factorized:

$$\lim_{\lambda\to\lambda_\tau}[(\lambda-\lambda_\tau)(\pi^3)^{-1}T_{2q'1q}(\tau,1,-\lambda,t)]$$
$$=\gamma_{2q'}{}^{(B)}(t,1,\tau)\gamma_{1q}{}^{(A)}(t,1,\tau). \quad (A3)$$

If we neglect the background term, the amplitude $T_{21,11}(s,t)$ can be written as

$$T_{21,11}(s,t)=\sum_{\tau=\pm1}\sum_{qq'}(1-\lambda_\tau{}^2)\gamma_{2q'}{}^{(B)}(t,1,\tau)$$
$$\times\gamma_{1q}{}^{(A)}(t,1,\tau)d_{211q'q}{}^{\tau,1,-\lambda_\tau}(\xi). \quad (A4)$$

From the symmetry properties of the amplitudes $T_{j'q'jq}(\tau,1,-\lambda,t)$ under space reflection,[5] we deduce

$$\gamma_{21}{}^{(B)}(t,1,\tau)=-\gamma_{2-1}{}^{(B)}(t,1,\tau), \quad (A5a)$$

$$\gamma_{11}{}^{(A)}(t,1,\tau)=\gamma_{1-1}{}^{(A)}(t,1,\tau). \quad (A5b)$$

Further,

$$d_{211q'q}{}^{\tau,1,-\lambda_\tau}(\xi)=\delta_{qq'}d_{211}{}^{q,-\lambda_\tau}(\xi). \quad (A5c)$$

Therefore, $T_{21,11}(s,t)$ is given by

$$T_{21,11}(s,t)=\sum_{\tau=\pm1}(1-\lambda_\tau{}^2)\gamma_{21}{}^{(B)}(t,1,\tau)\gamma_{11}{}^{(A)}(t,1,\tau)$$
$$\times[d_{211}{}^{1,-\lambda_\tau}(\xi)-d_{211}{}^{-1,-\lambda_\tau}(\xi)]. \quad (A6)$$

But asymptotically we know

$$d_{211}{}^{1,-\lambda_\tau}(\xi)\propto s^{\lambda_\tau-1},$$
$$d_{211}{}^{-1,-\lambda_\tau}(\xi)\propto s^{\lambda_\tau-3}.$$

Therefore, we can neglect the second term in (A6).

Evaluating $d_{211}{}^{1,-\lambda_\tau}(\xi)$ from the expression given in Ref. 1, we find

$$d_{211}{}^{1,-\lambda_\tau}(\xi)\approx\frac{\sqrt{15}}{\lambda_\tau+1}(\cosh\xi)^{\lambda_\tau-1}$$

and so

$$T_{21,11}(s,t)\approx-(\sqrt{15})\sum_{\tau=\pm1}[\lambda_\tau(t)-1]\gamma_{21}{}^{(B)}(t,1,\tau)$$
$$\times\gamma_{11}{}^{(A)}(t,1,\tau)(\cosh\xi)^{\lambda_\tau(t)-1}, \quad (A7)$$

where

$$\cosh\xi\approx\frac{s-u}{2[(m_\rho{}^2+m_\pi{}^2)(m_N{}^2+m_\Delta{}^2)]^{1/2}} \quad (A8)$$

along the forward direction.

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Old-New Model for the Nucleon Resonances*

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A simple phenomenological model of nucleon resonances is presented in which a resonance state is assumed to be a bound state of a point pseudoscalar particle and a point spin-$\frac{1}{2}$ particle. Associated with each resonant state is a nonlocal field operator which obeys generalized "Schrödinger" equations of motion. Assuming a relativistic harmonic-oscillator approximation for the "potential," a mass-spin relationship for members of a Regge trajectory can be derived, i.e., $m_R{}^2=a+bj$ (the Regge formalism, however, is not employed here). Using the field operators in a phenomenological (Lorentz-invariant) Lagrange function, the decays of members of a Regge trajectory into a pion and a nucleon are calculated in a systematic way, with no more than two unknown parameters per trajectory to be determined from experiment (usually only one is needed). The resulting expressions for the partial decay widths are not an unreasonable approximation to experiment. One interesting feature of this model is the Gaussian-like form factors that appear in the partial decay widths. These form factors are a consequence of the nonlocal interaction assumed here; they appear naturally and are not introduced *ad hoc*. Another interesting feature—and, at the same time, a check on the calculations—is the fact that a good approximation to the πNN coupling constant can be derived, given the experimental partial decay width $\Gamma(N_{1/2}{}^*(1688)\to\pi N)$. The number obtained is $g_{\pi NN}{}^2/4\pi=14.2$.

I. INTRODUCTION

BECAUSE of the lack of a reliable dynamical theory of the baryon resonance system, more phenomenological approaches have often been used in calculating high-energy resonance reactions. One approach frequently used is the "isobar model,"[1] in which the decay of one higher-energy level to another is described by an effective vertex in which a baryon state is represented by a Rarita-Schwinger field.[2] At least one phenomenological coupling constant is used to characterize the decay; if only one such parameter is necessary (as in

* A more elementary version of this model appears in a thesis submitted by the author to the faculty of the graduate school of the University of Minnesota in partial fulfillment of the requirements for the degree of Doctor of Philosophy, 1966 (unpublished).

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[1] See, e.g., P. Carruthers, Phys. Rev. **152**, 1345 (1966); J. G. Rushbrooke, *ibid.* **143**, 1345 (1966); D. M. Brudnoy, *ibid.* **145**, 1229 (1966).

[2] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941). For an explicit representation of the wave functions, see Ref. 1.

the case $J \to \frac{1}{2}+0$), then it can be evaluated from the experimental decay width. Information concerning possible regularities in the system, if at all possible in this approach, are numerical at best.[3] In addition, vertex form factor terms ("barrier penetration factors") can be introduced only in an *ad hoc* manner, and their origin is thus unclear.[3]

The possible existence of systematics in the resonance system can be discerned from present experimental data, even though the quantum numbers and decay widths of many of the excited levels presently identified have not been solidly ascertained. Nevertheless, those states belonging to the same Regge trajectory appear to obey the relationship[4] $m_R^2 \propto J$, where m_R is the mass of the state and J its spin; in addition, the slopes of the lines for different trajectories are nearly equal (see Fig. 1). This very simple relationship is reasonably satisfied for resonance energies up to about 3.5 BeV, and already speculation has been made concerning higher-energy regions.[5]

Although a "true" understanding of the resonance system may well turn out to be dependent on highly sophisticated mathematical machinery and, perhaps, even on physical principles now unknown to us, it is tempting to conjecture that some of the salient features might be described by a simple phenomenological model. It is such a model that is proposed in this paper.

In a sense the model is a reversion to old-fashioned methods and ideas. As explained in Sec. II, each state is assumed to be a composite of a point pion and a point spin-$\frac{1}{2}$ nucleon bound in a state of total angular momentum J. A nonlocal field operator associated with each state is constructed, similar to the field operator proposed many years ago by Yukawa,[6] to describe the nucleon as a particle with structure. Equations of motion are imposed on each field projected onto the physical state which it represents. In Sec. III appropriate approximations are made and the mass relationship $m_R^2 \propto J$ emerges. In Sec. IV and in Sec. V manifestly covariant Lagrange functions are constucted to describe the coupling of "families" of states (Regge trajectories) to the nucleon and pion. From these Lagrange functions, the partial decay widths of a family of states decaying into pion and nucleon are calculated. In Sec. IV, it is assumed that the nucleon itself is a member of the resonance system; in Sec. V, it is assumed that it is an independent point particle. In both calculations no more than two parameters (and usually only one parameter) are needed to obtain the partial widths for each family of states. It is interesting

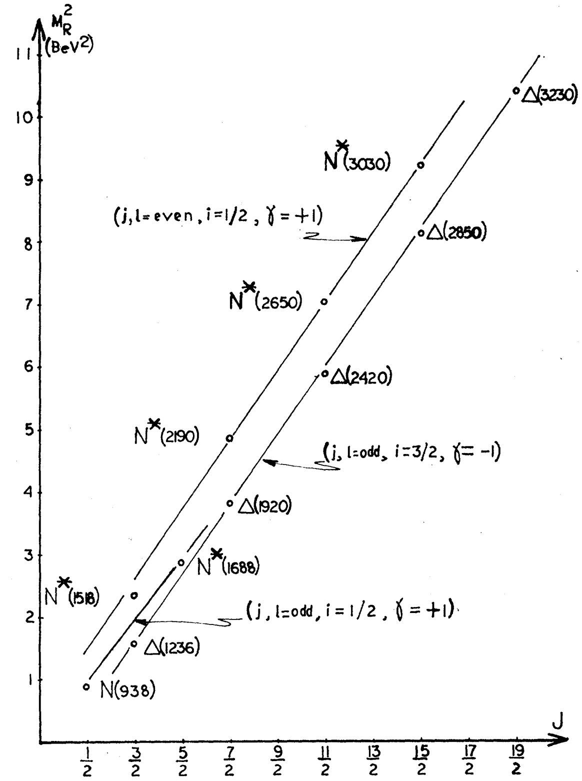

FIG. 1. m_R^2 versus J for the three resonance families (Regge trajectories) (j, l=odd, $i=\frac{3}{2}$, $\gamma=-1$), (j, l=odd, $i=\frac{1}{2}$, $\gamma=+1$), and (j, l=even, $i=\frac{1}{2}$, $\gamma=+1$). The spin and parity assignments are the same as those of V. Barger and D. Cline, Ref. 4.

that both calculations give quite similar results, in reasonable agreement with experiment (see Tables II–VII). As an additional check on the calculations of Sec. IV, the conventional πNN coupling constant can be derived to good approximation ($g_{\pi NN}^2/4\pi \approx 14.2$) given the experimental partial decay width of $N_{1/2}^*(1688) \to \pi N$.

II. GENERAL THEORY

The central assumption of this model is that an excited nucleon state can be described as a bound state of a point pseudoscalar particle and a point spin-$\frac{1}{2}$ particle of positive parity. These latter need not be "real" physical particles, such as the 138-MeV pion or the 938-MeV nucleon, and indeed for the purposes of the calculations presented in this paper, the masses of these constituent particles need not be known.

Each resonance state is characterized at least by the following quantities: (a) a rest mass m_R; (b) a spin and z component of spin j and m_j, respectively; (c) isotopic spin i, which can be either $\frac{1}{2}$ or $\frac{3}{2}$; (d) an intrinsic parity or, equivalently, the relative orbital angular momentum of the core particles in the resonance rest system. This latter parameter we denote by l, where $l=j-\frac{1}{2}$ or

[3] P. Carruthers and J. Shapiro, Phys. Rev. **159**, 1456 (1967).

[4] V. Barger and D. Cline, Phys. Rev. Letters **16**, 913 (1966); **16**, 1135 (1966); V. Barger, Rev. Mod. Phys. **40**, 129 (1968).

[5] See V. Barger, Rev. Mod. Phys. **40**, 129 (1968), especially Sec. III.

[6] H. Yukawa, Phys. Rev. **77**, 219 (1950); **77**, 849 (1950). The "relative time" function in Yukawa's field is a δ function $\delta(p\cdot r)$, resulting from the Born reciprocity principle [M. Born, Nature **136**, 952 (1935)].

$l=j+\frac{1}{2}$ depending on the intrinsic parity of the state $P_j=-(-1)^l$.

The states are created (annihilated) by the particle operators $A_{jm_jli\alpha}{}^\dagger(\mathbf{p})$ $(A_{jm_jli\alpha}(\mathbf{p}))$ and the antiparticle operators $B_{jm_jli\alpha}{}^\dagger(\mathbf{p})$ $(B_{jm_jli\alpha}(\mathbf{p}))$ which obey the conventional anticommutation relations

$$\begin{aligned}\{A_{jm_jli\alpha}(\mathbf{p}),A_{j'm_j'l'i'\alpha'}(\mathbf{p}')\} &=\{B_{jm_jli\alpha}(\mathbf{p}),B_{j'm_j'l'i'\alpha'}(\mathbf{p}')\}\\ &=\{A_{jm_jli\alpha}(\mathbf{p}),B_{j'm_j'l'i'\alpha'}(\mathbf{p}')\}\\ &=\{A_{jm_jli\alpha}(\mathbf{p}),B_{j'm_j'l'i'\alpha'}{}^\dagger(\mathbf{p}')\}=0, \qquad (1)\\ \{A_{jm_jli\alpha}(\mathbf{p}),A_{j'm_j'l'i'\alpha'}{}^\dagger(\mathbf{p}')\} &=\{B_{jm_jli\alpha}(\mathbf{p}),B_{j'm_j'l'i'\alpha'}{}^\dagger(\mathbf{p}')\}\\ &=(E_R/m_R)\delta_{jj'}\delta_{m_jm_j'}\delta_{ll'}\delta_{\alpha\alpha'}\delta_{ii'}\delta^{(3)}(\mathbf{p}-\mathbf{p}').\end{aligned}$$

The subscript α is a collective label for all other quantum numbers not mentioned above but which might be needed to characterize a resonance state. E_R is the resonance energy and m_R its rest mass.

Associated with each resonance is a field operator, a function of the space-time coordinates of the constituent point particles, which describes the internal motion of the latter as well as the free-particle motion of the particle as a whole.[6] The free-particle motion is described by a plane wave $e^{ip\cdot X}$ of four-momentum p, at the generalized "center of mass"

$$X_\mu\equiv\frac{m_0(x_0)_\mu+m_{1/2}(x_{1/2})_\mu}{m_0+m_{1/2}}.$$

Here, m_0 and $m_{1/2}$ are the masses of the core pseudoscalar and spin-$\frac{1}{2}$ particles, respectively, and $(x_0)_\mu$ and $(x_{1/2})_\mu$ their respective space-time coordinates. The internal motion is represented by a function dependent on the generalized "relative coordinates" $r_\mu\equiv(x_{1/2})_\mu-(x_0)_\mu$. We will refer to this operator as a "wave function" or a "wave operator."

The wave operator may be written

$$\psi_{jli\alpha}(X,r)\equiv\sum_{m_j}\int d^4p\ \delta(p^2+m_R{}^2)\theta(p_0)\times[e^{-ip\cdot X}f_{jm_jli\alpha}(p,r)A_{jm_jli\alpha}(\mathbf{p})+\text{antiparticle}], \qquad (2)$$

where $\theta(p_0)$ is the step function

$$\begin{aligned}\theta(p_0)&\equiv1 \quad \text{for} \quad p_0>0\\ &\equiv0 \quad \text{for} \quad p_0<0\end{aligned}$$

and $f_{jm_jli\alpha}(p,r)$ is a function containing the details of the internal behavior. A linear sum of these operators is constructed and associated with all particles in the resonance spectra

$$\Psi(X,r)\equiv\sum_{j,l,i,\alpha}b_{jli\alpha}\psi_{jli\alpha}(X,r), \qquad (3)$$

where the $b_{jli\alpha}$ are constants.

