Existence and Analyticity in s of the Solution to Non-Fredholm Bethe-Salpeter Equations for Scattering of Spin-$\frac{1}{2}$ Particles*

JOHN CONNELL†
Department of Physics, Michigan State University, East Lansing, Michigan 48823
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A large class of non-Fredholm Bethe-Salpeter equations for the scattering Green's function of spin-$\frac{1}{2}$ particles is considered. Many commonly treated equations belong to this class; an example is the quantum-electrodynamical (QED) electron-positron Bethe-Salpeter (BS) equation in the ladder approximation. It is proven that a solution of these equations by successive iterations exists. A domain of analyticity of the solution in the s plane is derived. For small coupling constant, this domain is very large. The domain shrinks to the point $s=0$ when the coupling constant is increased to a limiting value (of order unity). A simple and general inequality is derived for the binding energy of any bound state in terms of the coupling constant. It is proven that Goldstein's pseudoscalar homogeneous solution to the QED equation at $s=0$ does *not* correspond to a bound state when the fine-structure constant is less than of order unity. The method of proof of the existence and analyticity is to use majorization to show that an infinite series of analytic functions is uniformly convergent. No Hilbert-space techniques are used, so no difficulties arise from the non-Fredholm nature of the BS equations.

I. INTRODUCTION

SOLUTIONS of model Bethe-Salpeter (BS) equations have been quite extensively investigated in the last few years. Some recent motives for this work have been the hope of a derivation from a relativistic dynamical model of such properties as Regge trajectories,[1,2] $O(4)$ invariance relations and daughter trajectories.[3,4] More generally, there is always hope that the properties of the solutions of the model BS equations which have been investigated so far may be shared by the solution to a relativistic quark model of the hadrons, if a realistic model is ever found.

Any BS equation is difficult to solve, because it is an integral equation in four dimensions. The usual procedure has been to consider an equation for the interaction of two scalar particles in the ladder approximation (that is, exchanging one or more scalar particles). Such an equation can be expressed in the form of a Fredholm equation and then the standard formulas of Fredholm theory can be exploited with the use of small coupling constants,[1] numerical integration,[4] or general arguments[3] to derive whatever results are required.

Unfortunately, a BS quark model, for example a model of a meson as a quark-antiquark pair, must start from a BS equation for spin-$\frac{1}{2}$ particles[5] since quarks are undoubtedly spin-$\frac{1}{2}$ if they exist. However, it is well known that BS equations for the interaction of two spin-$\frac{1}{2}$ particles usually *cannot* be reduced to Fredholm form.[2,6] An example is Eq. (2) below, the quantum-electrodynamical (QED) equation for the interaction of an electron and positron in the single-photon-exchange ladder approximation.[6] The non-Fredholm property of such equations prevents use of the standard techniques which have been described above. Indeed, Fredholm BS equations for the interaction of scalar particles appear to differ from the non-Fredholm equations for the interaction of spin-$\frac{1}{2}$ particles in at least three ways, as follows.

The usual BS equations for a scattering Green's function of scalar particles have the following properties:

(i) An iteration solution always exists when the coupling constant is sufficiently small, since the Fredholm kernel has a finite norm.

(ii) The existence of a solution to the homogeneous equation at a particular value of the c.m. energy (for fixed coupling constant) implies that the scattering Green's function has a pole in the c.m. energy variable there, which is to say that (by the definition of the word) there is a *bound state* at that energy.

(iii) If the solution is a sum of planar diagrams (as it is in the ladder approximation), the solution has no Regge cuts.[7]

On the other hand, the sole example of electron-position scattering in the ladder approximation [Eq. (2) below] shows that non-Fredholm spin-$\frac{1}{2}$ BS equations may differ in all three of these properties, as follows:

(i) Conditions for the existence of a solution have not been established.

(ii) The existence of a homogeneous solution at a certain energy does not imply the existence of a bound

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[1] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962).

