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Conductivity Tensor of a Collisional Plasma in a Magnetic Field

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A general method is developed for the calculation of the conductivity tensor, collisional contribution included, associated with a wave in a plasma in a magnetic field. The method is based on an iterative procedure and is applicable to any kinetic equation. With the Landau collision term a result valid to all orders in $k^2\lambda_D^2$ is given in integral form, and the electron current contribution is calculated explicitly to order $k^2\lambda_D^2$. The leading contributions resulting from the ion motion are also obtained.

I. INTRODUCTION

The dispersion relation for a small amplitude electromagnetic wave may be written in terms of the following determinant

$$|(\omega^2 - c^2k^2)\delta_{ij} + c^2k_ik_j + 4\pi i\omega\sigma_{ij}| = 0 , \qquad (1)$$

where ω is the frequency, k_i the wave vector, and σ_{ij} the conductivity tensor associated with the wave. Neglecting two-particle correlations (or "collisions"), the conductivity tensor for a uniform plasma in a constant magnetic field has been calculated by a number of authors.[1–10] No solution of comparable generality exists to the problem of determining the collisional contribution to the conductivity tensor. It is to this problem that we shall address ourselves here.

The calculations in Refs. 1–10 are based on the Vlasov equation (the collisionless Boltzmann equation). This is justified as a first approximation in the sense that the effective collision frequency is usually small compared to the wave and/or cyclotron frequencies. For some purposes, however, it is essential that the collisional damping rate, though small, be known. In that case one must go beyond the Vlasov equation, and retain the appropriate collision integral in the kinetic equation. Since the collision term is usually small, an iterative procedure[11–15] is used to solve the linearized kinetic equation. The collisional contribution to the conductivity tensor is first expressed as an integral over an arbitrary collision integral. Subsequently we specialize with the Landau collision term. A result valid to all orders in $k^2\lambda_D^2$ and first order in the collision parameter is given in integral form. For the electron current contribution to the collisional-conductivity tensor, the integrals are carried out explicitly to first order in $k^2\lambda_D^2$ (i.e., the lowest-order thermal correction to the cold plasma result is obtained). The leading contribution resulting from the ion motion is also calculated. All the particles are assumed to be nonrelativistic, a restriction that is easily lifted.[16]

II. CONDUCTIVITY TENSOR FOR ARBITRARY COLLISION INTEGRAL

We consider a neutral plasma consisting of s species of charged particles in a constant magnetic field $\vec{B}_0$. The distribution function of the ith species (mass $m^{(i)}$, charge $q^{(i)}$) is assumed to deviate by

$$f^{(i)}\exp[i(\vec{k}\cdot\vec{r}-\omega t] \qquad (2)$$

from a Maxwellian distribution $f_0^{(i)}$ at the (common) temperature T, where

$$f_0^{(i)}=n^{(i)}(\beta^{(i)}/\pi)^{\frac{3}{2}} \times\exp(-\beta^{(i)}v^{(i)2}),\quad \beta^{(i)}\equiv m^{(i)}/2KT. \qquad (3)$$

In what follows we shall suppress the species superscript whenever possible. The distribution f satisfies the linearized kinetic equation,[1-10]

$$\frac{\partial f}{\partial\phi}+i(a-b\cos\phi)f=\frac{q\vec{E}}{m\Omega}\cdot\frac{\partial f_0}{\partial\vec{v}}-\frac{C(f)}{\Omega}. \qquad (4)$$

Here $\vec{E}$ is the wave electric field and $C(f)$ the linearized collision term. We have fixed the coordinate system as follows:

$$\vec{B}_0=B_0\vec{e}_z,\quad \vec{k}=k_x\vec{e}_x+k_z\vec{e}_z, \qquad (5)$$

and introduced cylindrical velocity coordinates,

$$\vec{v}=w\cos\phi\vec{e}_x+w\sin\phi\vec{e}_y+v_z\vec{e}_z, \qquad (6)$$

as well as the shorthand notation

$$\Omega\equiv qB_0/mc,\quad a\equiv(\omega-k_zv_z)/\Omega,\quad b\equiv k_xw/\Omega. \qquad (7)$$