Two equations of motion are imposed on $\Psi(X,r)$ projected onto an arbitrary state of the system. The first is the free-particle Dirac equation

$$\left(\gamma_\mu\frac{\partial}{\partial X_\mu}-m_R\right)\langle0|\Psi(X,r)|jm_jli\alpha;\mathbf{p}_R\rangle=0. \qquad (4)$$

This equation simply states that we have chosen double the needed two components to describe spin $\frac{1}{2}$; hence, the additional two are really superfluous. The second, and more important, equation is[7]

$$\begin{aligned}[(\partial^2/\partial X^2)+(\partial^2/\partial r^2)]\langle0|\Psi(X,r)|jm_jli\alpha;\mathbf{p}_R\rangle\\ =V(r)\langle0|\Psi(X,r)|jm_jli\alpha;\mathbf{p}_R\rangle, \qquad (5)\end{aligned}$$

where $V(r)$ is a Lorentz-invariant function of the relative coordinates only, i.e., of r^2 only. Using Eq. (2) in the above gives

$$\begin{aligned}[m_R{}^2+\nabla_{\mathbf{r}}{}^2-(\partial^2/\partial r_0{}^2)]f_{jm_jli\alpha}(p_R,r)\\ =V(r^2)f_{jm_jli\alpha}(p_R,r). \qquad (6)\end{aligned}$$

It will now be shown that in order to satisfy the above requirements, $f_{jm_jli\alpha}$ may be written in the following form:

$$\begin{aligned}f_{jm_jli\alpha}(p_R,r)=\sum_\sigma C(l\tfrac{1}{2}j;m_j-\sigma\,\sigma)u^{(\sigma)}(p_R)\\ \times Y_{l,m_j-\sigma}(\hat{L}^{-1}(p_R)\cdot r)\,|\mathbf{L}^{-1}(p_R)\cdot r|^l\\ \times h_{jli\alpha}(|\mathbf{L}^{-1}(p_R)\cdot r|,L^{-1}{}_{4\mu}(p_R)\cdot r_\mu)\\ \times\left[\frac{4\pi2^l(l!)^2}{(2l+1)!}\right]^{1/2}. \qquad (7)\end{aligned}$$

Here $C(l\frac{1}{2}j;m_j-\sigma\,\sigma)$ are Clebsch-Gordan coefficients[8]; $u^{(\sigma)}(p_R)$ is the Dirac four-spinor

$$u^{(\sigma)}(p_R)=\frac{1}{[2m_R(E_R+m_R)]^{1/2}}\begin{pmatrix}(m_R+E_R)\chi^{(\sigma)}\\ \boldsymbol{\sigma}\cdot\mathbf{p}_R\chi^{(\sigma)}\end{pmatrix},$$

where $\boldsymbol{\sigma}$ are the 2×2 Pauli matrices, and $\chi^{(\sigma)}$ the two-component Pauli spinors; $L^{-1}{}_{\mu\nu}(p_R)$ is the Lorentz boost transformation, which takes the vector $(p_R)_\mu$ into its rest frame,

$$\begin{aligned}L^{-1}{}_{mn}(p_R)&=\delta_{mn}+\frac{(p_R)_m(p_R)_n}{m_R(E_R+m_R)},\\ L^{-1}{}_{m4}(p_R)&=-L^{-1}{}_{4m}(p_R)=i(p_R)_m/m_R,\\ L^{-1}{}_{44}(p_R)&=E_R/m_R;\end{aligned}$$

$Y_{l,m_j-\sigma}(\hat{L}^{-1}(p_R)\cdot r)$ are the spherical harmonics[8] of order l and argument

$$\hat{L}^{-1}(p_R)\cdot r\equiv\frac{L^{-1}{}_{s\mu}(p_R)\cdot r_\mu}{[L^{-1}{}_{k\tau}(p_R)\cdot r_\tau L^{-1}{}_{k\nu}(p_R)\cdot r_\nu]^{1/2}},$$

[7] This equation for most intents and purposes is equivalent to the generalized Schrödinger equation $(2m_0)^{-1}\partial^2/\partial x_0{}^2+(2m_{1/2})^{-1}\times\partial^2/\partial x_{1/2}{}^2$, which is $[(2m_0+m_{1/2})]^{-1}\partial^2/\partial X^2+[(m_0+m_{1/2})/2m_0m_{1/2}]\partial^2/\partial r^2$. Using this latter equation would only change the definition of the parameter β in Eq. (21); however, it would introduce another parameter into Eq. (27).

[8] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

and the function $h_{jli\alpha}(|\mathbf{L}^{-1}(p_R)\cdot r|,L^{-1}{}_{4\mu}(p_R)\cdot r_\mu)$ is an unknown function of the two variables $|\mathbf{L}^{-1}(p_R)\cdot r| = [L^{-1}{}_{k\mu}(p_R)\cdot r_\mu L^{-1}{}_{k\nu}(p_R)\cdot r_\nu]^{1/2}$ and $L^{-1}{}_{4\mu}(p_R)\cdot r_\mu$, which is to be determined from $V(r^2)$ in Eq. (6).

First, Eq. (7) satisfies the Dirac equation $(p\!\!\!/_R+m_R) \times f_{jm_jli\alpha}(p_R,r)=0$, where $p\!\!\!/_R\equiv i\gamma_\mu(p_R)_\mu$. Second, both $|\mathbf{L}^{-1}(p_R)\cdot r|$ and $L^{-1}{}_{4\mu}(p_R)\cdot r_\mu$ are manifest Lorentz-invariants,

$$|\mathbf{L}^{-1}(p_R)\cdot r| = \left(r^2+\frac{(p_R\cdot r)^2}{m_R^2}\right)^{1/2},$$

$$L^{-1}{}_{4\mu}(p_R)\cdot r_\mu = -i\frac{(p_R\cdot r)}{m_R}.$$

In the limit $|\mathbf{p}_R|\to 0$, $f_{jm_jli\alpha}$ becomes

$$\sum_\sigma C(l\tfrac{1}{2}j;\, m_j-\sigma\,\sigma)u^{(\sigma)}(0)Y_{l,m_j-\sigma}(\hat{r})|\mathbf{r}|^l \times\left(\frac{4\pi 2^l(l!)^2}{(2l+1)!}\right)^{1/2} h_{jli\alpha}(|\mathbf{r}|,r_4),$$

which is similar in form to a nonrelativistic wave function associated with a particle of spin $\frac{1}{2}$ coupled to angular momentum l to form a state of spin j in a potential field.[9] In our case, l assumes a comparable physical meaning in this limit, as required. $f_{jm_jli\alpha}(p_R,r)$ can be written in a different, perhaps more familiar, way,

$$f_{jm_lli\alpha}(p_R,r)=r_{\alpha_1}\cdots r_{\alpha_l}u_{\alpha_1\cdots\alpha_l}{}^{(m_j)}(p_R) \times h_{jli\alpha}[|\mathbf{L}^{-1}(p_R)\cdot r|,L^{-1}{}_{4\mu}(p_R)\cdot r_\mu], \quad (7')$$

where $u_{\alpha_1\cdots\alpha_l}{}^{(m_j)}(p_R)$ is a Rarita-Schwinger-type tensor spinor,[2] i.e.,

$$\begin{aligned} u_{\alpha_1\cdots\alpha_l}{}^{(m_j)}(p_R) = \sum_{m_1,\cdots,m_l,\sigma} & C(112;\, m_1\, m_2-m_1)\cdots \\ & \times C(l-1\,1l;\, m_{l-1}\, m_l-m_{l-1}) \\ & \times C(l\tfrac{1}{2}j;\, m_l\sigma m_j)u^{(\sigma)}(p_R) \\ & \times \epsilon_{\alpha_1}{}^{(m_1)}(p_R)\epsilon_{\alpha_2}{}^{(m_2-m_1)}(p_R)\cdots \\ & \times \epsilon_{\alpha_l}{}^{(m_l-m_{l-1})}(p_R), \quad (8) \end{aligned}$$

where $C(n1\,n+1;\, m_n\, m_{n+1}-m_n)$ are Clebsch-Gordan coefficients[8] and $\epsilon_{\alpha_n}{}^{(m_v)}(p_R)$ are four-dimensonal spherical vectors[8] defined by

$$\epsilon_{\alpha_n}{}^{(m_v)}(0)\equiv\begin{pmatrix}\boldsymbol{\varepsilon}^{(m_v)}\\ 0\end{pmatrix},$$

$$\boldsymbol{\varepsilon}^{(1)}=-\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\\ 0\end{bmatrix},\quad \boldsymbol{\varepsilon}^{(0)}=\begin{bmatrix}0\\ 0\\ 1\end{bmatrix},\quad \boldsymbol{\varepsilon}^{(-1)}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\\ 0\end{bmatrix}.$$

Equation (7a) is obtained from Eq. (8) by the use of the equations[10]

$$\sum_{m_1} Y_{l_1m_1}(\hat{x})Y_{l_2,m-m_1}(\hat{x})C(l_1l_2l;\, m_1\, m-m_1) = \left(\frac{(2l_1+1)(2l_2+1)}{4\pi(2l+1)}\right)^{1/2} C(l_1l_2l;\,00)Y_{lm}(\hat{x})$$

and[11]

$$C(l_1l_2l;\,00)=\left(\frac{(2l_1)!(2l_2)!}{(2l)!}\frac{(l!)^2}{(l_1!)^2(l_2!)^2}\right)^{1/2},$$

where $l_1+l_2=l$. $\psi_{jli\alpha}(X,r)$ is, thus, a scalar product of l number of r_α's with a nonpoint Rarita-Schwinger-type field,

$$\begin{aligned} \psi_{jli\alpha}(X,r)=r_{\alpha_1}\cdots r_{\alpha_l}\sum_{m_j}\int d^4p\,\theta(p_0)\delta(p^2+m_R^2) & \\ \times[e^{-ip\cdot X}u_{\alpha_1\cdots\alpha_l}{}^{(m_j)}(p) & \\ \times h_{jli\alpha}([r^2+(p\cdot r)^2/m_R^2]^{1/2},\, -ip\cdot r/m_R) & \\ \times A_{jm_jli\alpha}(\mathbf{p})+\text{antiparticle}], & \end{aligned}$$

where the spinor index has been excluded for convenience. In this form it is not difficult to see that all the fields $\psi_{jli\alpha}(X,r)$ will have the same transformation properties under a proper Lorentz transformation Λ.[12] The particle-annihilation operators, for example, transform as[13]

$$U(\Lambda)A_{jm_jli\alpha}(\mathbf{p})U^{-1}(\Lambda)=\sum_{m_j'} A_{jm_j'li\alpha}(\Lambda\mathbf{p})D_{m_jm_j'}{}^{(j)*}(R),$$

where $U(\Lambda)$ is a unitary operator which acts on the Hilbert space of physical states, and $D_{m_jm_j'}{}^{(j)}(R)$ is the representation of the Wigner rotation R for spin j. Thus, the νth spinor component of a resonance field operator transforms as

$$\begin{aligned} U(\Lambda)[\psi_{jli\alpha}(X,r)]_\nu U^{-1}(\Lambda) &= \sum_{m_j} r_{\alpha_1}\cdots r_{\alpha_l}\int d^4p\,\delta(p^2+m_R^2)\theta(p_0)\{u_{\alpha_1\cdots\alpha_l}{}^{(m_j)}(p)_\nu \\ &\quad \times h_{jli\alpha}([r^2+(p\cdot r)^2/m_R^2]^{1/2},\, -ip\cdot r/m_R)e^{-ip\cdot X}\sum_{m_j'} A_{jm_j'li\alpha}(\Lambda\mathbf{p})D_{m_jm_j'}{}^{(j)*}(R)+\text{antiparticle}\} \\ &= \sum_{m_j} r_{\alpha_1}\cdots r_{\alpha_l}\int d^4(\Lambda p)\delta((\Lambda p)^2+m_R^2)\theta((\Lambda p)_0)[u_{\alpha_1\cdots\alpha_l}{}^{(m_j)}(\Lambda^{-1}\Lambda p)_\nu \\ &\quad \times h_{jli\alpha}([(\Lambda r)^2+(\Lambda p\cdot\Lambda r)^2/m_R^2]^{1/2},\, -i\Lambda p\cdot\Lambda r/m_R)e^{-i\Lambda p\cdot\Lambda X}\sum_{m_j'} A_{jm_j'li\alpha}(\Lambda\mathbf{p})D_{m_jm_j'}{}^{(j)*}(R)+\text{antiparticle}]. \end{aligned}$$

[9] Ref. 8, Chap. IX.
[10] Ref. 8, Chap. IV, Eq. (4.32).
[11] A. Messiah, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1962), Vol. II, Appendix C.
[12] The arguments used here are those of S. Weinberg, Phys. Rev. **133**, B1318 (1964); see especially Sec. VIII.
[13] See, e.g., S. Gasiorowicz, *Elementary Particle Physics* (John Wiley & Sons, Inc., New York, 1966).