[2] A. R. Swift and B. W. Lee, Phys. Rev. **131**, 1857 (1963).

[3] G. Domokos, Phys. Rev. **159**, 1387 (1967).

[4] V. Chung and D. R. Snider, Phys. Rev. **162**, 1639 (1967).

[5] H. M. Lipinski and D. R. Snider, Phys. Rev. **176**, 2054 (1968).

[6] G. Tiktopoulos, J. Math. Phys. **6**, 573 (1965).

[7] R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge University Press, Cambridge, England, 1966), Sec. 3.8. This statement does not apply to diagrams whose vertices have more than three particles (see Sec. 3.5 of reference).

state at that energy, since the Fredholm alternative does not hold.[2] In fact, Goldstein[8] and Kummer[9] have given solutions at zero total 4-momentum to the homogeneous part of the QED equation (2) below, and it has not been rigorously shown whether or not those solutions correspond to zero-mass bound states.

(iii) An infinite sum of planar diagrams can have Regge cuts. Lee and Swift[2] and Willey[10] have found them in the crossed channel forward scattering solution to spin-$\frac{1}{2}$ BS equations in the ladder approximation.

This paper will give some answers to questions raised by points (i) and (ii) above. (i) When does a solution to the scattering equation for spin-$\frac{1}{2}$ particles exist? (ii) When does a homogeneous solution correspond to a bound state? No discussion will be given of point (iii), the question of when Regge cuts exist.

It will be shown that conditions on the kernel and propagators of a spin-$\frac{1}{2}$ BS equation can be given, under which the iteration solution exists and is analytic in s (the invariant square of the c.m. energy). The size of the domain of analyticity in the s plane depends, of course, on the strength of the coupling constants in the equation. Then within this domain of analyticity, question (i) is answered: The solution exists. Also question (ii) is answered there: There are *no* bound states since the solution is analytic in s.

The technique used is majorization. The majorizations are not very stringent and appear to be applicable to quite a large class of spin-$\frac{1}{2}$ model BS equations; in particular, they apply to the QED electron-positron ladder-approximation equation already metioned [Eq. (2) below]. The procedure is to show that for suitable values of s the iteration solution to the BS equation, which is term-by-term analytic in s, can be majorized term by term by quantities independent of s. When the resulting series converges, a theorem on the uniform convergence of an infinite series of analytic functions implies that the original iteration solution is analytic in s. The conditions on s, for all the foregoing to occur, determine the domain of analyticity of the iteration solution in the s plane. This method avoids any attempt to use Fredholm techniques.

The proof of these results requires several steps, though it is quite easy. To help the reader not to get lost on the way, an outline of the paper is given in Sec. II. A complete summary of the problem, assumptions, results and the steps in the proof is given there.

It should be emphasized here that the solution to a BS equation will be defined as the *iteration solution*, when it exists, that is, when it converges. If the value of s is such that the iteration solution does not converge, then s must be given a value such that the iteration does converge. Then the resultant iteration solution may be analytically continued in s to the required value of s. This definition is in accordance with the idea that the solution to a BS equation should indeed be an infinite sum of Feynman diagrams. If the coupling constant is so great that there is no value of s such that the iteration solution exists, then it will be said, by definition, that no solution exists at all.

In the rest of this section the results of this paper will be put into context by giving a brief review of some of the rather scarce work on spin-$\frac{1}{2}$ BS equations.

As has been mentioned, Goldstein[8] found a homogeneous solution to the QED equation (2) at zero 4-momentum, soon after the BS equation was invented.[11] See topic (ii) above for further discussion of its meaning. Kummer[9] later added more homogeneous solutions to Goldstein's.