With the Bernstein[2] integrating factor, the kinetic equation, as shown by McBride,[15] can be recast into the form

$$f\equiv f_1+f_c=f_1-\Omega^{-1}\exp[i(b\sin\phi-a\phi)] \times\int_{\pm\infty}^{\phi}d\phi' C(f)\exp[i(a\phi'-b\sin\phi')]. \qquad (8)$$

Here f_1 is the solution to the collisionless equation, and may be written in the form

$$f_1=f_0\rho_lE_l/KT, \qquad (9)$$

where $\rho_l(\vec{v},\Omega)$ is given by[1-10]

$$[\rho_x,\rho_y,\rho_z]\Big|=iq\sum_{n=-\infty}^{\infty}[nwJ_n(b)/b,\, iwJ_n'(b),\, v_zJ_n(b)] \times\exp[i(b\sin\phi-n\phi)]/(a-n)\Omega. \qquad (10)$$

Equation (8) is in a convenient form for iteration. The solution to first order in the collision parameter is simply obtained by replacing $C(f)$ by $C(f_1)$. We shall carry the exact expression for f_c along a bit further, however.

The conductivity tensor σ_{kl} is obtained by calculating the current density

$$\sigma_{kl}E_l=\sum_{i=1}^{s}q^{(i)}\int d^3v^{(i)}v_k^{(i)}f^{(i)}, \qquad (11)$$

where $f^{(i)}\equiv f_1^{(i)}+f_c^{(i)}$ satisfies Eq. (8). The collisionless part $\sigma_{kl}^{(1)}$, obtained by integrating over $f_1^{(i)}$ only, has been determined previously.[1-10] It gives rise to the well known phenomenon of resonant damping[17] (Landau damping, cyclotron damping). We have nothing new to say with regard to the collisionless damping; our concern here is the additional damping due to collisions. The collisional part of the conductivity tensor $\sigma_{kl}^{(c)}$ is given by

$$\sigma_{kl}^{(c)}E_l=\sum_{i=1}^{s}q^{(i)}\int d^3v^{(i)}v_k^{(i)}f_c^{(i)}. \qquad (12)$$

Substituting for f_c from Eq. (8), expanding $\exp(ib\sin\phi)$ into

$$\sum_{-\infty}^{\infty}J_n(b)\exp(in\phi),$$

and integrating by parts in the ϕ variable, we arrive at the result

$$\sigma_{kl}^{(c)}E_l=\sum_{i=1}^{s}\int d^3v^{(i)}\tilde{\rho}_k^{(i)}C^{(i)}(f^{(i)}). \qquad (13)$$

Here $\tilde{\rho}_k(\vec{v},\Omega)\equiv\rho_k(\vec{v},-\Omega) \qquad (14)$

is the vector given in Eq. (10) with Ω replaced by $-\Omega$. With Eq. (13) we have succeeded in expressing the conductivity tensor as a direct integral over $C(f)$ rather than over f_c. This represents an important simplification.

From this point on we shall limit the discussion to terms of first order in the collision parameter, i.e., replace $C(f)$ by $C(f_1)$ in Eq. (13). We also recognize explicitly the dependence on all s distribution functions in writing

$$C^{(i)}(f_1^{(i)}) \equiv \sum_{j=1}^{s} C^{(ij)}(f_1^{(i)};f_1^{(j)}). \tag{15}$$

We recall that $C^{(ij)}$ has been linearized in $f_1^{(i)}$ and $f_1^{(j)}$, so that in substituting for f_1 from Eq. (9), we are permitted to extract the common velocity independent factor E_l/KT from the $C^{(ij)}$ terms. The resultant $\sigma_{kl}^{(c)}$, valid for an arbitrary collision integral, is

$$\sigma_{kl}^{(c)} = (KT)^{-1} \sum_{i=1}^{s} \sum_{j=1}^{s} \int d^3v^{(i)} \bar{\rho}_k^{(i)} \times C^{(ij)}(\rho_l^{(i)} f_0^{(i)}; \rho_l^{(j)} f_0^{(j)}). \tag{16}$$