We put $\Lambda p\equiv p'$, and note that $L(\Lambda^{-1}p')=\Lambda^{-1}L(p')R$, where $L(\Lambda^{-1}p')$ and $L(p')$ are the boost Lorentz transformations of their respective arguments. Hence,

$$\epsilon_\alpha{}^{(m_v)}(\Lambda^{-1}p')=L_{\alpha\alpha'}(\Lambda^{-1}p')\epsilon_{\alpha'}{}^{(m_v)}(0)=\Lambda_{\alpha\sigma}{}^{-1}L_{\sigma\rho}(p')R_{\rho\alpha'}\epsilon_{\alpha'}{}^{(m_v)}(0)=\Lambda^{-1}{}_{\alpha\sigma}L_{\sigma\rho}(p')D_{m_v m_v'}{}^{(1)}(R)\epsilon_\rho{}^{(m_v')}\,,$$

where $\epsilon_\alpha{}^{(m_v)}(p)$ is the spherical four vector defined above.[14] Hence[15]

$$u_{\alpha_1\cdots\alpha_l}{}^{(m_j)}(\Lambda^{-1}p')_\nu=\Lambda^{-1}{}_{\alpha_1\sigma_1}\cdots\Lambda^{-1}{}_{\alpha_l\sigma_l}L_{\sigma_1\rho_1}(p')\cdots L_{\sigma_l\rho_l}(p')D_{\nu\rho}{}^{(1/2)}(\Lambda^{-1})D_{\rho\tau}{}^{(1/2)}(L(p'))\sum_{\mu_1,\cdots,\mu_l,\sigma'}C(112;\mu_1\,\mu_2-\mu_1)\cdots$$
$$\times C(l-1\,1l;\,\mu_{l-1}\,\mu_l-\mu_{l-1})C(l\tfrac{1}{2}j;\,\mu_l\sigma')u_\tau{}^{(\sigma')}(0)\epsilon_{\rho_1}{}^{(\mu_1)}(0)\cdots\epsilon_{\rho_l}{}^{(\mu_l-\mu_{l-1})}(0)D_{m_j\,\mu_l+\sigma'}{}^{(j)}(R)$$

and[16]

$$\sum_{m_j}D_{m_jm_j'}{}^{(j)*}(R)D_{m_j\,\mu_l+\sigma'}{}^{(j)}(R)=\delta_{m_j'\,\mu_l+\sigma'}\,.$$

Here $D^{(1/2)}(\Lambda)$ is a representation of the Lorentz transformation for spin $\frac{1}{2}$. Thus,

$$U(\Lambda)(\psi_{jli\alpha}(X,r))_\nu U^{-1}(\Lambda)=\sum_{m_j}r_{\alpha_1}\cdots r_{\alpha_l}\Lambda^{-1}{}_{\alpha_1\sigma_1}\cdots\Lambda^{-1}{}_{\alpha_l\sigma_l}D_{\nu\rho}{}^{(1/2)}(\Lambda^{-1})\int d^4p\;\delta(p^2+m_R{}^2)\theta(p_0)$$
$$\times[u_{\alpha_1\cdots\alpha_l}{}^{(m_j)}(p)_\rho h_{jli\alpha}([(\Lambda r)^2+(p\cdot\Lambda r/m_R)^2]^{1/2},\,-ip\cdot\Lambda r/m_R)e^{-ip\cdot\Lambda X}A_{jm_jli\alpha}(\mathbf{p})$$
$$+\text{antiparticle}]$$
$$=D_{\nu\rho}{}^{(1/2)}(\Lambda^{-1})(\psi_{jli\alpha}(\Lambda X,\Lambda r))_\rho\,.$$

Hence also,

$$U(\Lambda)\Psi(X,r)U^{-1}(\Lambda)=D^{(1/2)}(\Lambda^{-1})\Psi(\Lambda X,\Lambda r)\,.$$

Note that under the parity transformation P, defined by

$$P|jm_jli\alpha;\,\mathbf{p}_R\rangle=\eta_P|jm_jli\alpha;\,-\mathbf{p}_R\rangle,\quad|\eta_P|=1$$

the field transforms as

$$U(P)\psi_{jli\alpha}(X,r)U^{-1}(P)=\eta_P(-1)^l\gamma_4\psi_{jli\alpha}(i_pX,i_pr)\,,$$

where $(i_px)_\alpha=(-1)^{\delta_{\alpha4}+1}x_\alpha$ and η_P is the intrinsic parity of the state. The latter, however, is $-(-1)^l$, so that all fields transform alike under parity even though the states themselves do not.

As a result of these transformation properties, the coupling of two nucleon resonances to another particle can be described by a Lagrange function similar to the one that describes the coupling of two spin-$\frac{1}{2}$ particles to that other particle. For example, the coupling of two spin-$\frac{1}{2}$ particles to a pseudoscalar is written $g_{1/2}\bar{\psi}_{1/2}\gamma_5\times\psi_{1/2}\phi_0$, $\psi_{1/2}$ being the spin-$\frac{1}{2}$ fields; hence, the coupling of spin j to spin j' and pseudoscalar is written $g\bar{\Psi}\gamma_5\Psi\phi_0$, Ψ being the resonance operator. In this latter example, $j'+\frac{1}{2}$ amplitudes are needed to describe the coupling ($j'\leqslant j$, say), assuming parity is conserved. These terms emerge automatically from the expression $\bar{\Psi}\gamma_5\Psi\phi_0$. If, in addition, the functions $h_{jli\alpha}$ are known, then there is only one unknown over-all parameter, and not $j'+\frac{1}{2}$ parameters, to be determined. The resulting expressions are not only more easily calculated, but they are also manifestly Lorentz-covariant. In this paper we shall be concerned only with j-$\frac{1}{2}$-0 coupling of nonstrange particles, so that in addition to the above-mentioned Lagrange function, we may also write $\bar{\Psi}\psi_{1/2}\phi_0$, with Ψ the resonance field, and $\psi_{1/2}$ the spin-$\frac{1}{2}$ point field. These matters are discussed in detail in Secs. IV and V.

Before concluding this section, we remark that the states are normalized so that there are $E/(2\pi)^3m$ particles per unit volume, i.e.,

$$\left\langle\int\rho(X)d^3X\right\rangle$$
$$\equiv\int d^3X\int dr\langle jm_jli\alpha;\,\mathbf{p}|\Psi^\dagger(X,r)\Psi(X,r)|jm_jli\alpha;\,\mathbf{p}\rangle$$
$$=|b_{jli\alpha}|^2\left(\frac{E}{m}\right)\left[\frac{2^2(l!)^2}{(2l+1)!}\right]\delta^{(3)}(0)$$

and, thus,

$$|b_{jli\alpha}|^2=\left[\frac{2^l(l!)^2}{(2l+1)!}\right]^{-1}.$$

III. HARMONIC-OSCILLATOR APPROXIMATION

Equation (6) can be written

$$\left[m_R{}^2+\nabla_{r'}{}^2-\frac{\partial^2}{\partial r_0'^2}\right]Y_{lm_l}(\hat{r}')|\mathbf{r}'|^lh_{jli\alpha}(|\mathbf{r}'|,r_4')$$
$$=V(r'^2)Y_{lm_l}(\hat{r}')|\mathbf{r}'|^lh_{jli\alpha}(|\mathbf{r}'|,r_4')\,,$$

where $r_\mu'\equiv L^{-1}{}_{\mu\nu}(p_R)\cdot r_\nu$. We attempt a solution to this equation by assuming a four-dimensional harmonic-oscillator approximation for $V(r'^2)$, $V(r'^2)=-V_0+V_1r'^2$ ($V_0>0$). Writing $h_{jli\alpha}(|\mathbf{r}'|,r_4')\equiv R(|\mathbf{r}'|)T(r_0')$, we

[14] This last equality can be found, e.g., in Ref. 8, Chap. IV, p. 64.
[15] Ref. 8, p. 58.
[16] Ref. 8, p. 59.

obtain

$$[-\nabla_{r'}^2+V_1|\mathbf{r}'|^2]Y_{lm_l}(\hat{r}')|\mathbf{r}'|^l R(|\mathbf{r}'|)T(r_0')$$
$$=\text{const}\,Y_{lm_l}(\hat{r}')|\mathbf{r}'|^l R(|\mathbf{r}'|)T(r_0')$$
$$=\left[m_R^2+V_0-\frac{\partial^2}{\partial r_0'^2}+V_1 r_0'^2\right]\times Y_{lm_l}(\hat{r}')|\mathbf{r}'|^l R(|\mathbf{r}'|)T(r_0'). \quad (9)$$

The first of these equations is the familiar three-dimensional harmonic-oscillator equation,[17] from which we obtain[17]

$$\text{const}=[2n_s+l-\tfrac{1}{2}](4\nu)\,,\quad 4\nu\equiv 2\sqrt{V_1} \quad (10)$$

where the integer n_s is the "principal quantum number" contained in the radial solution[17]

$$R_{n_s l}(|\mathbf{r}'|)=\left(\frac{2(2\nu)^{l+3/2}(n_s-1)!}{[\Gamma(n_s+l+\frac{1}{2})]^3}\right)^{1/2} e^{-\nu|\mathbf{r}'|^2}\times L_{n_s+l-1/2}{}^{l+1/2}(2\nu|\mathbf{r}'|^2)$$
$$=(-1)^{l+1/2}\left(\frac{2(2\nu)^{l+3/2}(n_s-1)!}{\Gamma(n_s+l+\frac{1}{2})}\right)^{1/2}\times\frac{\Gamma(n_s+l+\frac{1}{2})}{\Gamma(l+\frac{3}{2})}e^{-\nu|\mathbf{r}'|^2}\times{}_1F_1(-n_s+1;\,l+\tfrac{3}{2};\,2\nu|\mathbf{r}'|^2). \quad (11)$$

Here $\Gamma(n_s+l+\frac{1}{2})$ is the gamma function,[18]

$$L_{n_s+l-1/2}{}^{l+1/2}(2\nu|\mathbf{r}'|^2)$$

is the associated Laguerre polynomial[19] of order (n_s-1) (hence $n_s\geqslant 1$), and ${}_1F_1(-n_s+1;l+\frac{3}{2};2\nu|\mathbf{r}'|^2)$ is the confluent hypergeometric function.[20] The constant factors in Eq. (11) have been chosen so that $\{|\mathbf{r}'|^l R_{ln_s}(|\mathbf{r}'|)\}$ are orthonormal,[19] i.e.,

$$\int_0^\infty R_{n_s l}^*(|\mathbf{r}'|)|\mathbf{r}'|^l R_{\bar{n}_s l}(|\mathbf{r}'|)|\mathbf{r}'|^l|\mathbf{r}'|^2 d(|\mathbf{r}'|)=\delta_{n_s\bar{n}_s}.$$

From the second equation in Eq. (9), the one-dimensional harmonic-oscillator equation[21]

$$[-m_R^2-V_0+(2n_s+l-\tfrac{1}{2})4\nu+(\partial^2/\partial r_0'^2)-V_1 r_0'^2]T(r_0')=0\,,$$

we obtain[21]

$$(2n_s+l-\tfrac{1}{2})(4\nu)-m_R^2-V_0=(n_t+\tfrac{1}{2})(4\nu)\,,$$

where the integer n_t, $n_t\geqslant 0$, is contained in the solution[21]

$$T_{n_t}(r_0')=\left(\frac{2\nu}{2^{n_t}n_t!\sqrt{\pi}}\right)^{1/2} e^{-\nu r_0'^2}H_{n_t}((2\nu)^{1/2}r_0')\,, \quad (12)$$

where $H_{n_t}((2\nu)^{1/2}r_0')$ is the Hermite polynomial[21] of order n_t. The constants in Eq. (12) have been chosen so that $\{T_{n_t}(r_0')\}$ are orthonormal,[21] i.e.,

$$\int_{-\infty}^{\infty} T_{n_t}^*(r_0')T_{\bar{n}_t}(r_0')dr_0'=\delta_{n_t\bar{n}_t}.$$

The formula obtained for the resonance mass spectra,

$$m_R^2=-V_0+4\nu[2n_s-n_t+l-1]\,, \quad (13)$$

relating the mass squared of a particle to its spin, displays the manifold degeneracy of the harmonic-oscillator states, i.e., those states for which $(2n_s-n_t+l-1)$ = const are all degenerate. We wish to remove as much of this degeneracy as possible and toward this aim we will make the following arguments.

First, we ask the question whether or not the nucleon itself is a member of the resonance spectra. If it is, and specifically if it is the ground state of the system, then it is represented by that field operator of spin $\frac{1}{2}$, isospin $\frac{1}{2}$, and parameter $l=1$, the latter resulting from parity $P_{\text{nucleon}}=-(-1)^l=1$. Since it is the ground state, we may assign it the lowest possible principal quantum numbers, $n_s=1$ and $n_t=0$. From Eq. (13) the choice $n_s=1$ is clear, but not so for the choice $n_t=0$. Nevertheless, we make this assumption and later we shall give other arguments in support of this choice (see discussion at the end of Sec. IV). If, on the other hand, the nucleon is not a member of the resonance spectra, but an independent particle, we shall represent it by the field operator[22]

$$\Psi(X)=\sum_m\int d^4p\,\theta(p_0)\delta(p^2+m_N^2)\times[u^{(m)}(p)e^{-ip\cdot X}A_m(\mathbf{p})+\text{antiparticle}],$$

which is the conventional point-particle field for spin $\frac{1}{2}$.

There is no *a priori* way of determining which viewpoint is preferable. Therefore calculations of, say, decay widths based on both will be presented in this paper. In discussing two-body decays, we will limit ourselves to the case of resonance interacting with nucleon and pion, i.e., excited state denoted by quantum numbers $(j,\,m_j,\,l,\,i,\,n_s,\,n_t,\,\alpha')$ decaying into point pion and (1) the ground state, of quantum numbers $(\frac{1}{2},\,m_{1/2},\,1,\,\frac{1}{2},\,1,\,0,\,\alpha')$ from the one point of view, or (2) a point nucleon, from the other viewpoint.

The second case is discussed in Sec. V; the former is discussed in this and the following sections. Thus, consider a pion impinging on the ground state, exciting it

[17] See, e.g., A. de-Shalit and I. Talmi, *Nuclear Shell Theory* (Academic Press Inc., New York, 1963), p. 40.
[18] I. N. Sneddon, *Special Functions of Mathematical Physics and Chemistry* (Oliver and Boyd, London, 1961).
[19] Ref. 18, Sec. 44.
[20] Ref. 18, Sec. 11.
[21] Ref. 18, Sec. 40.
[22] Ref. 13, Chap. 2.