Lee and Swift[2] examined the $N\bar{N}$ scattering equation with exchange of pseudoscalar particles in the ladder approximation. They managed to sum the singular (non-Fredholm) part of the iteration solution under several approximations using a technique due to R. F. Sawyer, and they found the crossed-channel forward scattering to be dominated by a Regge cut. Willey[10] has found the *exact* form of an additive component of the solution to the QED equation (2) below, at zero 4-momentum, and his solution is also dominated by a Regge cut.

Neither Lee and Swift[2] nor Willey[10] put a cutoff in their equations or used any other artifice to change their equation into a Fredholm equation and create a Hilbert space. Such artifices have been used sometimes. Under the assumption that the homogeneous solutions to the QED equation (2) (slightly transformed) are square integrable, Tiktopoulos[6] has proven, by Hilbert-space methods, that the spectrum of bound-state energy eigenvalues of Eq. (2) approaches that of the Coulomb problem to lowest order in the fine structure constant. Unfortunately, there is no proof of his crucial initial assumption that the solutions lie in a Hilbert space. Quite recently, Lipinski and Snider[5] have examined the $N\bar{N}$ BS equation with scalar exchange in the ladder approximation with s small but not zero. Their interest is in possible Regge trajectories of the pion. In order to use conventional perturbation theory (with s as the small parameter) they introduced a cutoff by *subtracting* from the kernel the exchange of a more massive particle of the same nature and coupling constant (that is, they added a "ghost" exchange to the kernel). Their resultant equation is Fredholm, and they can derive numerical answers from it.[11a]

No technique seems to have been invented yet for solving spin-$\frac{1}{2}$ BS equations for $s\neq 0$ without using such artifices as these. On the other hand, the use of

[8] J. S. Goldstein, Phys. Rev. **91**, 1516 (1953).
[9] W. Kummer, Nuovo Cimento **31**, 219 (1964).
[10] R. S. Willey, Phys. Rev. **153**, 1364 (1967).
[11] E. E. Salpeter and H. A. Bethe, Phys. Rev. **84**, 1232 (1951); J. Schwinger, Proc. Natl. Acad. Sci. U. S. **37**, 452 (1951); **37**, 456 (1951); M. Gell-Mann and F. Low, Phys. Rev. **84**, 350 (1951).
[11a] *Footnote added in proof*. In addition to Ref. 5, a cutoff single-particle-exchange BS equation for spin-$\frac{1}{2}$ particles has been considered by P. Narayanaswamy and A. Pagnamenta, Nuovo Cimento **53A**, 635 (1968). They obtain some numerical solutions.

cutoffs or other assumptions to create a Hilbert space may completely change the nature of the equation and the solution. Therefore, the discovery of a new technique is needed. The results of this paper, while by no means established by any useful new technique, do show that in discussing a spin-$\frac{1}{2}$ BS equation with $s\neq 0$, it is not *always* necessary to start by changing the problem to create a Hilbert space.

For completeness, a result of Yatsun[12] will be mentioned. He has shown that an iteration solution exists to the BS equation for massless leptons interacting by massive spin-1 exchange (which, of course, is not renormalizable) in the ladder approximation, in the particular case when the 4-momentum of each of the final particles is zero. The conditions are that the coupling constant lie in a certain domain of the complex plane. Yatsun's result is remarkable because the tensor part of the spin-1 meson propagator causes the kernel to be so divergent that even the proof of the present paper would not apply to his equation.

II. SUMMARY OF PROBLEM, HYPOTHESES, RESULTS, AND PROOF

A. Problem

The problem is to show when iteration solutions to the spin-$\frac{1}{2}$ BS equations below exist.

