algebra, and $\mathcal{J}^D$ commutes with $\varphi_r{}^D$. The time dependence of $\mathcal{J}^D$ and φ^D is that of $\theta_{\mu\nu}{}^S$ and $\theta_{\mu\nu}{}^F(\varphi_r{}^D)$, respectively. Thus $\mathcal{J}^D$ carries the symmetric time dependence, while $\varphi_r{}^D$ are free fields. Finally, the Poincaré generators are constructed from $\theta_{\mu\nu}{}^S+\theta_{\mu\nu}{}^F(\varphi^D)$, so that the $\mathcal{J}$'s are four-vectors. We can say, then, that the interaction $\theta_{\mu\nu}{}^I$ is a function of the original Sugawara currents together with some canonical free fields.

To assure Lorentz invariance, $\theta_{\mu\nu}{}^T$ must satisfy (4.1) in the Heisenberg picture. However, since the Schwinger condition contains no time derivatives, it must also be true in the interaction picture. In general, $\theta_{\mu\nu}{}^F\equiv\theta_{\mu\nu}{}^S(\mathcal{J}^D)+\theta_{\mu\nu}{}^F(\varphi^D)$ satisfies the Schwinger condition by itself; we thus have the following restriction on $\theta_{\mu\nu}{}^I(\mathcal{J}^D,\varphi_r{}^D)$:

$$[\theta_{00}{}^F(\mathbf{x}),\theta_{00}{}^I(\mathbf{y})]+[\theta_{00}{}^I(\mathbf{y}),\theta_{00}{}^F(\mathbf{x})]+[\theta_{00}{}^I(\mathbf{x}),\theta_{00}{}^I(\mathbf{y})]=i[\theta_{0i}{}^I(\mathbf{x})+\theta_{0i}{}^I(\mathbf{y})]\times\partial_i{}^{(x)}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \quad (4.3)$$

Now we can make a systematic investigation of possible perturbations. From here on we are in the interaction picture, so we suppress the label D. Suppose first that there are no extra (canonical) fields. As a subcase, suppose also that $\theta_{00}{}^I$ is a polynomial in the currents. Then Sugawara's reasoning can be repeated to show that either $\theta_{00}{}^I$ is quadratic or it is an infinite polynomial. What about interactions involving derivatives of currents like $\partial_\mu\mathcal{J}_\nu\partial^\mu\mathcal{J}^\nu$? Using $\partial^\mu\mathcal{J}_{a\mu}=0$, $\partial_\mu\mathcal{J}_{a\nu}-\partial_\nu\mathcal{J}_{a\mu}=(1/2C)C_{abc}[\mathcal{J}_{b\mu},\mathcal{J}_{c\nu}]_+$ to eliminate time derivatives, we learn that such interactions are effectively of higher order in $\mathcal{J}$, and not consistent, unless, again, one goes to infinite polynomials. Another way of saying this is via the S matrix. If we assume $S=T\exp[-i\int d^4x\,\theta_{00}{}^I(x)]$, we find that for any but quadratic interactions, we need add infinite numbers of contact terms to maintain Lorentz invariance to all orders. Thus, excluding infinite polynomials, the general form of the perturbed Sugawara theory—with no additional canonical fields— is the mass-mixing model.

We can say a little about infinite polynomial theories. Assuming for the moment that such an interaction can be constructed at all, then the Heisenberg-picture currents will be infinite polynomials in the interaction-picture currents. This is because the terms that must be added to $\mathcal{J}$ to obtain a J that transforms as a vector in the Heisenberg picture will be proportional to the Schwinger terms in $[\theta_{00}{}^I(\mathcal{J}),\mathcal{J}_0]$. In fact, for the theories described in this paper, one has

$$J_i(x)=\mathcal{J}_i(x)+i\int d^3y(x_i-y_i)[\theta_{00}{}^I(\mathbf{y},t),\mathcal{J}_0(x)]. \quad (4.4)$$

For the general case, see Ref. 19. Thus, in the Heisenberg picture, both the energy-momentum tensor and the commutators of the currents will contain infinite polynomials in the currents.

What about theories with other (in general canonical) fields? It is, of course, possible to introduce interactions of the form $\mathcal{J}^\mu C_\mu$, where C_μ is some independent (canonical) vector field operator (W field, electromagnetic field, lepton weak current, i.e., the "hybrid" theories of Ref. 2, etc.), or $\Delta\theta_{\mu\nu}=g_{\mu\nu}\sigma$ (the scalar and pseudoscalar densities of Ref. 2). What about other interactions of the form, say, $(\mathcal{J}_\mu C^\mu)^2$? The commutator of this with itself contains terms of fourth order in $\mathcal{J}$, and so on; again we have to go to infinite polynomials. Similar troubles are found with forms like $\mathcal{J}_\mu\mathcal{J}_\nu{}^*G^{\mu\nu}$, $\partial_\mu\mathcal{J}_\nu\partial^\mu C^\nu$, etc. In particular then, if infinite polynomials are to be avoided, nonminimal electromagnetic coupling[24] is ruled out.

In summary, the exclusion of infinite polynomials restricts the form of the perturbing Hamiltonian to be $\lambda_{ab}{}^1\mathcal{J}_{a\mu}\mathcal{J}^{b\mu}+\lambda_{ab}{}^2\mathcal{J}_{a\mu}C^{b\mu}+\sigma+$contact terms.[25] The internal symmetry of these perturbations (i.e., the structure of $\lambda_{ab}{}^{1,2}$, etc.) is unrestricted. Moreover, the σ approach appears to be the only way of introducing semistrong interactions without modifying the original algebra.