III. CONDUCTIVITY TENSOR FROM THE LANDAU EQUATION

A. General Results

In this section we shall calculate the conductivity tensor with the Landau collision integral.[18–20,8] The validity of this collision term for a plasma in a magnetic field will be discussed in Sec. IV. Inserting the linearized Landau term into Eq. (16), and integrating by parts in the $v_m^{(i)}$ variable, we obtain immediately

$$\sigma_{kl}^{(c)} = -\frac{2\pi}{KT} \sum_{i=1}^{s} \sum_{j=1}^{s} \ln\Lambda^{(ij)} \times (q^{(i)}q^{(j)}/m^{(i)})^2 \int d^3v^{(i)} \times \int d^3v^{(j)} f_0^{(i)} f_0^{(j)} \times \frac{\partial \bar{\rho}_k^{(i)}}{\partial v_m^{(i)}} \left(\frac{\delta_{mn}}{g} - \frac{g_m g_n}{g^3} \right) \times \left(\frac{\partial \rho_l^{(i)}}{\partial v_n^{(i)}} - \frac{m^{(i)}}{m^{(j)}} \frac{\partial \rho_l^{(j)}}{\partial v_n^{(j)}} \right). \tag{17}$$

The vector g_m is the relative velocity

$$g_m \equiv v_m^{(i)} - v_m^{(j)}, \tag{18}$$

and $\ln\Lambda^{(ij)} (= \ln\Lambda^{(ji)})$ is the usual Coulomb logarithm of the ratio of maximum to minimum impact parameter. Recalling the definition of $\bar{\rho}_k$ in terms of ρ_k as given by Eq. (14), we verify explicitly that the conductivity tensor has the required symmetry,

$$\sigma_{kl}^{(c)}(\Omega) = \sigma_{lk}^{(c)}(-\Omega). \tag{19}$$

This symmetry is obvious for the contribution from the first term in the last parenthesis in Eq. (17), and is easily checked for the remaining term by considering jointly the (i,j) and (j,i) contributions. With the symmetry of Eq. (19), it is clear that only 6 of the 9 components of $\sigma_{kl}^{(c)}$ need be calculated.

B. The Electron Current

The dominant contribution to the conductivity tensor in a plasma comes from the electron current. If we neglect the ion motion altogether, $\sigma_{kl}^{(c)}$ may be written as the sum of an electron-ion collision term and an electron-electron collision term,

$$\sigma_{kl}^{(c)} = \sigma_{kl}^{(\text{ei})} + \sigma_{kl}^{(\text{ee})}, \tag{20}$$

where

$$\sigma_{kl}^{(\text{ei})} = -2\pi Z n_e e^4 \ln\Lambda^{(\text{ei})}/KT m_e^2 \int d^3v f_0 \frac{\partial \bar{\rho}_k}{\partial v_m} \left(\frac{\delta_{mn}}{v} - \frac{v_m v_n}{v^3} \right) \frac{\partial \rho_l}{\partial v_n} \tag{21}$$

$$\sigma_{kl}^{(\text{ee})} = -2\pi e^4 \ln\Lambda^{(\text{ee})}/KT m_e^2 \int d^3v \int d^3v' f_0(\vec{v}) f_0(\vec{v}') \times \frac{\partial \bar{\rho}_k}{\partial v_m} \left(\frac{\delta_{mn}}{g} - \frac{g_m g_n}{g^3} \right) \left(\frac{\partial \rho_l}{\partial v_n} - \frac{\partial \rho_l}{\partial v_n'} \right). \tag{22}$$

In Eqs. (21) and (22) all the quantities ($f_0, \rho_k, g_m \equiv v_m - v'_m$, etc.) are electron quantities except for Z, the "average" ion charge, which is defined as follows,

$$Zn_e e^2 \equiv \sum_{i=1}^{s-1} n^{(i)}(q^{(i)})^2 \tag{23}$$

Note that for a neutral plasma we also have the condition

$$\sum_{i=1}^{s-1} n^{(i)} q^{(i)} = n_e e \,.$$

What remains to be done is to carry out the velocity integrals in Eqs. (21) and (22). Turning to Eq. (21), and using cylindrical velocity coordinates, Eq. (6), we find that the ϕ integration can be carried out exactly and that the double sum of Bessel functions reduces to a single sum. The remaining double integral over v_z and w is more troublesome. However, if the following conditions are satisfied