The case of fermion-antifermion scattering will be considered for definiteness, but the proof will apply equally well to the fermion-fermion case. The general BS equation for the Green's function $T_{\alpha\beta;\rho\sigma}(p,q;K)$ (see Fig. 1 for the notation), is illustrated in Fig. 2(a). In full it is

$$T_{\alpha\beta;\rho\sigma}(p,q;K)=I_{\alpha\beta;\rho\sigma}(p,q;K)+\int\frac{d^4k}{(2\pi)^4}I_{\alpha\beta;\rho\theta}(p,k;K)$$
$$\times G_{\delta\epsilon}(k+\tfrac{1}{2}K)G_{\lambda\theta}(k-\tfrac{1}{2}K)T_{\epsilon\beta;\lambda\sigma}(k,q;K). \quad (1)$$

An example from QED which will be used throughout as an illustration is the equation for electron-positron scattering in the ladder approximation [Fig. 2(b)]:

$$T_{\alpha\beta;\rho\sigma}(p,q;K)=-ie^2\gamma^{\mu}{}_{\alpha\beta}\gamma_{\mu\sigma\rho}(1/(p-g)^2)$$
$$+\int\frac{d^4k}{(2\pi)^4}\frac{1}{(p-k)^2}$$
$$\times\left[\gamma^{\mu}\frac{1}{\gamma\cdot(k+\frac{1}{2}K)+m}\right]_{\alpha\epsilon}\left[\frac{1}{\gamma\cdot(k-\frac{1}{2}K)+m}\gamma_{\mu}\right]_{\lambda\rho}$$
$$\times T_{\epsilon\beta;\lambda\sigma}(k,q;K). \quad (2)$$

In the above examples (1) and (2), Wick rotation on the variables p^0, q^0, k^0 is assumed to be allowed and already to have been performed. Then $k^0=ik_4$, k_4 real, etc. Also d^4k has been taken to mean $d\mathbf{k}dk_4$. The metric used is defined by $a\cdot b=\mathbf{a}\cdot\mathbf{b}-a^0b^0=\mathbf{a}\cdot\mathbf{b}+a_4b_4$. Then $k^2\geq 0$ always. The total energy-momentum K may be complex. Define $s=-K^2$.

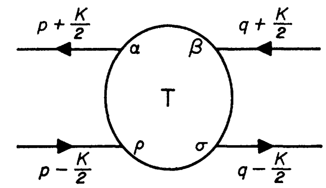

FIG. 1. The Feynman diagram for the Green's function $i^{-2}T_{\alpha\beta;\rho\sigma}(p,q;K)$.

B. Hypotheses

Assume that Wick rotation, as performed above, is possible, and that it can be reversed at the end without changing any analyticity properties of T as a function of s.

If the BS equation (1) is written symbolically as $T=I+IGGT$, then the iteration solution is of course $T=I+IGGI+IGGIGGI+\cdots$. Assume, as described in Sec. I, that the solution to a BS equation is the iteration solution if it exists for some value of s, and that it is the analytic continuation in s of the iteration solution for other values of s.

Let m be the mass of each fermion. The fermion propagator will have the Källén-Lehmann representation[13]

$$G(p)=\int_0^{\infty}d\kappa^2\left[\frac{\rho_1(\kappa^2)}{\gamma\cdot p+\kappa-i\epsilon}+\frac{\kappa\rho_2(\kappa^2)}{p^2+\kappa^2-i\epsilon}\right]. \quad (3)$$

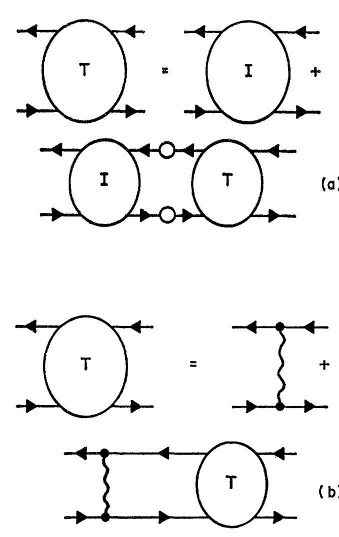

FIG. 2. (a) The general BS equation (1). (b) The QED BS equation (2).

[12] V. A. Yatsun, AN Ukrainskoi SSR, Inst. Teor. Fiz., 1968 (to be published).