TABLE I. Nucleon-resonance table, from Ref. 33.

l	j^P	i	γ	m_R (MeV)	m_R^2 (BeV)^2
1	$\frac{3}{2}+$	$\frac{3}{2}$	$-$	1236	1.53
3	$\frac{7}{2}+$	$\frac{3}{2}$	$-$	1950	3.80
5	$\frac{11}{2}+$(?)	$\frac{3}{2}$	$-$	2420	5.86
7	$\frac{15}{2}+$(?)	$\frac{3}{2}$	$-$	2850	8.12
9	$\frac{19}{2}+$(?)	$\frac{3}{2}$	$-$	3230	10.4
2	$\frac{3}{2}-$	$\frac{1}{2}$	$+$	1525	2.33
4	$\frac{7}{2}-$	$\frac{1}{2}$	$+$	2200	4.84
6	$\frac{11}{2}-$(?)	$\frac{1}{2}$	$+$	2650	7.02
8	$\frac{15}{2}-$(?)	$\frac{1}{2}$	$+$	3030	9.18
1	$\frac{1}{2}+$	$\frac{1}{2}$	$+$	938	0.88
3	$\frac{5}{2}+$	$\frac{1}{2}$	$+$	1690	2.86
1	$\frac{1}{2}+$	$\frac{1}{2}$	$+$	1470	2.16
0	$\frac{1}{2}-$	$\frac{1}{2}$	$-$	1550	2.4
2	$\frac{5}{2}-$	$\frac{1}{2}$	$-$	1680	2.82
0	$\frac{1}{2}-$	$\frac{1}{2}$	$-$	1710	2.92
0	$\frac{1}{2}-$	$\frac{3}{2}$	$-$	1640	2.69

into a $(j, m_j, l, i, n_s, n_t, \alpha')$ level. In this process no less than $(l-1)$ units of angular momentum are introduced into the nucleon system by the pion. Neglecting the recoil of the more massive excited state,[23] the operator connecting the nucleon states connects states containing $(2n_s+l-2)$ three-dimensional harmonic-oscillator quanta and one three-dimensional harmonic-oscillator quantum. Thus, if the lowest-order term in $|\mathbf{p}_\pi|$ is to contribute to the decay width, then $l-1\geqslant 2n_s+l-3$, or $n_s\leqslant 1$. Thus $n_s=1$.

As regards n_t, the "time principal quantum number," we may argue that, in the same limit, it must be equal to the value of the ground-state quantum number. The matrix element in the decay amplitude, to lowest order, contains $\int T_{n_t}^*(r_0)T_0(r_0)dr_0$, which, by the orthonormality condition, gives δ_{n_t0}.

The above arguments, of course, are not compelling proofs; they are, nonetheless, some justification for restricting the states to $(n_s,n_t)=(1,0)$. With this restriction Eq. (13) then becomes

$$m_R^2=-V_0+4\nu(l+1). \qquad (13')$$

We define a "family"[24] of states to be those of equal parity, equal isospin, and equal "γ parity".[25] This latter quantity distinquishes between states whose $\mathbf{l}$ and $\frac{1}{2}$ angular-momentum vectors couple antiparallel or parallel; the former are defined to have positive γ parity, the latter negative γ parity.[25] If we wish to consider the mass spectra for a particular family of states, we must remove the degeneracies remaining in Eq. (13′) due to the spin-, isospin-, and parity-independent choice made for $V(r^2)$.

We assume that the following are small perturbation terms and are added to $V(r^2)$:

$$\sum_{j,i,P} V_{ij}{}^P(r^2)\Pi_j\Pi_iP\,,$$

where Π_j and Π_i are projection operators for spin and isospin, respectively, and P is the parity operator. Specifically,

$$\Pi_{j=l+1/2}=(l+1+2\mathbf{L}\cdot\mathbf{S})/(2l+1)\,,$$
$$\Pi_{j=l-1/2}=(l-2\mathbf{L}\cdot\mathbf{S})/(2l+1)\,,$$
$$\Pi_{i=3/2}=\tfrac{1}{3}(2\mathbf{1}\cdot\mathbf{T}+2)\,,$$
$$\Pi_{i=1/2}=\tfrac{1}{3}(1-2\mathbf{1}\cdot\mathbf{T})\,,$$

where $\mathbf{L}$ is the angular-momentum operator, $\mathbf{S}$ the spin operator, $\mathbf{T}$ the isospin-$\frac{1}{2}$ and $\mathbf{1}$ the isospin-1 operators, respectively. The simplest choice for $V_{ji}{}^P$ is to take all equal to constants, thereby leaving the slopes of the $m_R^2\propto j$ lines equal for all families. This is only approximately correct. For the $(P=+1,\ i=\frac{3}{2},\ \gamma=-1)$ family (see Table I), $4\nu=113.5\times10^4$ MeV^2 and $V_0=74\times10^4$ MeV^2; for the $(P=-1,\ i=\frac{1}{2},\ \gamma=+1)$ family, $4\nu=113.5\times10^4$ MeV^2 and $V_0=88\times10^4$ MeV^2; however, for the $(P=+1, i=\frac{1}{2}, \gamma=+1)$ family, $4\nu=98\times10^4$ MeV^2 and $V_0=108\times10^4$ MeV^2. A term $2\nu br^2$ may be added to the above function; b is calculated to be -15.5×10^4 MeV^2.

Apart from the levels included in the above discussion, there are other energies at which πN partial-wave amplitudes become purely imaginary, suggesting the existence of other resonance levels (see Table I). It is not clear, however, if or how these possible levels can be interpreted within the framework of this model. For those states whose spin, parity, isospin, and γ parity are identical to those already given, it might be conjectured that their principal quantum numbers are different from those assigned, i.e., different from $(n_s,n_t)=(1,0)$. We may speculate that there are, say, $(n_s,n_t)=(2,1)$ levels, whose mass squared is, thus, (4ν) MeV^2 above those of the $(n_s,n_t)=(1,0)$ family. Taking $4\nu=113.5\times10^4$ MeV^2 and the ground state, $m_N^2=88\times10^4$ MeV^2, this new P_{11} level has $m_R\approx1425$ MeV.

IV. DECAY OF AN EXCITED STATE INTO PION AND NUCLEON

In general, the coupling of two nucleon states to a pion may be represented by the Lagrange density function

$$\mathcal{L}(X,r)=\bar{\Psi}(X,r)\gamma_5\Psi(X_N,r)\phi(X_\pi)+\text{H.c.}\,, \qquad (14)$$

where $\Psi(X,r)$ is the resonance field operator at center-of-mass point X, $\psi(X_N,r)$ the resonance field operator

[23] See the Appendix for a more detailed treatment of these arguments.

[24] A "family of states" is the same as a Regge trajectory. In our formalism, however, there are no "trajectories," only states; hence, we prefer the name "families."

[25] P. Carruthers, Ref. 1, and Ref. 3. The convention of the first paper was changed in the second; that of the later paper is adapted here.

at center-of-mass point X_N, and $\phi(X_\pi)$ the pseudoscalar point pion operator[26] at point X_π. Here

$$X_N = X + (m_\pi/m_N+m_\pi)(X_N - X_\pi),$$
$$X_\pi = X - (m_N/m_N+m_\pi)(X_N - X_\pi),$$

and

$$X_N - X_\pi = \beta r,$$

where β is a constant depending on the masses of the nucleon and pion, m_N and m_π, respectively. The matrix γ_5 is inserted in order to conserve parity, since the two resonance fields transform alike under parity (see Sec. II).

Specifically, if we are interested in the coupling of a *family of states* to a nucleon and a pion, we write

$$M_{m_j m_{1/2}} \equiv g\langle j m_j l_i \gamma;\, \mathbf{p}_R | \int dX dr\, \bar{\Psi}(X,r)\gamma_5 \Psi(X_N,r) \times \phi(X_\pi) | \tfrac{1}{2} m_{\frac{1}{2}} 1 \tfrac{1}{2} +;\, \mathbf{p}_N | 0;\, \mathbf{p}_\pi \rangle, \quad (15)$$

where the coupling parameter g is assumed to be the same for all members of the same family; $\mathbf{p}_R$ is the resonance three momentum, $\mathbf{p}_N$ and $\mathbf{p}_\pi$ the nucleon and pion three momentum, respectively.

Using Eqs. (2) and (7) in the above, we obtain

$$M_{m_j m_{1/2}} = g(2\pi)^4 \delta^{(4)}(p_R - p_N - p_\pi) \sum_{\sigma\sigma'} C(l_i \tfrac{1}{2} j;\, m_j - \sigma\, \sigma) \bar{u}^{(\sigma)}(p_R) \gamma_5 C(1\tfrac{1}{2}\tfrac{1}{2};\, m_{\frac{1}{2}} - \sigma'\, \sigma') u^{(\sigma')}(p_N) F_{\sigma\sigma'}(p_R, p_N, p_\pi), \quad (16)$$

where

$$F_{\sigma\sigma'} \equiv \int d^4r\, Y_{l_i, m_j-\sigma}{}^*(\hat{L}^{-1}(p_R)\cdot r) |\mathbf{L}^{-1}(p_R)\cdot r|^{l_i} h_{j l_i \gamma}{}^*(|\mathbf{L}^{-1}(p_R)\cdot r|, L^{-1}{}_{4\mu}(p_R)\cdot r_\mu) e^{i\beta p^{\mathrm{rel}}\cdot r} \times Y_{1, m_{1/2}-\sigma'}(\hat{L}^{-1}(p_N)\cdot r) |\mathbf{L}^{-1}(p_N)\cdot r| h_{\frac{1}{2}1\frac{1}{2}+}(|\mathbf{L}^{-1}(p_N)\cdot r|, L^{-1}{}_{4\nu}(p_N)\cdot r_\nu). \quad (17)$$

$F_{\sigma\sigma'}$ is, in a sense, a "vertex form factor." In Eq. (17) we have defined the four-vector

$$p_\mu{}^{\mathrm{rel}} = \frac{m_\pi}{m_N + m_\pi}(p_N)_\mu - \frac{m_N}{m_N + m_\pi}(p_\pi)_\mu.$$

In the center-of-momentum system, i.e., $\mathbf{p}_R = 0$, the vector $\mathbf{p}^{\mathrm{rel}}$ becomes $\mathbf{p}^{\mathrm{rel}} = \mathbf{p}_N = -\mathbf{p}_\pi \equiv \mathbf{p}$. In the harmonic-oscillator approximation, restricting the states to those with $(n_s, n_t) = (1,0)$, $F_{\sigma\sigma'}$ becomes

$$F_{\sigma\sigma'} = N_{l_i} \int d^4r\, Y_{l_i, m_j-\sigma}{}^*(\hat{r}) |\mathbf{r}|^{l_i} e^{-\nu|\mathbf{r}|^2} e^{-\nu r_0^2} e^{i\beta \mathbf{p}\cdot\mathbf{r}} Y_{1, m_{1/2}-\sigma'}(\hat{L}^{-1}(p)\cdot r) |\mathbf{L}^{-1}(p)\cdot r| e^{-\nu|\mathbf{L}^{-1}(p)\cdot r|^2} e^{-\nu(p\cdot r/m)^2} e^{-\beta p_0^{\mathrm{rel}} r_0}, \quad (17')$$

where

$$N_{l_i} = \frac{4}{\pi}\left(\frac{2(2\nu)^{l_i+5} 2^{2l_i} l_i!}{3(2l_i+1)!}\right)^{1/2} (-1)^{l_i}.$$

The calculation of $F_{\sigma\sigma'}$ is straightforward but lengthy, and the details are given in the Appendix. Here we cite the result [Eq. (A12)]:

$$F_{\sigma\sigma'} = N_{l_i}\left(\frac{3}{4\pi}\right)^{1/2} \sum_{\bar{l}} C(1\bar{l}l_i;\, m_{1/2}-\sigma'\, m_j - \sigma - m_{1/2} + \sigma') C(1 l_i \bar{l};\, 00) Y_{\bar{l}, m_j-\sigma-m_{1/2}+\sigma'}{}^*(\hat{p}) \times \exp\left[-\beta^2 \frac{m_N{}^2}{8\nu}\left(\frac{p_0{}^{\mathrm{rel}}}{p_0}\right)^2\right] \left(\frac{\pi}{2\nu}\right)^{1/2} \left(\frac{m_N}{p_0}\right) \left\{ \frac{i^{l_i-1} x^{l_i-1} e^{-x^2}}{(2\nu)^{(1/2)(l_i+4)}} \times \left[-\left(1 - \frac{m_N}{p_0}\right)(x^2 - \tfrac{1}{2}l_i) + \frac{x^2}{\alpha}\frac{m_N}{p_0}\left(\frac{p_0{}^{\mathrm{rel}}}{p_0}\right)\right] - (4\pi)^{1/2} \langle r^{l_i+1}\rangle_{\bar{l}l_i} \right\}$$

$$\equiv N_{l_i}\left(\frac{3}{4\pi}\right)^{1/2} \sum_{\bar{l}} C(1\bar{l}l_i;\, m_{1/2}-\sigma'\, m_j - \sigma - m_{1/2} + \sigma') C(1\bar{l}l_i;\, 00) Y_{\bar{l}, m_j-\sigma-m_{1/2}+\sigma'}{}^*(\hat{p}) \times \exp[-\beta^2 (m_N{}^2/8\nu)(p_0{}^{\mathrm{rel}}/p_0)^2] F_{\bar{l}l_i 1}, \quad (18)$$

where we have defined

$$x \equiv \frac{\beta\alpha|\mathbf{p}|}{(8\nu)^{1/2}}, \quad \alpha \equiv 1 - \frac{p_0{}^{\mathrm{rel}}}{p_0},$$

and

$$\langle r^{l_i+1}\rangle_{\bar{l}l_i} \equiv i^{\bar{l}} \int_0^\infty r^{l_i+3} e^{-2\nu r^2} j_{\bar{l}}(\alpha\beta|\mathbf{p}|r) dr,$$

[26] Ref. 13, Chap. 1.