$$|k_x/\Omega\sqrt{\beta}\,| \ll 1\,, \tag{24}$$

$$|k_z/(\omega - n\Omega)\sqrt{\beta}\,| \ll 1\,, \tag{25}$$

we can expand ρ_h in a power series in kv. We find that ρ_h has the structure,

$$\rho_h = A_{hi}v_i + B_{hij}kv_iv_j + D_{hijk}k^2v_iv_jv_k + \cdots\,, \tag{26}$$

where the coefficients are independent of v_i and of k. For $\tilde{\rho}_h$ we have a similar expansion with coefficients $\tilde{A}_{hi}$, etc. With those expansions the velocity integrals in Eqs. (21) and (22) can clearly be performed. We shall retain terms to second order in k, which actually represent the lowest-order corrections to the "cold plasma" results, since terms linear in k integrate to zero.

The ion-electron collision integral gives a $k = 0$ term bilinear in the $\tilde{A}$ and A coefficients, and a k^2 term involving the products $\tilde{A}D$, $\tilde{D}A$, and $\tilde{B}B$. The electron-electron collision integral gives no $k = 0$ contribution. Physically this results from the fact that electron-electron collisions do not alter the mean electron velocity. We do get an electron-electron term of order k^2, however, which is bilinear in the $\tilde{B}$ and B coefficients. This term is equal to the corresponding electron-ion $\tilde{B}B$ contribution divided by $Z\sqrt{2}$, provided we ignore the difference between $\ln\Lambda^{(ei)}$ and $\ln\Lambda^{(ee)}$. In the Landau equation the value of Λ is in any case ambiguous in the sense that it depends on the impact parameter cutoffs. The precise value of Λ must be determined with a completely convergent collision term, e.g., "Balescu-Lenard" plus "Boltzmann" minus "Landau." The uncertainty in Λ, however, is not very important if $\ln\Lambda \gg 1$. In this paper we shall choose Λ to be defined as follows:

$$\Lambda \equiv 12\pi n_e \lambda_D^{\,3}\,, \tag{27}$$

where $\lambda_D \equiv (KT/4\pi n_e^2)^{1/2}$ is the Debye length.

Carrying out the integrals in Eqs. (21) and (22) to second order in k, we obtain

$$\sigma_{kl}^{(c)} = (2/\pi)^{\frac{1}{2}} Z\omega_p^{\,3} \ln\Lambda K_{kl}/4\pi\omega^2\Lambda\,, \tag{28}$$

where $\omega_p \equiv (4\pi n_e e^2/m_e)^{1/2}$ is the plasma frequency, and where K_{kl} is given by

$$K_{xx} = \omega^2(\omega^2 + \Omega^2)/(\omega^2 - \Omega^2)^2 + (1 + 3/5\sqrt{2}\,Z)k_z^{\,2}\omega^2(\omega^4 + 6\omega^2\Omega^2 + \Omega^4)/\beta(\omega^2 - \Omega^2)^4$$
$$+ \frac{k_x^{\,2}\omega^2}{5\beta}\left(\frac{6(\omega^2 + 2\Omega^2)}{(\omega^2 - \Omega^2)^2(\omega^2 - 4\Omega^2)} + \frac{(1 + 1/\sqrt{2}\,Z)(4\omega^4 + 31\omega^2\Omega^2 + 28\Omega^4)}{(\omega^2 - \Omega^2)^2(\omega^2 - 4\Omega^2)^2}\right)\,,$$

$$K_{yy} = \omega^2(\omega^2 + \Omega^2)/(\omega^2 - \Omega^2)^2 + (1 + 3/5\sqrt{2}\,Z)k_z^{\,2}\omega^2(\omega^4 + 6\omega^2\Omega^2 + \Omega^4)/\beta(\omega^2 - \Omega^2)^4$$
$$+ \frac{k_x^{\,2}}{5\beta}\left(\frac{2\omega^2(\omega^2 + 14\Omega^2)}{(\omega^2 - \Omega^2)^2(\omega^2 - 4\Omega^2)} + \frac{(1 + 1/\sqrt{2}\,Z)(3\omega^6 + 40\omega^4\Omega^2 + 4\omega^2\Omega^4 + 16\Omega^6)}{(\omega^2 - \Omega^2)^2(\omega^2 - 4\Omega^2)^2}\right)\,,$$