[13] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper and Row Publishers, Inc., New York, 1961), Sec. 17b.

Define a constant g_0, which arises in the propagator majorization (Sec. IV), by

$$\frac{1}{g_0}=(2\pi)^{-1}\int_0^\infty d\kappa^2[\rho_1(\kappa^2)+\rho_2(\kappa^2)]. \tag{4}$$

Assume that this integral exists. (It usually does; see Sec. IV.) Note that if $G(p)=1/(\gamma\cdot p+m)$, as in the QED example (2), then $g_0=2\pi$.

Assume that the kernel $I(p,q;K)$ is analytic in s below the elastic threshold at $s=4m^2$.

A simple norm $\|\ \|$ for finite-dimensional matrices will be introduced in Sec. III. Assume that the kernel $I_{\alpha\beta;\rho\sigma}(p,q;K)$, treated as a 16×16 dimensional matrix, can be majorized according to

$$\|I(p,q;K)\|\leq g^2/(p-q)^2 \tag{5}$$

for all p,q with g a positive constant. [Recall $(p-q)^2\geq 0$.]

In the QED example (2), the kernel is certainly analytic in s, and the majorization (5) is satisfied with $g=2e$, as will be shown in Sec. III.

Kernels definitely excluded by these requirements [the analyticity in s, and the majorization (5)] are those which contain low-mass annihilation diagrams, such as in Fig. 3(a), and those containing diagrams derived from nonrenormalizable field theories, such as the massive $J=1$ exchange shown in Fig. 3(b). It is possible that the requirements might be satisfied in QED to all orders of perturbation theory when annihilation diagrams do not occur, as in electron-electron scattering. This question is discussed in Sec. III, where the majorization (5) is described in detail.

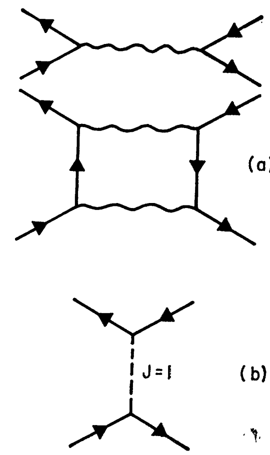

FIG. 3. (a) Low-mass annihilation diagrams, which do not obey the kernel analyticity requirement in s. (b) Massive $J=1$ exchange, which does not obey the kernel majorization (8).

C. Results

The result found is that the iteration solution $T(p,q;K)$ to the BS equation (1) exists, and is analytic in s, in a certain domain $D(g/g_0)$ of the s plane. This domain is specified by a relation between $s/4m^2$ and g/g_0, Eq. (35) below in Sec. VII. g must be less than g_0. When g approaches g_0, the domain $D(g/g_0)$ shrinks to the single point $s=0$. The domain $D(g/g_0)$ is sketched in Fig. 4 for various values of g/g_0.

When s is real the condition (35) simplifies, and shows that the part of the real s axis lying in $D(g/g_0)$ is given by

$$-4m^2(g_0^2/g^2-1)<s<4m^2(1-g/g_0)^2. \tag{6}$$

The right-hand side of Eq. (6) immediately gives a very general inequality for the binding energy of any bound state that may appear in the solution T of the BS equation (1). If the bound-state mass is M, then the binding energy B is $2m-M$. Since the Green's function T will have a pole in s at $s=M^2$, the analyticity condition (6) implies that $M>2m(1-g/g_0)$, which is to say

$$B<2mg/g_0. \tag{7}$$

Therefore, when any model BS equation obeys the hypotheses described above, the inequality (7) gives an upper bound on the possible binding energy of its bound states. Conversely, if a certain binding energy is required, the inequality (7) places a lower bound on the possible coupling constants which can give it.