$j_{\bar{l}}(kr)$ being the spherical Bessel function of order $\bar{l}$.[27] Using this result in Eq. (16), we obtain

$$M_{m_jm_{1/2}}=g(2\pi)^4\delta^{(4)}(p_R-p_N-p_\pi)N_{l_i}\left(\frac{1}{4\pi}\right)^{1/2}\left(\frac{1}{2m_N(E_N+m_N)}\right)^{1/2}\exp\left[-\beta^2\frac{m_N^2}{8\nu}\left(\frac{p_0^{\rm rel}}{p_0}\right)^2\right]|\mathbf{p}|6(2l_i+1)$$
$$\times C(l_i\tfrac{1}{2}j;\, m_j-m_{1/2}\, m_{1/2})Y_{l_im_j-m_{1/2}}{}^*(\hat{p})\sum_{\bar{l}}[C(1l_i\bar{l};\,00)]^2X(\tfrac{1}{2}l_ij;\,\tfrac{1}{2}1\tfrac{1}{2};\,1\bar{l}l_i)F_{\bar{l}l_i1}, \quad (19)$$

where $X(\frac{1}{2}l_ij;\,\frac{1}{2}1\frac{1}{2};\,1\bar{l}l_i)$ is the 9-j symbol.[28] This result can be obtained from the formula[29]

$$\bar{u}^{(\sigma)}(0)\gamma_5u^{(\sigma')}(p)=|\mathbf{p}|(4\pi)^{1/2}[2m_N(m_N+E_N)]^{-1/2}C(1\tfrac{1}{2}\tfrac{1}{2};\,\sigma-\sigma'\,\sigma')Y_{1,\sigma-\sigma'}{}^*(\hat{p}),$$

and from Eq. (11.31) of Ref. 8. Now[30]

$$\sum_{\bar{l}}[C(1l_i\bar{l};\,00)]^2X(\tfrac{1}{2}l_ij;\,\tfrac{1}{2}1\tfrac{1}{2};\,1\bar{l}l_i)=1/6(2l_i+1)$$

and

$$\sum_{\bar{l}}[C(1l_i\bar{l};\,00)]^2X(\tfrac{1}{2}l_ij;\,\tfrac{1}{2}1\tfrac{1}{2};\,1\bar{l}l_i)\langle r^{l_i+1}\rangle_{\bar{l}l_i}=\begin{cases}-\dfrac{1}{6(2l_i+1)}\left(\dfrac{1}{4\pi}\right)^{1/2}\dfrac{i^{l_i-1}x^{l_i-1}e^{-x^2}}{(2\nu)^{(1/2)(l_i+4)}}(x^2+\tfrac{1}{2}), & j=l_i-\tfrac{1}{2}\\ -\dfrac{1}{6(2l_i+1)}\left(\dfrac{1}{4\pi}\right)^{1/2}\dfrac{i^{l_i-1}x^{l_i-1}e^{-x^2}}{(2\nu)^{(1/2)(l_i+4)}}(x^2-l_i), & j=l_i+\tfrac{1}{2}\end{cases}$$
$$\equiv-\frac{1}{6(2l_i+1)}\left(\frac{1}{4\pi}\right)^{1/2}\frac{i^{l_i-1}x^{l_i-1}e^{-x^2}}{(2\nu)^{(1/2)(l_i+4)}}S_{jl_i}(x).$$

These equations can be obtained by using Eq. (A10) of the Appendix.

Thus we arrive at the relatively simple result for the decay amplitude,

$$M_{m_jm_{1/2}}=g(2\pi)^4\delta^{(4)}(p_R-p_N-p_\pi)N_{l_i}\tfrac{1}{2}[2m_N(m_N+E_N)]^{-1/2}|\mathbf{p}|C(l_i\tfrac{1}{2}j;\,m_j-m_{1/2}\,m_{1/2})Y_{l_i,m_j-m_{1/2}}{}^*(\hat{p})$$
$$\times\exp\left[-\beta^2\frac{m_N^2}{8\nu}\left(\frac{p_0^{\rm rel}}{p_0}\right)^2\right]\frac{i^{l_i-1}x^{l_i-1}e^{-x^2}}{(2\nu)^{(1/2)(l_i+5)}}\frac{m_N}{p_0}\left[-\left(1-\frac{m_N}{p_0}\right)(x^2-\tfrac{1}{2}l_i)+\frac{1}{\alpha}x^2\frac{m_N}{p_0}\left(\frac{p_0^{\rm rel}}{p_0}\right)+S_{jl_i}(x)\right]. \quad (20)$$

The partial decay width for the two-body decay is[31]

$$\Gamma_{l_i}{}^j=\frac{1}{(2\pi)^5}\int\frac{m_N}{2p_{\pi0}p_{N0}}\frac{1}{2j+1}\sum_{m_jm_{1/2}}|M_{m_jm_{1/2}}|^2d^3p_Nd^3p_\pi$$
$$=\frac{g^2}{4\pi}|\mathbf{p}|\frac{|\mathbf{p}|^2}{m_R(m_N+p_0)}\frac{2^{2l_i}l_i!}{(2l_i+1)!}\frac{(x^2)^{l_i-1}e^{-2x^2}}{(p_0/m_N)^2}\exp\left[-\beta^2\frac{m_N^2}{4\nu}\left(\frac{p_0^{\rm rel}}{p_0}\right)^2\right]|\mathfrak{F}_{l_i}{}^{(j)}(x)|^2, \quad (21)$$

where

$$\mathfrak{F}_{l_i}{}^{(j)}(x)\equiv\left[-\left(1-\frac{m_N}{p_0}\right)(x^2-\tfrac{1}{2}l_i)+\frac{x^2}{\alpha}\frac{m_N}{p_0}\left(\frac{p_0^{\rm rel}}{p_0}\right)+S_{jl_i}(x)\right]$$

and we have redefined g to include constant factors not explicitly appearing in Eq. (21).

The decay width Eq. (21) contains a number of interesting features. For decreasingly small $|\mathbf{p}|$, $\Gamma_{l_i}{}^j$ goes as $|\mathbf{p}|^{2l_i+1}$, as it should,[32] but for increasing $|\mathbf{p}|$ it need not increase in spite of the rapidly increasing factors $|\mathbf{p}|^3x^{2l_i-2}|\mathfrak{F}_{l_i}{}^{(j)}(x)|^2$. The Gaussian e^{-2x^2} and the two factors $(m/p_0)^22^{2l_i}l_i!/(2l_i+1)!$ decrease in such a way that the net effect on $\Gamma_{l_i}{}^j$, for appropriate β, is a relatively slow decrease (see Tables II–IV).

The constant β is determined from the experimental data. We choose the first two members of the (j, l= odd, $i=\frac{3}{2}$, $\gamma=-1$) family, the $\Delta_{3/2}(1240)$ and $\Delta_{3/2}(1920)$ resonances. Inserting the proper kinematic and spin factors into Eq. (21), we obtain

$$\mathfrak{R}(\beta^2)\equiv\frac{\Gamma_1{}^{3/2}(\beta^2)}{\Gamma_3{}^{7/2}(\beta^2)}=\frac{3.41}{(\beta^2)^2}e^{1.025\beta^2}\left(\frac{0.025\beta^2-0.985}{0.226\beta^2-2.67}\right)^2. \quad (22)$$

[27] Ref. 18, Chap. IV.
[28] Ref. 18, Chap. XI.
[29] G. Källén, *Elementary Particle Physics* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1964), p. 109.
[30] Ref. 8, Chap. XI.
[31] Ref. 13, Chap. 9.
[32] Ref. 11, Vol. I, Chap. X.

TABLE II. Partial-decay widths $\Gamma(\Delta\to\pi N)$ for the (j, l=odd, $i=\frac{3}{2}$, $\gamma=-1$) family, assuming the nucleon is the ground state of the resonance system.

l	j^P	m_R (MeV)	i	γ	$\Gamma_{l_i}{}^j$, Eq. (21) (MeV) $\beta^2=1.09$	$\beta^2=2.4$	$\Gamma_{l_i}{}^j$, experimental (MeV) Data[a]			Average[b]
1	$\frac{3}{2}+$	1236	$\frac{3}{2}$	−	120[c]	120[d]	120[e]	124[f]	125.1[g]	120
3	$\frac{7}{2}+$	1920	$\frac{3}{2}$	−	90	90	124[h] 80[k] 85[g]	146[i] 103[k]	70[j] 80[l]	88
5	$\frac{11}{2}+$(?)	2420	$\frac{3}{2}$	−	24	20	13[m] 31[p]	35[n] 45[q]	32[o] 17[r]	34
7	$\frac{15}{2}+$(?)	2850	$\frac{3}{2}$	−	5	3.5	13[n] 24[q]	11[o] 3.8[r]	12[p]	12
9	$\frac{19}{2}+$(?)	3230	$\frac{3}{2}$	−	1.3	0.55	2.6[n] 1.1[q]	1.3[o] 0.11[r]	1.3[p]	2

[a] The experimental partial widths as listed in Ref. 33. This includes, essentially, the work of the last five years. Whenever the spin assignment of a level is not certain, it is usually the quantity $(j+\frac{1}{2})(\Gamma_{el}/\Gamma)$ that is given by experiment, and not (Γ_{el}/Γ). Thus the partial widths listed here are calculated from experiment by assuming our spin-parity assignments.
[b] Reference 33.
[c] $g^2/4\pi|_{\beta^2=1.09}=43$.
[d] $g^2/4\pi|_{\beta^2=2.4}=50$.
[e] M. G. Olsson, Phys. Rev. Letters **14**, 118 (1965).
[f] G. Gidal, A. Kernan, and S. Kim, Phys. Rev. **141**, 1261 (1966).
[g] C. Lovelace, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (Wiley-Interscience, Inc., New York, 1968).
[h] G. Holder and G. Ebel, Nucl. Phys. **48**, 470 (1963).
[i] T. J. Devlin, J. Solomon, and G. Bertsch, Phys. Rev. Letters **14**, 1031 (1965).
[j] P. J. Duke, D. P. Jones, M. A. R. Kemp, P. G. Murphy, J. D. Prentice, J. J. Thresher, H. H. Atkinson, C. R. Cox, and K. S. Heard, Phys. Rev. Letters **15**, 468 (1965).
[k] A. Yokosawa, S. Suwa, R. E. Hill, R. J. Esterling, and N. E. Booth, Phys. Rev. Letters **16**, 714 (1966).
[l] P. Bareyre, C. Bricman, and G. Villet, Phys. Rev. **165**, 1730 (1968).
[m] A. N. Diddens, E. W. Jenkins, T. F. Kycia, and K. F. Riley, Phys. Rev. Letters **10**, 262 (1963).
[n] A. Citron, W. Galbraith, T. F. Kycia, B. A. Leontic, R. H. Phillips, A. Rousset, and P. H. Sharp, Phys. Rev. **144**, 1011 (1966).
[o] V. Barger and M. Olsson, Phys. Rev. **151**, 1123 (1966).
[p] V. Barger and D. Cline, Phys. Rev. **155**, 1792 (1967).
[q] F. N. Dikmen, Phys. Rev. Letters **18**, 798 (1967).
[r] S. W. Kormanyos, A. D. Krisch, J. R. O'Fallon, K. Ruddick, and L. G. Ratner, Phys. Rev. **164**, 1661 (1967).

The ratio $\Gamma_1{}^{3/2}/\Gamma_3{}^{7/2}\approx\frac{4}{3}$ is taken from experiment[33]; the four values of β^2 which will satisfy $\mathcal{R}(\beta^2)\approx\frac{4}{3}$ are $\beta^2=1.09$, $\beta^2=2.4$, and two solutions $\beta^2\approx 38.8$. Only the first two values will give reasonable results because $\beta^2\approx 38.8$ in the exponential causes $\Gamma_{l_i}{}^j$ to decrease much too rapidly (for $\beta^2=38.779531$ such that $\mathcal{R}(\beta^2)=\frac{4}{3}$, $\Gamma(1240)/\Gamma(2420)\approx 10^{17}$). Tables II–IV lists the partial decay widths for those families of resonances for which we have at least two possible members. It is seen that for both values of β^2, Eq. (21) is a reasonably accurate approximation to the experimental partial widths. It is interesting to note that had the choice $n_l=1$ been made instead of $n_l=0$, the resulting expression for the decay width would have been somewhat similar to Eq. (21) for the low-lying resonances, except for an additional over-all factor of $(m/p_0)^2$. The resulting expression

TABLE III. Partial-decay widths $\Gamma(N^*\to\pi N)$ for the (j, l=even, $i=\frac{1}{2}$, $\gamma=+1$) family, assuming the nucleon is the ground state of the resonance system.

l	j^P	m_R (MeV)	i	γ	$\Gamma_{l_i}{}^j$, Eq. (21) (MeV) $\beta^2=1.08$	$\beta^2=2.4$	$\Gamma_{l_i}{}^j$, experimental (MeV) Data[a]			Average[b]
2	$\frac{3}{2}-$	1518	$\frac{1}{2}$	+	63[c]	63[d]	67[e] 66[g]	60[e]	65[f]	63
4	$\frac{7}{2}-$	2190	$\frac{1}{2}$	+	39	42	60[h]	66[i]	90[f]	75
6	$\frac{11}{2}-$(?)	2650	$\frac{1}{2}$	+	11	9	26[j] 17[m]	32[k] 18[n]	28[l]	27
8	$\frac{15}{2}-$(?)	3030	$\frac{1}{2}$	+	3.5	1.7	2.4[j] 0.8[m]	4.4[k]	2.8[l]	2.5

[a] The experimental partial widths as listed in Ref. 33. This includes, essentially, the work of the last five years. Whenever the spin assignment of a level is not certain, it is usually the quantity $(j+\frac{1}{2})(\Gamma_{el}/\Gamma)$ that is given by experiment, and not (Γ_{el}/Γ). Thus the partial widths listed here are calculated from experiment by assuming our spin-parity assignments.
[b] Reference 33.
[c] $g^2/4\pi|_{\beta^2=1.09}=160$.
[d] $g^2/4\pi|_{\beta^2=2.4}=82$.
[e] P. Bareyre, C. Bricman, and G. Villet, Phys. Rev. **165**, 1730 (1968).
[f] C. Lovelace, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (Wiley-Interscience Publishers, Inc., New York, 1968).
[g] P. Bareyre, C. Bricman, A. V. Stirling, and G. Villet, Phys. Letters **18**, 342 (1965); B. H. Brandsen, P. J. O'Donnell, and R. G. Moorhouse, Phys. Rev. **139**, B1566 (1955).
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[i] A. Yokosawa, S. Suwa, R. E. Hill, R. J. Esterling, and N. E. Booth, Phys. Rev. Letters **16**, 714 (1966).
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[k] V. Barger and M. Olsson, Phys. Rev. **151**, 1123 (1966).
[l] V. Barger and D. Cline, Phys. Rev. **155**, 1792 (1967).
[m] F. N. Dikmen, Phys. Rev. Letters **18**, 798 (1967).
[n] S. W. Kormanyos, A. D. Krisch, J. R. O'Fallon, K. Ruddick, and L. G. Ratner, Phys. Rev. **164**, 1661 (1967).

[33] A. Rosenfeld, N. Barash-Schmidt, A. Barbaro-Galtieri, L. Price, P. Söding, C. Wohl, M. Roos, and W. Willis, Rev. Mod. Phys. **40**, 77 (1968); see especially "Baryons Table," p. 83.