$$K_{xy} = -K_{yx} = i\omega\Omega\left[\frac{2\omega^2}{(\omega^2-\Omega^2)^2} + \frac{4(1+3/5\sqrt{2}\,Z)k_z{}^2\omega^2(\omega^2+\Omega^2)}{\beta(\omega^2-\Omega^2)^4}\right. \tag{29}$$

$$\left. + \frac{k_x{}^2}{5\beta}\left(\frac{8(2\omega^2+\Omega^2)}{(\omega^2-\Omega^2)^2(\omega^2-4\Omega^2)} + \frac{(1+1/\sqrt{2}Z)(19\omega^4+28\omega^2\Omega^2+16\Omega^4)}{(\omega^2-\Omega^2)^2(\omega^2-4\Omega^2)^2}\right)\right] ,$$

$$K_{xz} = K_{zx} = [k_x k_z/5\beta(\omega^2-\Omega^2)^3]\,[(5+1/\sqrt{2}\,Z)\omega^4 + (14+13/\sqrt{2}\,Z)\Omega^2\omega^2 - (3+\sqrt{2}/Z)\Omega^4] ,$$

$$K_{yz} = -K_{zy} = -[i\Omega k_x k_z/5\beta\omega(\omega^2-\Omega^2)^3]\,[(15+7/\sqrt{2}\,Z)\omega^4 + (2+7/\sqrt{2}\,Z)\Omega^2\omega^2 - (1+\sqrt{2}/Z)\Omega^4] ,$$

$$K_{zz} = 1 + 2k_z{}^2(1+\sqrt{2}/5Z)/\beta\omega^2 + (k_x{}^2/5\beta)[2/(\omega^2-\Omega^2) + 3(1+1/\sqrt{2}\,Z)(\omega^2+\Omega^2)/(\omega^2-\Omega^2)^2] .$$

Here β is defined by Eq. (3), and the reader is reminded that in accord with Eq. (7), Ω is negative. In Eq. (29) the terms containing the factor Z^{-1} arise from the electron-electron collision integral, Eq. (22). The remaining terms, in particular all the $k=0$ contributions, come from the electron-ion collision integral, Eq. (21).

Note that the assumption stated in Eq. (25) implies that $\sigma_{kl}{}^{(c)}$ as given in Eqs. (28) and (29) is not correct for ω within $\sim k_z(KT/m)^{1/2}$ of the cyclotron frequency and multiples thereof. In order to study the collisional damping in these frequency intervals, the v_z integration in Eqs. (21) and (22) would have to be carried out exactly. In just these intervals, however, the collision-free damping will be dominant. For that reason, Eq. (25) is not a very serious restriction as far as $\sigma_{kl}{}^{(c)}$ is concerned.

For the convenience of those who may wish to study the dispersion relation, Eq. (1), in detail, we will write down the collisionless contribution [1-10] to the conductivity tensor $\sigma_{kl}{}^{(1)}$ to second order in k, again valid only under the conditions of Eqs. (24) and (25). From the electron current we obtain,

$$\sigma_{kl}{}^{(1)} = i\omega_p{}^2 T_{kl}/4\pi\omega , \tag{30}$$

where T_{kl} is given by

$$T_{xx} = [\omega^2/(\omega^2-\Omega^2)]\,[1 + k_z{}^2(\omega^2+3\Omega^2)/2\beta(\omega^2-\Omega^2)^2 + 3k_x{}^2/2\beta(\omega^2-4\Omega^2)] ,$$

$$T_{yy} = [\omega^2/(\omega^2-\Omega^2)]\,[1 + k_z{}^2(\omega^2+3\Omega^2)/2\beta(\omega^2-\Omega^2)^2 + k_x{}^2(\omega^2+8\Omega^2)/2\beta\omega^2(\omega^2-4\Omega^2)] ,$$

$$T_{xy} = -T_{yx} = [i\omega\Omega/(\omega^2-\Omega^2)]\,[1 + k_z{}^2(3\omega^2+\Omega^2)/2\beta(\omega^2-\Omega^2)^2 + 3k_x{}^2/\beta(\omega^2-4\Omega^2)] ,$$