If the QED equation (2), which is being used as an example, does have the Coulomb binding energy $B=\frac{1}{4}\alpha^2 m$ in its lowest bound state when the fine structure constant $\alpha=e^2/4\pi$ is small (which has apparently never really been proven[6]), then (7) is verified in this model. Indeed, $g=2e$ as will be shown in Sec. III, and $g_0=2\pi$ as was shown in part B of this section, and when α is small it is certainly true that $\frac{1}{4}\alpha^2 m<2m2e/2\pi$.

The left-hand side of the inequalities (6) is rather surprising. When g is nearly as great as g_0, the method of this paper has only succeeded in showing the iteration solution to be analytic for quite a small range of negative s. (The small range of analyticity for positive s in this case is not surprising, because when g is nearly as great as g_0, a deep bound state of mass $M\ll 2m$ may occur.) On the other hand, when the coupling

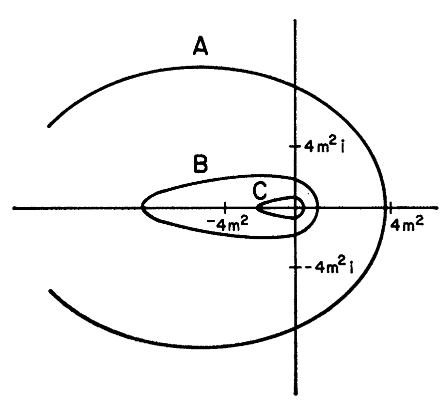

FIG. 4. Sketch of the boundaries of the proven domain of analyticity of $T(p,q;K)$ in the s plane. A: g/g_0 small. B: $g/g_0=\frac{1}{2}$. C: g nearly equal to g_0.

constant in the BS equation is small, as in the QED model equation (2), then the magnitude of the left-hand side of Eq. (6) is very large indeed since g/g_0 can be small, and so the domain of analyticity extends very far out on the negative s axis. See the sketches of the domain $D(g/g_0)$ in Fig. 4.

The inequality (7) proves that Goldstein's solution to the homogeneous part of the QED equation (2) does not correspond to a bound state when α is small (Sec. VII).

D. Proof

The method used is to show that the kernel I and the product GG of the propagators can be analytic in s while the iteration solution $T=I+IGGI+IGGIGGI+\cdots$ to Eq. (1) converges uniformly in s. When all this happens, T will be analytic in s.

In Sec. III a discussion is given of when the kernel $I(p,q;K)$ may be majorized by a function $g^2/(p-q)^2$ which is independent of K, and hence of s. This majorization is assumed to be possible.

In Sec. IV it is shown that the propagators $G(k\pm\frac{1}{2}K)$ may be majorized by a function $L/(k^2+\mu^2)^{1/2}$, L a constant, if s obeys a certain condition and if the integral (4) over the propagator spectral functions, which defines the constant g_0, exists. The majorizing function is again independent of s.

In Sec. V it is shown that when the majorizing functions are substituted for I and G in the iteration solution, the resulting series converges if L, g, and g_0 have a certain relation.

The analyticity in s of I and G is discussed in Sec. VI. I is assumed to be analytic in s in the domain of the s plane of interest. (It will be, if there are no annihilation diagrams.) G is proven to be analytic in that domain.

All the conditions arising in the parts of the proof occurring in the previous sections are collected in Sec. VII to give relations between the variables $s/4m^2$ and g/g_0 which define a domain, called $D(g/g_0)$, in the s plane. In this domain, the iteration solution $T=I+IGGI+\cdots$ is analytic in s because it is a uniformly convergent series of analytic functions. The relations defining $D(g/g_0)$ are given in Eq. (35), and the dependence on g/g_0 of the size and shape of $D(g/g_0)$ is shown in Fig. 4.

III. KERNEL MAJORIZATION

The hypothesis

$$\|I_{\alpha\beta;\rho\sigma}(p,q;K)\|\leq[g^2/(p-q)^2] \tag{8}$$

will now be explained, and illustrated by the QED equation (2).