TABLE IV. Partial decay widths $\Gamma(N^* \to \pi N)$ for the (j, l=odd, $i=\frac{1}{2}$, $\gamma=+1$) family, assuming the nucleon is the ground state of the resonance system.

l	j	m_R (MeV)	i	γ	Γ_{l_i}, Eq. (21) (MeV) $\beta^2=1.09$	$\beta^2=2.4$	Γ_{l_i}, experimental (MeV) Data[a]	Average[b]
1	$\frac{1}{2}^+$	938	$\frac{1}{2}$	+				
3	$\frac{5}{2}^+$	1688	$\frac{1}{2}$	+	85[c]	85[d]	124[e] 146[f] 70[g] 80[h] 103[i] 80[i] 85[j]	85

[a] The experimental partial widths as listed in Ref. 33. This includes, essentially, the work of the last five years. Whenever the spin assignment of a level is not certain, it is usually the quantity $(j+\frac{1}{2})(\Gamma_{el}/\Gamma)$ that is given by experiment, and not (Γ_{el}/Γ). Thus the partial widths listed here are calculated from experiment by assuming our spin-parity assignments.
[b] Reference 33.
[c] $g^2/4\pi|_{\beta^2=1.09}=820$.
[d] $g^2/4\pi|_{\beta^2=2.4}=250$.
[e] G. Holder and G. Ebel, Nucl. Phys. **48**, 470 (1963).
[f] T. J. Devlin, J. Solomon, and G. Bertsch, Phys. Rev. Letters **14**, 1031 (1965).
[g] P. J. Duke, D. P. Jones, M. A. R. Kemp, P. G. Murphy, J. D. Prentice, J. J. Thresher, H. H. Atkinson, C. R. Cox, and K. S. Heard, Phys. Rev. Letters **15**, 468 (1965).
[h] A. Yokosawa, S. Suwa, R. E. Hill, R. J. Esterling, and N. E. Booth, Phys. Rev. Letters **16**, 714 (1966).
[i] P. Bareyre, C. Bricman, and G. Villet, Phys. Rev. **165**, 1730 (1968).
[j] C. Lovelace, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (Wiley-Interscience, Inc., New York, 1968).

$\mathcal{R}(\beta^2)=\Gamma_1^{3/2}(\beta^2)/\Gamma_3^{7/2}(\beta^2)$, for reasonable values of β^2, will not permit a value of β^2 for which $\mathcal{R}(\beta^2)<\sim 2$. This adds confidence in our original choice.

We may compare the expression obtained for the partial decay width, Eq. (21), with that obtained from an isobar-model calculation. This latter expression is[1]

$$\Gamma_{\text{i.m.}}^{\ j}=\frac{g_{\text{i.m.}}^2}{4\pi}\frac{2^n(n!)^2}{(2n+1)!}\frac{E_N\pm M_N}{m_R}\left(\frac{|\mathbf{p}|}{m_\pi}\right)^{2n+1}m_\pi\,,\qquad j=l_i\pm\tfrac{1}{2}$$

where $n=j-\frac{1}{2}$ and m_π is the mass of the pion; the subscript (i.m.) designates "isobar model". Especially interesting is the case $j=l_i-\frac{1}{2}$. Putting $\Gamma_{l_i}^{\ j=l_i-1/2}=\Gamma_{\text{i.m.}}^{\ j=l_i-1/2}$, we obtain

$$\frac{g_{\text{i.m.}}^2}{4\pi}=\frac{g^2}{4\pi}\frac{(2n+1)!}{(2l_i+1)!}\frac{2^{2l_i}l_i!}{2^n(n!)^2}\left(\frac{x^2}{|\mathbf{p}|^2/m_\pi}\right)^{l_i-1}\times\frac{e^{-2x^2}}{(p_0/m_N)^2}\exp\left[-\beta^2\frac{m_N^2}{4\nu}\left(\frac{p_0^{\text{rel}}}{p_0}\right)^2\right]|\mathfrak{F}_{l_i}^{(j)}(x)|^2.\quad(23)$$

For the (j, l=odd, $i=\frac{1}{2}$, $\gamma=+1$) family, for which the nucleon is the first member, we have obtained the results (see Table IV) $g^2/4\pi|_{\beta^2=1.09}=820$, $g^2/4\pi|_{\beta^2=2.4}=250$,

$$g_{\pi^0 nN^{*0}}^2/4\pi|_{\beta^2=1.09}=820/3\approx 273\,,$$
$$g_{\pi^0 nN^{*0}}^2/4\pi|_{\beta^2=2.4}=250/3\approx 83\,.$$

These numbers are calculated from the experimental partial decay width $\Gamma(1688)=85$ MeV. We formally extend these results to the case of πNN coupling by putting $l_i=1$ in Eq. (23) and taking the limit $m_R\to m_N$. In that limit we obtain

$$g_{\text{i.m.}}^2/4\pi|_{\beta^2=1.09}=45.6\,,\quad g_{\text{i.m.}}^2/4\pi|_{\beta^2=2.4}=14.2\,.$$

With the latter value of β^2, $\beta^2=2.4$, we have obtained a good approximation of the πNN coupling constant $(g_{\text{i.m.}}|_{l_i=1,m_R=m_N}=g_{\pi NN})$.

Conversely, we could take the πNN coupling constant as given, and obtain a prediction for the experimental value of $\Gamma_{l_i=3}^{\ j=5/2}(1688)$.

V. DECAY OF AN EXCITED STATE INTO PION AND POINT NUCLEON

We now consider the nucleon to be a point particle represented by the field

$$\Psi(x)=\sum_m\int d^4p\;\delta(p^2+m_N^2)\theta(p_0)\times[u^{(m)}(p)A_m(p)e^{-ipx}+\text{antiparticle}],$$

and the Lagrange density function describing the coupling $J-\frac{1}{2}-0$ can be written

$$\mathcal{L}(X,r)=\bar{\Psi}(X,r)\psi(X_N)\phi(X_\pi)+\text{H.c.},\qquad(24)$$

where $\Psi(X,r)$ is the resonance operator at the center-of-mass point X, and $\psi(x_N)$ the point pion operator at point x_n; it is assumed that $x_N-x_\pi=r$. In this case no matrix γ_5 is needed for parity conservation because $\Psi(X,r)\xrightarrow{P}-\gamma_4\Psi(i_pX,i_pr)$.

The matrix element describing the decay of a family of states into π and N, characterized by the parameter g, is

$$M_{m_jm_{1/2}}=g\langle jm_jl_ii\gamma;\,\mathbf{p}_R|\int\bar{\Psi}(X,r)\psi\left(X+\frac{m_\pi}{m_N+m_\pi}r\right)\phi\left(X+\frac{m_N}{m_N+m_\pi}r\right)dXdr|\tfrac{1}{2}m_{1/2}1\tfrac{1}{2}+;\,\mathbf{p}_N|0;\,\mathbf{p}_\pi\rangle$$
$$=g(2\pi)^4\delta^{(4)}(p_R-p_N-p_\pi)\sum_\sigma C(l_i\tfrac{1}{2}j;\,m_j-\sigma\,\sigma)\bar{u}^{(\sigma)}(p_R)u^{(m_{1/2})}(p_N)F_\sigma(p_R,p_N,p_\pi)\,,\qquad(25)$$

where

$$F_\sigma\equiv\int d^4r\,Y_{l_i,m_j-\sigma}^*(\hat{L}^{-1}(p_R)\cdot r)\,|\mathbf{L}^{-1}(p_R)\cdot r|^{l_i}h_{jl_i\gamma}^*(|\mathbf{L}^{-1}(p_R)\cdot r|,L^{-1}{}_{4\mu}(p_R)\cdot r_\mu)e^{ip^{\text{rel}}\cdot r}$$

is the "form factor" expression. In this case the form factors are simply the Fourier transforms of the "wave functions." Again, we have defined

$$p_\mu^{\text{rel}}\equiv\frac{m_\pi}{m_N+m_\pi}(p_N)_\mu-\frac{m_N}{m_N+m_\pi}(p_\pi)_\mu\,.$$

In the harmonic-oscillator approximation in the center-of-momentum system ($\mathbf{p}_R=0$),

$$F_\sigma=\int Y_{l_i,m_j-\sigma}^*(\hat{r})|\mathbf{r}|^{l_i}R_{1l_i}^*(|\mathbf{r}|)e^{i\mathbf{p}\cdot\mathbf{r}}d\mathbf{r}\int d\mathbf{r}_0 T_0^*(r_0)e^{-i\mathbf{p}_0^{\mathrm{rel}}\cdot\mathbf{r}_0}.$$

This calculation is immeasurably simpler than the previous one. We easily obtain[34]

$$F_\sigma=\left(\frac{2^2(2\nu)^{l_i+3/2}2^{2l_i}l_i!}{\pi(2l_i+1)!}\right)^{1/2}(-1)^{l_i-1/2}\sum_{lm_l}\int d\mathbf{r}\,Y_{l_im_j-\sigma}^*(\hat{r})|\mathbf{r}|^{l_i}e^{-\nu|\mathbf{r}|^2}i^l j_l(|\mathbf{p}||\mathbf{r}|)Y_{lm_l}^*(\hat{r})Y_{lm_l}^*(\hat{p})e^{-(p_0^{\mathrm{rel}})^2/4\nu}$$

$$=\frac{(-1)^{l_i-1/2}}{\pi}\left(\frac{2^{2l_i}l_i!}{(2l_i+1)!}\right)^{1/2}e^{-(p_0^{\mathrm{rel}})^2/4\nu}Y_{l_im_j-\sigma}^*(\hat{p})i^{l_i}\left(\frac{|\mathbf{p}|}{\sqrt{(2\nu)}}\right)^{l_i}\left(\frac{1}{\sqrt{(2\nu)}}\right)^3 e^{-|\mathbf{p}|^2/4\nu}. \quad (26)$$

Therefore, the partial-decay width is

$$\Gamma_{l_i}{}^j=\frac{g^2}{4\pi}\frac{2^{2l_i}l_i!}{(2l_i+1)!}\frac{E_N+m_N}{m_R}e^{-|\mathbf{p}|^2/2\nu}e^{-(p_0^{\mathrm{rel}})^2/2\nu}\times\left(\frac{|\mathbf{p}|^2}{2\nu}\right)^{l_i}|\mathbf{p}|, \quad (27)$$

where, again, g has been redefined to include all constant factors not appearing explicitly in Eq. (27). Similar to the previous calculation, $\Gamma_{l_i}{}^j$ goes as $|\mathbf{p}|^{2l_i+1}$ for small $|\mathbf{p}|$, as it should,[32] and it does not increase with increasing $|\mathbf{p}|$ (and l_i) because of the Gaussian factor and the l_i factor, $2^{2l_i}l_i!/(2l_i+1)!$. Tables V–VII give the numerical results.

We cannot use Eq. (27) to derive the πNN coupling constant as we did previously by taking the limiting case of $m_R\to m_N$, because of our assumption here that the nucleon does not belong to the resonance spectra.

VI. SUMMARY AND DISCUSSION

Because of the simplifying features of this model, beginning from the central assumption that a resonance state can be represented as a bound state of point particles, together with the harmonic-oscillator approximation for the "potential," a framework is provided in which such physical quantities as mass spectra and decay widths can be calculated in a systematic way. As it turns out, in effect, it is unimportant to the calculation of the partial-decay width whether or not the nucleon itself is a member of the bound-state spectra (see Tables II–VII). The arguments in support of the choice $(n_s,n_t)=(1,0)$, however, do depend on the assumption that the nucleon is the ground state; similar arguments are not given for the point-particle assumption. As regards these "principal quantum numbers," it has implicitly been assumed that the states $(n_s,n_t)\neq(1,0)$ are superfluous states not corresponding to physical resonance states, although we cannot discount the possibility that they may well correspond to other subfamilies of states (see discussion at the end of Sec. IV). These latter, however, are represented by decay amplitudes whose lowest power of $|\mathbf{p}|$ is not $|\mathbf{p}|^l$ but rather $|\mathbf{p}|^{l+2}$ or higher (see Appendix).

The results obtained for the physical decay widths lend encouragement for the use of these methods, for they predict a relatively slow decrease of $\Gamma_{l_i}{}^j$ with increasing energy. Whether this is a singular feature of the harmonic-oscillator potential or is a feature of a more general set of relativistic potentials is not clear to us at this time. Equally encouraging is the fact that a good

TABLE V. Partial-decay widths $\Gamma(\Delta\to\pi N)$ for the (j, l=odd, $i=\frac{3}{2}$, $\gamma=-1$) family, assuming the nucleon is a point particle not belonging to the resonance system.

l	j^P	m_R (MeV)	i	γ	$\Gamma_{l_i}{}^j$, Eq. (27) (MeV)
1	$\frac{3}{2}^+$	1236	$\frac{3}{2}$	−	120[a]
3	$\frac{7}{2}^+$	1920	$\frac{3}{2}$	−	85
5	$\frac{11}{2}^+$(?)	2420	$\frac{3}{2}$	−	24
7	$\frac{15}{2}^+$(?)	2850	$\frac{3}{2}$	−	10
9	$\frac{19}{2}^+$(?)	3230	$\frac{3}{2}$	−	2.2

[a] $g^2/4\pi=6$.

TABLE VI. Partial-decay widths $\Gamma(N^*\to\pi N)$ for the (j, l=even, $i=\frac{1}{2}$, $\gamma=+1$) family, assuming the nucleon is a point particle not belonging to the resonance system.

l	j^P	m_R (MeV)	i	γ	Γ_{l_i}, Eq. (27) (MeV)
2	$\frac{3}{2}^-$	1518	$\frac{1}{2}$	+	63[a]
4	$\frac{7}{2}^-$	2180	$\frac{1}{2}$	+	34
6	$\frac{11}{2}^-$(?)	2650	$\frac{1}{2}$	+	9
8	$\frac{15}{2}^-$(?)	3030	$\frac{1}{2}$	+	4

[a] $g^2/4\pi=4.53$.

TABLE VII. Partial-decay widths $\Gamma(N^*\to\pi N)$ for the (j, l=odd, $i=\frac{1}{2}$, $\gamma=+1$) family, assuming the nucleon is a point particle not belonging to the resonance system.

l	j^P	m_R (MeV)	i	γ	Γ_{l_i}, Eq. (27) (MeV)
1	$\frac{1}{2}^+$	938	$\frac{1}{2}$	+	
3	$\frac{5}{2}^+$	1688	$\frac{1}{2}$	+	85[a]

[a] $g^2/4\pi=18.7$.

[34] Ref. 18, Sec. 33, Eq. (32.15).

approximation to the πNN coupling constant can be derived from our equations in the limit $m_R \to m_N$, namely, $g_{\pi NN}^2/4\pi \approx 14.2$.