In conclusion we want to say a few words about a "finite" current theory, in the spirit of Lee[11] and of Gell-Mann *et al.*[12] Because we have essentially the usual freedom in introducing scalar bosons and W mesons, the program of Ref. 12 can be carried out with relatively little change, yielding a finite (off-diagonal) weak interactions. To adapt Lee's approach, we would have to introduce octets of W mesons and identify these with the real currents. Although this is probably not in the spirit of the game, it may be interesting.

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APPENDIX: COVARIANT PASSAGE TO INTERACTION PICTURE

In this Appendix we shall reword the analysis of Sec. II in a manifestly covariant form. We first derive a manifestly covariant form for the commutation relations of the currents. Assume that the currents $J_{a\mu}$ satisfy the equal-time commutation relations of the mass-mixing model (2.2), with $D_{ab}=\delta_{ab}\lambda_a$. By performing an explicit Lorentz transformation one can then construct the commutators on an arbitrary spacelike surface σ. For simplicity, we shall only con-

[24] Nonminimal couplings of spin-1 fields to one another seem always to lead to infinite polynomials. See, e.g., T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962).

[25] A possibility for determining the internal symmetry may lie in requiring, say, the symmetric algebra of fields in the Heisenberg picture, by virtue of finding a complicated cancellation among, say, all the perturbations.

sider flat spacelike surfaces. Such a spacelike plane σ is determined by specifying the constant timelike vector n^μ ($n^\mu n_\mu = 1$) normal to the plane. We obtain the manifestly covariant algebra on σ,

$$\begin{aligned}\delta[(x-y)\cdot n][J_{a\mu}(x),J_{b\nu}(y)] = iC_{abc}[n_\mu J_{c\nu}(x)\lambda_b\lambda_c^{-1} \\ + n_\nu J_{c\mu}(x)\lambda_a\lambda_c^{-1} + n_\mu n_\nu n_\lambda J^{c\lambda}(x)(1-\lambda_b\lambda_c^{-1}-\lambda_a\lambda_c^{-1})] \\ \times\delta^{(4)}(x-y) + i\delta_{ab}\lambda_a(n_\mu D_\nu + n_\nu D_\mu)\delta^{(4)}(x-y)\,, \quad \text{(A1)}\end{aligned}$$

where $D_\mu \equiv \partial_\mu - n_\mu n_\lambda \partial^\lambda$ has only components tangential to σ. When $n^\mu = (1,0,0,0)$, the algebra reduces to the usual form (2.2). Alternatively, when $\lambda_a = 1$, we have the covariant algebra of the symmetric theory. The interaction can be removed from the algebra on σ by defining a new current,

$$\mathcal{J}_{a\mu}(x,n) \equiv (1/\lambda_a)J_{a\mu} + (1-1/\lambda_a)n_\mu n_\nu J^{a\nu}. \quad \text{(A2)}$$

$\mathcal{J}_{a\mu}(x,n)$ satisfies the symmetric covariant algebra on σ, and is a Lorentz-vector, now explicitly frame-dependent, i.e., has explicit n dependence. The generator of displacements normal to σ is

$$\mathcal{H}(\sigma) = n^\mu P_\mu = \int n^\mu \theta_{\mu\nu}(x) n^\nu \delta(x\cdot n) d^4x. \quad \text{(A3)}$$

The interaction picture is designed to remove the dependence on the interaction from this displacement, but it will also remove the frame dependence of $\mathcal{J}_{a\mu}$. As a function of $\mathcal{J}$, $\theta_{\mu\nu}$ splits into two parts, a free tensor $\theta_{\mu\nu}{}^S(\mathcal{J})$ and an interaction tensor $\theta_{\mu\nu}{}^I(\mathcal{J},n)$. In this case,

$$\theta_{00}{}^I(\mathcal{J},n) = \frac{1-\lambda_a}{2C}\left(\mathcal{J}_{a\mu}\mathcal{J}^{a\mu} + \frac{1-\lambda_a}{2C\lambda_a}(n_\mu \mathcal{J}^{a\mu})^2\right). \quad \text{(A4)}$$

Note that $\theta_{00}{}^I(\mathcal{J},n)$ is itself a frame-dependent tensor. This is, of course, always the case except when $\theta_{\mu\nu}{}^I = g_{\mu\nu}\otimes$(scalar). The unitary transformation that removes the interaction from the "time" generator satisfies

$$(n\cdot\partial)U(\sigma) = -i\mathcal{H}^I(\sigma)U(\sigma)\,,$$

$$\mathcal{H}^I(\sigma) = \int d^4x\, \delta(x\cdot n)\theta_{00}{}^I(\mathcal{J},n). \quad \text{(A5)}$$

$U(\sigma)$ can be solved for in the usual manner. It has the explicit frame dependence of $\theta_{00}{}^I(\mathcal{J},n)$. In fact, its frame dependence is just what is needed to make $\mathcal{J}_{a\mu}{}^D(x)$ a frame-independent four-vector [where the interaction-picture Lorentz generators are constructed with the *frame-independent* $\theta_{00}{}^S(\mathcal{J}^D)$]. As emphasized, e.g., in Ref. 13, such procedures are natural for theories with derivative coupling or spin-1 fields. Evidently, there is no loss of generality in restricting ourselves, as in the text, to the flat surfaces $n^\mu = (1,0,0,0)$.