$$T_{xz} = T_{zx} = k_x k_z\omega^2/\beta(\omega^2-\Omega^2)^2 , \tag{31}$$

$$T_{yz} = -T_{zy} = -i\Omega k_x k_z(3\omega^2-\Omega^2)/2\beta\omega(\omega^2-\Omega^2)^2 ,$$

$$T_{zz} = 1 + 3k_z{}^2/2\beta\omega^2 + k_x{}^2/2\beta(\omega^2-\Omega^2) .$$

We obtain a similar contribution from each ion species with the obvious changes in mass, charge, and density. In expanding $\sigma_{kl}{}^{(1)}$, we do of course throw away the collisionless damping. In order to include the collisionless damping, or to study wave propagation in the frequency intervals violating Eq. (25), we must use the exact $\sigma_{kl}{}^{(1)}$, which is well known [1-10] and will not be displayed here.

The dispersion relation for an electron plasma wave propagating at an arbitrary direction with respect to $\vec{B}_0$ is now given by Eq. (1) with σ_{ij} equal to the sum of the terms given by Eqs. (28) and (30). In writing out the determinant of Eq. (1), it must be remembered that only terms linear in the collision parameter $\Lambda^{-1}\ln\Lambda$ should be retained, since $\sigma_{ij}{}^{(c)}$ is correct only to this order.

C. Effect of Ion Motion

The effect of ion motion on $\sigma_{ij}{}^{(c)}$ will be negligible except at low frequencies. There are really two effects: the electron current calculated in the previous section will be modified, and, in addition, we must include the ion currents. In Eq. (17) we have given the complete result for a plasma consisting of electrons and $s-1$ species of ions. To keep the discussion of reasonable length, we shall now limit ourselves to a single species of ion with mass m_i, charge Ze, and density n_e/Z.

The contribution due to ion-ion collisions has in effect already been calculated. It is identical to the electron-electron contribution [i.e., the terms containing the factor Z^{-1} in Eq. (29)] except for the obvious changes in mass, charge, and density. From Eqs. (28) and (29) we see that for high frequencies, $\omega \gg \Omega_i$, the (like species) ion-ion term is of order $(m_e/m_i)^{5/2}$ as compared to the electron-electron terms. A factor of $m_i^{-3/2}$ comes from $\omega_p{}^3$ and a factor of $m_i{}^{-1}$ from β^{-1} (collisions between ions of different types would contribute $k=0$ terms of order $(m_e/m_i)^{3/2}$ relative to the electron terms). Because of the $(\omega^2-\Omega_i{}^2)$ and $(\omega^2-4\Omega_i{}^2)$ factors in the denominators, however, the ion-ion terms must be included at low frequencies, $\omega \lesssim \Omega_i$.

Turning to the effect of the ion motion on the electron-ion collision terms in Eq. (17), we shall consider only the leading contributions, namely those which arise from the $k=0$ parts of $\tilde{\rho}_k$ and ρ_l. We discuss first the contributions coming from the first term in the last parenthesis of Eq. (17). For the electron current this is the term that has been written down in Eq. (21) after setting $\vec{g}$ equal to $\vec{v}^{(e)}$. Retaining the exact $\vec{g}$, however, does not alter the frequency dependence of the result obtained previously; in particular there are no new resonances at multiples of the ion cyclotron frequency. We merely obtain small corrections to Eq. (21) in the mass ratio m_e/m_i which are negligible at *all* frequencies. The corresponding $k=0$ term in the ion current may be obtained from the $k=0$ part of Eqs. (28) and (29) by multiplying by $(Zm_e/m_i)^2$ and replacing Ω_e by Ω_i. Combining these terms with the leading electron and ion current contributions from the second term in the last parenthesis of Eq. (17), we finally obtain the following additions to K_{kl} as given in Eq. (29):

$$K_{xx}{}'=K_{yy}{}'=\frac{Z^2m_e{}^2\omega^2(\omega^2+\Omega_i{}^2)}{m_i{}^2(\omega^2-\Omega_i{}^2)^2}+\frac{2Zm_e\omega^2(\omega^2+\Omega_i\Omega_e)}{m_i(\omega^2-\Omega_i{}^2)(\omega^2-\Omega_e{}^2)}\,,$$

$$K_{xy}{}'=-K_{yx}{}'=\frac{2iZ^2m_e{}^2\omega^3\Omega_i}{m_i{}^2(\omega^2-\Omega_i{}^2)^2}+\frac{2iZm_e\omega^3\Omega_e}{m_i(\omega^2-\Omega_i{}^2)(\omega^2-\Omega_e{}^2)}\,. \tag{32}$$