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APPENDIX

Derivation of Equation (18)

First, we derive an expression for $|\mathbf{L}^{-1}(p)\cdot r|\,Y_{1m_1}(\hat{L}^{-1}(p)\cdot r)$ in terms of $\mathbf{r}$ and $\mathbf{p}$. $L^{-1}{}_{k\mu}(p)\cdot r_\mu$ can be written $L^{-1}{}_{k\mu}(p)\cdot r_\mu = r_k+\rho_k$, $\rho_k \equiv p_k[\mathbf{p}\cdot\mathbf{r}-(m_N+p_0)r_0]/m_N(m_N+p_0)$, where we have used the explicit representation of the boost transformation as given in Sec. II. In general, we can write[35]

$$|\mathbf{L}^{-1}(p)\cdot r|^l Y_{lm_l}(\hat{L}^{-1}(p)\cdot r) = \sum_{LM_L}(-1)^{l-L}\left(\frac{4\pi(2l+1)!}{(2l-2L+1)!(2L+1)!}\right)^{1/2} C(L\,l-L\,l;\, m_L\, m_l-m_L)$$
$$\times|\boldsymbol{\varrho}|^{l-L}Y_{l-L,m_l-m_L}(\hat{p})\,|\mathbf{r}|^L Y_{L,m_L}(\hat{r}), \quad \text{(A1)}$$

where $C(L\,l-L\,l;\, m_L\, m_l-m_L)$ are Clebsch-Gordan coefficients, and $\hat{\rho}=\hat{p}$. Equation (A1) is derived by considering the expansion of $e^{i\mathbf{k}\cdot\mathbf{r}'}=e^{i\mathbf{k}\cdot\mathbf{r}}e^{i\mathbf{k}\cdot\rho}$ ($\mathbf{r}'=\mathbf{r}+\boldsymbol{\varrho}$) into partial waves, multiplying by $|\mathbf{k}|^{-l}Y_{lm_l}(\hat{k})$ on both sides, integrating over $\hat{k}$, and then taking the limit $|\mathbf{k}|\to 0$. The radial part of the vector $\boldsymbol{\varrho}$ can be written

$$|\boldsymbol{\varrho}|^{l-L}=\left(\frac{|\mathbf{p}|}{m_N}\right)^{l-L}\sum_{s=0}^{l-L}C_s{}^{(l-L)}P_s(\hat{p}\cdot\hat{r}). \quad \text{(A2)}$$

$P_s(\hat{p}\cdot\hat{r})$ is the Legendre polynomial of order s.[36] It is straightforward to show that

$$C_s{}^{(l-L)}=\frac{2s+1}{2}\pi r_0{}^{l-L}(-1)^s\frac{(l-L)!}{2^{l-L}}\left(\frac{|\mathbf{p}|\,|\mathbf{r}|}{(m_N+p_0)r_0}\right)^s$$
$$\times\frac{{}_2F_1((s-l+L+1)/2;\,(s-l+L)/2;\,s+\frac{3}{2};\,[|\mathbf{p}|\,|\mathbf{r}|/(m_N+p_0)r_0]^2)}{\Gamma((l-L-s+1)/2)\Gamma((l-L-s)/2+1)\Gamma(s+\frac{3}{2})}, \quad \text{(A3)}$$

where Γ is the gamma function[18] and ${}_2F_1$ the hypergeometrc function[18] (which in the above is a polynomial of order $\frac{1}{2}(l-L-s)$ or $\frac{1}{2}(l-L-s-1)$, whichever is integral). Hence,

$$|\mathbf{L}^{-1}(p)\cdot r|^l Y_{lm_l}(\hat{L}^{-1}(p)\cdot r)=\sum_{L,s}\sum_{\lambda,\lambda',m_\lambda}(-1)^{\lambda'}W(\lambda L\lambda'\,l-L;\,sl)C(s\,l-L\,\lambda';00)C(sL\lambda;\,00)\left(\frac{4\pi(2l+1)!}{(2L)!(2l-2L)!}\right)^{1/2}$$
$$\times\left(\frac{|\mathbf{p}|}{m_N}\right)^{l-L}|\mathbf{r}|^L C_s{}^{(l-L)}C(\lambda\lambda' l;\, m_\lambda\, m_l-m_\lambda)Y_{\lambda m_\lambda}(\hat{r})Y_{\lambda',m_l-m_\lambda}(\hat{p}), \quad \text{(A4a)}$$

where $W(\lambda L\lambda'\,l-L;\,sl)$ is a Racah coefficient.[37] Equation (A4) is obtained by using[8]

$$P_s(\hat{p}\cdot\hat{r})=\frac{4\pi}{2s+1}\sum_{m_s}Y_{sm_s}{}^*(\hat{p})Y_{sm_s}(\hat{r}),$$

$$Y_{l_1m_{l_1}}(\hat{x})Y_{l_2m_{l_2}}(\hat{x})=\sum_\lambda\left(\frac{(2l_1+1)(2l_2+1)}{4\pi(2\lambda+1)}\right)^{1/2}C(l_1l_2\lambda;\, m_{l_1}m_{l_2})C(l_1l_2\lambda;\,00)Y_{\lambda,m_{l_1}+m_{l_2}}(\hat{x})$$

and

$$\sum_\beta C(abe;\alpha\beta)C(edc;\alpha+\beta\,\gamma-\alpha-\beta)C(bdf;\beta\,\gamma-\alpha-\beta)=[(2e+1)(2f+1)]^{1/2}C(afc;\alpha\,\gamma-\alpha)W(abcd;\,ef).$$

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[35] M. Danos and L. C. Maximon, J. Math. Phys. **6**, 766 (1965), Eq. (21).
[36] Ref. 18, Chap. III.
[37] Ref. 18, Chap. VI.

For the special case $l=1$,

$$|\mathbf{L}^{-1}(p)\cdot r|\,Y_{1m_1}(\hat{L}^{-1}(p)\cdot r)=\sum_{\lambda,\lambda',m_\lambda}(-1)^{\lambda'}[3(4\pi)]^{1/2}[W(\lambda 0\lambda' 1;01)C(01\lambda';00)C(00\lambda;00)(|\mathbf{p}|/m_N)r_0$$
$$+W(\lambda 1\lambda' 0;01)C(00\lambda';00)C(01\lambda;00)|\mathbf{r}|-W(\lambda 0\lambda' 1;11)C(11\lambda';00)C(10\lambda;00)$$
$$\times(|\mathbf{p}|^2|\mathbf{r}|/m_N(m_N+p_0)]C(\lambda\lambda' 1;m_\lambda\, m_1-m_\lambda)Y_{\lambda m_\lambda}(\hat{r})Y_{\lambda',m_1-m_\lambda}(\hat{p})$$
$$\equiv\sum_{\lambda,\lambda',m_\lambda}(-1)^{\lambda'}(f_1^{\lambda\lambda'}r_0+f_2^{\lambda\lambda'}|\mathbf{r}|)C(\lambda\lambda' 1;m_\lambda\, m_1-m_\lambda)Y_{\lambda m_\lambda}(\hat{r})Y_{\lambda',m_1-m_\lambda}(\hat{p}).\qquad\text{(A4b)}$$

Also, we can write

$$e^{-\nu|\mathbf{L}^{-1}(p)\cdot r|^2}e^{-\nu[L^{-1}{}_{0\mu}(p)\cdot r_\mu]^2}=e^{-\nu r^2}e^{-2\nu(L^{-1}{}_{0\mu}(p)\cdot r_\mu)^2}=e^{-\nu|\mathbf{r}|^2}e^{+\nu r_0^2}e^{-2\nu(p\cdot r/m_N)^2}.$$

Hence $F_{\sigma\sigma'}$ becomes

$$F_{\sigma\sigma'}=N\sum_{\lambda,\lambda',m_\lambda}(-1)^{\lambda'}C(\lambda\lambda' 1;m_\lambda\, m_1-m_\lambda)Y_{\lambda',m_1-m_\lambda}(\hat{p})\int d\mathbf{r}dr_0\; Y_{\lambda m_\lambda}(\hat{r})(f_1^{\lambda\lambda'}r_0+f_2^{\lambda\lambda'}|\mathbf{r}|)Y_{l_i m_{l_i}}{}^*(\hat{r})$$
$$\times|\mathbf{r}|^{l_i}e^{-2\nu|\mathbf{r}|^2}\,{}_1F_1(-n_s+1;\,l_i+\tfrac{3}{2};\,2\nu|\mathbf{r}|^2)H_{n_t}((2\nu)^{1/2}r_0)e^{i\beta\mathbf{p}\cdot\mathbf{r}}e^{-\beta p_0^{\mathrm{rel}}r_0}e^{-2\nu(p\cdot r/m)^2},\qquad\text{(A5)}$$

where $m_{l_i}\equiv m_j-\sigma$ and $m_1\equiv m_{1/2}-\sigma'$; we have assumed that for the ground state $(n_s,n_t)=(1,0)$. N includes all normalization constants.

The time integration is, thus,

$$T_n{}^{n_t}\equiv\int dr_0 H_{n_t}((2\nu)^{1/2}r_0)e^{-i\beta p_0^{\mathrm{rel}}\cdot r_0}\exp\left[-2\nu\left(\frac{p_0}{m_N}r_0-\frac{\mathbf{p}\cdot\mathbf{r}}{m_N}\right)^2\right]r_0{}^n,$$

where $n=0$ or 1.

$$T_n{}^{n_t}=\left(\frac{1}{(2\nu)^{1/2}}\right)^{n+1}\int dx_0 H_{n_t}(x_0)e^{-ip_0'x_0}e^{-(ax_0-b)^2}x_0{}^n,$$

where

$$(2\nu)^{1/2}r_0=x_0,\quad a=p_0/m_N,\quad b=(2\nu)^{1/2}\mathbf{p}\cdot\mathbf{r}/m_N,\quad p_0'=\beta p_0^{\mathrm{rel}}/(2\nu)^{1/2}.$$

With

$$H_{n_t}(x_0)=\sum_{r=0/1}^{n_t}h_r{}^{(n_t)}x_0{}^r,$$

(summation from either 0 or 1 to n_t)

$$T_n{}^{n_t}=\left(\frac{1}{(2\nu)^{1/2}}\right)^{n+1}\sum_{r=0/1}^{n_t}h_r{}^{(n_t)}\left(i\frac{d}{dp_0'}\right)^{r+n}\int dx_0 e^{-ip_0'x_0}e^{-(ax_0-b)^2}$$
$$=\left(\frac{1}{(2\nu)^{1/2}}\right)^{n+1}\sum_{r=0/1}^{n_t}h_r{}^{(n_t)}\left(i\frac{d}{dp_0'}\right)^{r+n}\frac{1}{a}e^{-ip_0'b/a}\int e^{-i(p_0'/a)y}e^{-y^2}dy$$
$$=\left(\frac{1}{(2\nu)^{1/2}}\right)^{n+1}\frac{\sqrt{\pi}}{a}\sum_{r=0/1}^{n_t}h_r{}^{(n_t)}\left(i\frac{d}{dp_0'}\right)^{r+n}e^{-i(b/a)p_0'}e^{-\frac{1}{4}p_0'^2/a^2}$$
$$\equiv\sum_{s=0}^{n_t+n}\bar{a}_s{}^{(n)}\left(\frac{(2\nu)^{1/2}}{m_N}|\mathbf{p}|\right)^s|\mathbf{r}|^s(\hat{p}\cdot\hat{r})^s\exp\left(-i\beta\frac{p_0^{\mathrm{rel}}}{p_0}\mathbf{p}\cdot\mathbf{r}\right)\exp\left[-\beta^2\frac{m_N^2}{8\nu}\left(\frac{p_0^{\mathrm{rel}}}{p_0}\right)^2\right]$$
$$\equiv\sum_{s=0}^{n_t+n}a_s{}^{(n)}|\mathbf{r}|^s(\hat{p}\cdot\hat{r})^s\exp\left(-i\beta\frac{p_0^{\mathrm{rel}}}{p_0}\mathbf{p}\cdot\mathbf{r}\right)\exp\left[-\beta^2\frac{m_N^2}{8\nu}\left(\frac{p_0^{\mathrm{rel}}}{p_0}\right)^2\right],\qquad\text{(A6)}$$

where we have put

$$a_s{}^{(n)}\equiv\bar{a}_s{}^{(n)}\left(\frac{(2\nu)^{1/2}}{m_N}|\mathbf{p}|\right)^s.$$

We define

$$\alpha\equiv 1-(p_0^{\mathrm{rel}}/p_0)=2\frac{m_N}{m_N+m_\pi}\frac{m_R^2}{m_R^2+m_N^2-m_\pi^2},$$

so that Eq. (A5) becomes

$$F_{\sigma\sigma'}=N\sum_{\lambda,\lambda',m_\lambda}(-1)^{\lambda'}C(\lambda\lambda'1;\,m_\lambda\,m_1-m_\lambda)Y_{\lambda',m_1-m_\lambda}(\hat{p})\sum_{s=0}^{n_t+1}\left\{\exp\left[-\beta^2\frac{m_N^2}{8\nu}\left(\frac{p_0^{\rm rel}}{p_0}\right)^2\right]\right\}\int d\mathbf{r}Y_{l_im_{l_i}}{}^*(\hat{r})$$

$$\times Y_{\lambda m_\lambda}(\hat{r})(\hat{p}\cdot\hat{r})^s|\mathbf{r}|^{l_i+s}e^{i\beta\alpha\mathbf{p}\cdot\mathbf{r}}{}_1F_1(-n_s+1;\,l_i+\tfrac{3}{2};\,2\nu|\mathbf{r}|^2)e^{-2\nu|\mathbf{r}|^2}(f_1{}^{\lambda\lambda'}a_s{}^{(1)}+f_2{}^{\lambda\lambda'}a_s{}^{(0)}|\mathbf{r}|),\quad\text{(A7)}$$

with

$$a_{s=n_t+1}{}^{(0)}\equiv0.$$

Now

$$(\hat{p}\cdot\hat{r})^s=\sum_{u=0/1}^{s}d_u{}^{(s)}P_u(\hat{p}\cdot\hat{r})=\sum_{u=0/1}^{s}d_u{}^{(s)}\frac{4\pi}{2u+1}\sum_{m_u}Y_{um_u}{}^*(\hat{p})Y_{um_u}(\hat{r}),$$

so that

$$\int d\hat{r}\,Y_{l_im_{l_i}}{}^*(\hat{r})Y_{\lambda m_\lambda}(\hat{r})e^{i\beta\alpha\mathbf{p}\cdot\mathbf{r}}(\hat{p}\cdot\hat{r})^s=\sum_{u=0/1}^{s}\sum_{l,l_1}d_u{}^{(s)}i^lj_l(\alpha\beta|\mathbf{p}||\mathbf{r}|)\left(\frac{(2\lambda+1)(2l+1)}{4\pi(2l_1+1)}\right)^{1/2}C(lul_1;\,00)$$

$$\times C(l_1ul;\,00)(-1)^{u-l_1-l_i}C(l_i\lambda l_1;\,00)C(\lambda l_1l_i;\,m_\lambda\,m_{l_i}-m_\lambda)Y_{l_1,m_{l_1}-m_\lambda}(\hat{p}),$$

where $j_l(k|\mathbf{r}|)$ is the spherical Bessel function of order l.[27]

Thus,

$$F_{\sigma\sigma'}=N\left(\frac{3}{4\pi}\right)^{1/2}\exp\left[-\beta^2\frac{m_N^2}{8\nu}\left(\frac{p_0^{\rm rel}}{p_0}\right)^2\right]\sum_{\lambda,\lambda'}\sum_{s=0}^{n_t+1}\sum_{u=0/1}^{s}\sum_{l,l_1,\bar{l}}d_u{}^{(s)}\left(\frac{(2\lambda+1)(2l+1)(2\lambda'+1)}{4\pi}\right)^{1/2}(-1)^{u+\lambda'+1}$$

$$\times C(lul_1;\,00)C(l_1ul;\,00)C(l_i\lambda l_1;\,00)C(\lambda'l_1\bar{l};\,00)W(1\lambda'l_il_1;\,\lambda\bar{l})C(1\bar{l}l_i;\,m_1\,m_{l_i}-m_1)$$

$$\times Y_{\bar{l},m_{l_i}-m_1}(\hat{p})(f_1{}^{\lambda\lambda'}a_s{}^{(1)}\langle r^{l_i+s}\rangle_{ll_i}+f_2{}^{\lambda\lambda'}a_s{}^{(0)}\langle r^{l_i+s+1}\rangle_{ll_i}),\quad\text{(A8)}$$

where

$$\langle r^m\rangle_{ll_i}\equiv i^l\int r^2dr\,r^m\,{}_1F_1(-n_s+1;\,l_i+\tfrac{3}{2};\,2\nu r^2)e^{-2\nu r^2}j_l(\alpha\beta|\mathbf{p}|r).$$

Using the definitions of $f_1{}^{\lambda\lambda'}$ and $f_2{}^{\lambda\lambda'}$ given by Eq. (A4b), we can write

$$S_1{}^{\lambda'l_1}\equiv\sum_\lambda(2\lambda+1)^{1/2}C(l_i\lambda l_1;\,00)W(1\lambda'l_il_1;\,\lambda\bar{l})f_1{}^{\lambda\lambda'}$$

$$=\sqrt{3}(4\pi)^{1/2}\frac{|\mathbf{p}|}{m_N}W(1\lambda'l_il_1;\,0\bar{l})W(00\lambda'1;\,01)C(01\lambda';\,00)\delta_{l_il_1};$$

$$S_1{}^{l_1}\equiv\sum_{\lambda'}(2\lambda'+1)^{1/2}(-1)^{\lambda'}C(\lambda'l_1\bar{l};\,00)S_1{}^{\lambda'l_1}$$

$$=-(4\pi)^{1/2}\frac{|\mathbf{p}|}{m_N}\frac{1}{(2l_1+1)^{1/2}}C(1l_i\bar{l};\,00)\delta_{l_il_1},$$

$$S_1\equiv\sum_{l_1}C(lul_1;\,00)C(l_1ul;\,00)S_1{}^{l_1}$$

$$=-(4\pi)^{1/2}\frac{|\mathbf{p}|}{m_N}\frac{1}{(2l_i+1)^{1/2}}C(lul_i;\,00)C(l_iul;\,00)C(1l_i\bar{l};\,00),$$

$$S_2{}^{\lambda'l_1}=\sum_\lambda(2\lambda+1)^{1/2}C(l_i\lambda l_1;\,00)W(1\lambda'l_il_1;\,\lambda\bar{l})f_2{}^{\lambda\lambda'}$$

$$=3(4\pi)^{1/2}C(l_i1l_1;\,00)W(1\lambda'l_il_1;\,1\bar{l})\left[W(11\lambda'0;\,01)C(00\lambda';\,00)-W(10\lambda'1;\,11)C(11\lambda';\,00)\frac{|\mathbf{p}|^2}{m_N(m_N+p_0)}\right],$$

$$S_2{}^{l_1}\equiv\sum_{\lambda'}(2\lambda'+1)^{1/2}(-1)^{\lambda'}C(\lambda'l_1\bar{l};\,00)S_2{}^{\lambda'l_1}$$

$$=(4\pi)^{1/2}\left[C(l_i1l_1;\,00)(2\bar{l}+1)^{-(1/2)}\delta_{l_1\bar{l}}+C(1l_il_1;\,00)(2l_1+1)^{-(1/2)}C(1l_i\bar{l};\,00)\frac{|\mathbf{p}|^2}{m_N(m_N+p_0)}\right],$$

$$S_2 \equiv \sum_{l_1} C(lul_1;00)C(l_1ul;00)S_2^{l_1}$$

$$=(4\pi)^{1/2}C(1l_i\bar{l};00)\left[(2\bar{l}+1)^{-(1/2)}C(lu\bar{l};00)C(\bar{l}ul;00)-\frac{1}{(2l_i+1)^{1/2}}\sum_{l_1}C(lul_1;00)C(l_1ul;00)C(l_i1l_1;00)\times C(l_11l_i;00)\frac{|\mathbf{p}|^2}{m_N(m_N+p_0)}\right]$$

$$=(4\pi)^{1/2}C(1l_i\bar{l};00)\left\{\frac{1}{(2\bar{l}+1)^{1/2}}C(lu\bar{l};00)C(\bar{l}ul;00)-\frac{1}{(2l_i+1)^{1/2}}\frac{|\mathbf{p}|^2}{m_N(m_N+p_0)}\frac{1}{2u+1}\times[uC(l\,u-1\,l_i;00)C(l_i\,u-1\,l;00)+(u+1)C(l\,u+1\,l_i;00)C(l_i\,u+1\,l;00)]\right\}.$$

Thus, Eq. (A8) becomes

$$F_{\sigma\sigma'}=N\left(\frac{3}{4\pi}\right)^{1/2}\sum_{\bar{l}}C(1\bar{l}l_i;m_1\,m_{l_i}-m_1)C(1\bar{l}l_i;00)Y_{\bar{l}m_{l_i}-m_1}{}^*(\hat{p})\exp\left[-\beta^2\frac{m_N{}^2}{8\nu}\left(\frac{p_0{}^{\mathrm{rel}}}{p_0}\right)^2\right](4\pi)^{1/2}\sum_l\sum_{s=0}^{n_l+1}\sum_{u=0/1}^{s}(-1)^u d_u{}^{(s)}$$

$$\times\left\{a_s{}^{(1)}\left(\frac{2l+1}{2l_i+1}\right)^{1/2}\frac{|\mathbf{p}|}{m_N}\langle r^{l_i+s}\rangle_{ll_i}C(lul_i;00)C(l_iul;00)+a_s{}^{(0)}\langle r^{l_i+s+1}\rangle_{ll_i}\left[\left(\frac{2l+1}{2l_i+1}\right)^{1/2}\frac{1}{2u+1}\frac{|\mathbf{p}|^2}{m_N(m_N+p_0)}\right.\right.$$

$$\times[uC(l\,u-1\,l_i;00)C(l_i\,u-1\,l;00)+(u+1)C(l\,u+1\,l_i;00)C(l_i\,u+1\,l;00)]$$

$$\left.\left.-\left(\frac{2l+1}{2\bar{l}+1}\right)^{1/2}C(lu\bar{l};00)C(\bar{l}ul;00)\right]\right\}. \quad \text{(A9)}$$

In the nonrelativistic (N.R.) limit, as $(|\mathbf{p}|/m)$ becomes decreasingly small, the dominant term is the $a_{s=0}{}^{(0)}$ term (hence $u=0$). Thus, in this limit

$$F_{\sigma\sigma'}{}^{\mathrm{N.R.}}=N\sqrt{3}\sum_l C(1ll_i;m_1\,m_{l_i}-m_1)Y_{l,m_{l_i}-m_1}{}^*(\hat{p})\beta^l|\mathbf{p}|^l\frac{1}{(2l+1)!!}\times C(1ll_i;00)i^l\int_0^\infty r^2dr\,r^{l_i+1+l}e^{-2\nu r^2}\,{}_1F_1(-n_s+1;l_i+\tfrac{3}{2};2\nu r^2).$$

l assumes the two values l_i-1 and l_i+1. If, for small $(|\mathbf{p}|/m)$, the amplitude is to go as $|\mathbf{p}|^{2l_i+1}$, then

$$\int_0^\infty r^2dr\,r^{l_i+1+l_i-1}e^{-2\nu r^2}\,{}_1F_1(-n_s+1;l_i+\tfrac{3}{2};2\nu r^2)\neq 0.$$

However, this integral is easily evaluated,

$$\int_0^\infty r^2dr\,r^{2l_i}e^{-2\nu r^2}\,{}_1F_1(-n_s+1;l_i+\tfrac{3}{2};2\nu r^2)\propto{}_2F_1(-n_s+1;l_i+\tfrac{3}{2};l_i+\tfrac{3}{2};1)$$

$$=\sum_{m=0}^{n_s-1}\frac{(-1)^m(n_s-1)!}{(n_s-1-m)!m!}$$

$$=1,\quad n_s=1$$

$$=0,\quad \text{otherwise}.$$

Here we have used the equation[38]

$$\int_0^\infty {}_pF_q\left[\begin{matrix}\alpha_1\cdots\alpha_p; & \pm b^2x^2\\ \beta_1\cdots\beta_q & \end{matrix}\right]e^{-a^2x^2}x^{\mu-1}dx=\frac{\Gamma(\frac{1}{2}\mu)}{2a^\mu}\,{}_{p+1}F_q\left[\begin{matrix}\alpha_1\cdots\alpha_p; & \frac{1}{2}\mu; & \pm b^2/a^2\\ \beta_1\cdots\beta_q & & \end{matrix}\right],$$

where ${}_pF_q$ is the generalized hypergeometric function.[38] In general, from Eq. (A9) we can see that the lowest power

[38] Ref. 18, p. 48.

of $|\mathbf{p}|$ in the decay amplitude is $|\mathbf{p}|^{l_i}$ (the power of $|\mathbf{p}|$ in $F_{\sigma\sigma'}$ is $|\mathbf{p}|^{l_i-1}$) if and only if $n_s=1$. We may, thus, restrict ourselves to this choice.

In Eq. (A6) for small $(|\mathbf{p}|/m)$, i.e., neglecting terms of $(|\mathbf{p}|^2/m^2)$, $T_0^{n_t}$ will be

$$\int dx_0 H_{n_t}(x_0) e^{-x_0^2} = \delta_{n_t 0}$$

and, hence, we make the choice $n_t=0$ (see the discussion at the end of Sec. IV).

With these choices, the integrals $\langle r^m \rangle_{ll_i}$ become[34]

$$\langle r^m \rangle_{ll_i} = i^l \int dr\, r^{m+2} e^{-2\nu r^2} j_l(\alpha\beta|\mathbf{p}|r) = i^l \left(\frac{1}{4\pi}\right)^{1/2} \left(\frac{\alpha\beta|\mathbf{p}|}{(8\nu)^{1/2}}\right)^l \exp(-\beta^2\alpha^2|\mathbf{p}|^2/8\nu)$$
$$\times \frac{\Gamma(\frac{1}{2}(l+m+2+1))}{(2\nu)^{(1/2)(m+3)}\Gamma(l+\frac{3}{2})} {}_1F_1(\tfrac{1}{2}l-\tfrac{1}{2}m;\, l+\tfrac{3}{2};\, \beta^2\alpha^2|\mathbf{p}|^2/8\nu). \quad \text{(A10)}$$

Hence, it is straightforward to obtain the following:

$$\sum_l \langle r^{l_i} \rangle_{ll_i} \left(\frac{2l+1}{2l_i+1}\right)^{1/2} \delta_{ll_i} = \left(\frac{1}{4\pi}\right)^{1/2} i^{l_i} \frac{x^{l_i} e^{-x^2}}{(2\nu)^{(1/2)(l_i+3)}}, \quad \text{(A11a)}$$

$$\sum_l \langle r^{l_i+1} \rangle_{ll_i} \left(\frac{2l+1}{2l_i+1}\right)^{1/2} C(l1l_i;\, 00) C(l_i1l;\, 00) = \left(\frac{1}{4\pi}\right)^{1/2} \frac{i^{l_i-1} e^{-x^2} x^{l_i-1}}{(2\nu)^{(1/2)(l_i+4)}} (x^2 - \tfrac{1}{2}l_i), \quad \text{(A11b)}$$

and we have defined $x \equiv \alpha\beta|\mathbf{p}|/(8\nu)^{1/2}$.

Also,

$$a_0^{(0)} = \frac{1}{(2\nu)^{1/2}} \frac{\sqrt{\pi}}{p_0/m_N},$$

$$a_0^{(1)} = \frac{\beta}{(2\nu)^{1/2}} \left(\frac{m_N}{p_0}\right) \left(\frac{p_0^{\text{rel}}}{p_0}\right) \frac{m_N}{(8\nu)^{1/2}} \frac{a_0^{(0)}}{i},$$

$$a_1^{(1)} = \frac{|\mathbf{p}|}{p_0} a_0^{(0)},$$

so that Eq. (A9) becomes

$$F_{\sigma\sigma'} = N_{l_i} \left(\frac{3}{4\pi}\right)^{1/2} \sum_{\bar{l}} C(1\bar{l}l_i;\, m_1\, m_{l_i}-m_1) Y_{\bar{l}m_{l_i}-m_1}{}^*(\hat{p}) \exp\left[-\beta^2 \frac{m_N{}^2}{8\nu} \left(\frac{p_0^{\text{rel}}}{p_0}\right)^2\right] C(1\bar{l}l_i;\, 00) \left(\frac{\pi}{2\nu}\right)^{1/2} \frac{m_N}{p_0} \left\{ \frac{i^{l_i-1} x^{l_i-1} e^{-x^2}}{(2\nu)^{(1/2)(l_i+4)}} \right.$$
$$\left. \times \left[\frac{\beta|\mathbf{p}|}{(8\nu)^{1/2}} x \left(\frac{m_N}{p_0}\right) \left(\frac{p_0^{\text{rel}}}{p_0}\right) + (x^2 - \tfrac{1}{2}l_i) \frac{|\mathbf{p}|^2}{m_N} \left(\frac{1}{p_0+m_N} - \frac{1}{p_0}\right)\right] - (4\pi)^{1/2} \langle r^{l_i+1} \rangle_{\bar{l}l_i} \right\}. \quad \text{(A12)}$$

This is just Eq. (18) of Sec. IV.