The corresponding $K_{zz}{}'$ term of $(Zm_e/m_i)^2 + 2Zm_e/m_i$ can be ignored for all frequencies.

In summary: at high frequencies $\omega \gg \Omega_i$ the neglect of ion motion is justified, and the results of Sec. B are applicable. At low frequencies $\omega \lesssim \Omega_i$ the dispersion relation is again obtained from Eq. (1) with σ_{ij} equal to the sum of $\sigma_{ij}{}^{(1)}$ and $\sigma_{ij}{}^{(c)}$, Eq. (28); but in Eq. (28) the K_{ij} is now to be taken as the sum of the terms in Eqs. (29) and (32) as well as the ion-ion collision terms identified in the second paragraph of this section.

IV. DISCUSSION

The collisional contribution to the conductivity tensor associated with a wave in a plasma in a magnetic field has been given in a convenient integral form, Eq. (16), valid for any collision integral. The corresponding result for the Landau collision term is given in Eq. (17). Neglecting the ion motion, the collisional conductivity, valid to all orders in $k^2\lambda_D{}^2$, is given by the integrals of Eqs. (21) and (22). These integrals are carried out completely to second order in the wave vector with the result as displayed in Eqs. (28) and (29).

The k independent and the $k_z{}^2$ parts of K_{xx}, K_{xy}, and K_{yy} lead to the same dispersion relation as that obtained by Buti.[14] In calculating the damping rate, however, Buti commits an error, as pointed out by McBride.[15] The k independent $k_z{}^2$, and $k_x{}^2$ parts of K_{zz} are all in agreement with the dispersion relations obtained by McBride.[15] We confirm his important observation that in a moment equation approach, the heat flow tensor cannot be neglected if all the k^2 terms are to be included. Earlier work relating to the K_{zz} term, including that reported in Refs. 11 and 13, is discussed by McBride[15] and found by him to be partially in error. We agree with his conclusions.

The $K_{xz}(=K_{zx})$ and $K_{yz}(=-K_{zy})$ terms have, to the best of our knowledge, not been considered previously nor have the $k_x{}^2$ parts of the K_{xx}, K_{yy}, and $K_{xy}(=-K_{yx})$ terms. Since we have calculated all the components of the conductivity tensor to order $k^2\lambda_D{}^2$, we are now in a position to study all the modes of wave propagation to that order, including those propagating at an arbitrary angle to the magnetic field. In all cases the collisional damping rate to order $k^2\lambda_D{}^2$ is obtained without difficulty from the general dispersion relation Eq. (1) with σ_{ij} given by Eqs. (28) and (30).

We have also calculated the relevant additions to the conductivity tensor resulting from the ion motion. As discussed in Sec. III C, these additions become significant at low frequencies, $\omega \lesssim \Omega_i$.

The results given in Eqs. (13) and (16) are valid for any kinetic equation. The later results are obtained with the Landau[18] collision integral. It is well known,[21] however, that the magnetic field modifies the collision term unless

$$\Omega \ll \omega_p. \quad (33)$$

The results obtained with the Landau equation are therefore subject to this restriction, Eq. (33). To extend our results we must use a collision integral valid for strong magnetic fields, such as the one derived by Rostoker.[21] We also recognize[22] that the Landau collision term, or even the Balescu-Lenard term,[20] is not strictly applicable for high frequencies $\omega > \omega_p$, where the one-particle distribution function and the pair correlation function vary on the same time scale. Starting with Eq. (16), we are now in the process of extending the results obtained in this paper and the accompanying paper by McBride and Pytte[16] to collision integrals of wider applicability.

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