Let the norm $\|A\|$ of any finite-dimensional matrix A be defined to be the greatest of the absolute values of the eigenvalues of A. Then if $\{\gamma^\mu,\gamma^\nu\}=-2g^{\mu\nu}$ (μ, $\nu=0$, 1, 2, 3), the following facts may be verified:

$$\|\gamma^\mu\|=1\,, \tag{9}$$

$$\|(\gamma\cdot k+m)^{-1}\|=(k^2+m^2)^{-1/2}\,, \tag{10}$$

if k is a real vector in Euclidean space (that is, $\mathbf{k}$, k_4 are real) and m is a real constant.

In the case of the QED example (2), it is evident that

$$\|I_{\alpha\beta;\rho\sigma}(p,q;K)\|=\|e^2\gamma^\mu{}_{\alpha\beta}(p-q)^{-2}\gamma_{\mu\rho\sigma}\| \leq e^2\|\gamma^\mu\|(p-q)^{-2}\|\gamma_\mu\| \tag{11}$$

as can easily be seen upon considering all possible vectors in the 16-dimensional space upon which I acts. Equations (11) and (9) show that the majorization (8) is satisfied for this QED example with

$$g=2e\,. \tag{12}$$

Also, the kernel is certainly analytic in s.

For what other kernels I will the majorization (8) be true? If I is a sum of single-particle exchanges, whether the particles have mass or not, then (8) will be true as long as the vertices are bounded and the propagators drop off as fast as $1/|t|$ for large $|t|$ (thus, excluding massive $J=1$ exchange).

A result of Johnson, Willey, and Baker[14] suggests that (8) might be true for some QED BS kernels in every order of perturbation theory. They show that in QED (under certain conditions), the kernel $I(p,q;K)$ for electron-position scattering has the property that $I(p,q;K)$ is finite in each order of the unrenormalized fine structure constant α_0, as long as both p and q are very large and either $p^2\ll q^2$ or $q^2\ll p^2$. Since I has the dimensions of $1/q^2$ or $1/p^2$, this result shows that $\|I(p,q;K)\|\leq\text{const}/q^2$ in the first case, and $\|I(p,q;K)\|\leq\text{const}/p^2$ in the second case. These results also hold for the BS kernel for electron-electron scattering, which contains no annihilation diagrams and so has the required analyticity in s below the elastic threshold. The asymptotic behavior described above, and the single-photon-exchange kernel of Eq. (2), both satisfy the majorization (8). Therefore, it is tempting to speculate that the results of Johnson, Willey, and Baker[14] can be applied to electron-electron scattering and generalized until (8) can be proven to hold in each order of perturbation theory for all p,q. (The total energy-momentum K would perhaps have restrictions on it similar to those which occur in the next section.) If such a general result were true, then the results of the present paper would be true in equal generality, as g_0 is also finite in each order of perturbation theory according to the treatment of Johnson, Baker, and Willey.

[14] K. Johnson, R. Willey, and M. Baker, Phys. Rev. **163**, 1699 (1967).

IV. PROPAGATOR MAJORIZATION

Let the four-vector K be an arbitrary complex vector, but the vector k be real in Euclidean space ($\mathbf{k}$, k_4 real). Then it will be shown that under certain conditions on K, to be derived, there exists a constant L such that for all k,

$$\left\|\frac{1}{\gamma\cdot(k+\frac{1}{2}K)+m}\right\|\leq\frac{L}{(k^2+\mu^2)^{1/2}},\quad 0<\mu<m. \quad (13)$$

The conditions for this to be true are unchanged when the sign of K is changed. Evidently $L\geqslant 1$.

Since the Källén-Lehmann representation (3) can be written[13]

$$G(p)=\int d\kappa^2\left[\frac{\rho_1(\kappa^2)+\frac{1}{2}\rho_2(\kappa^2)}{\gamma\cdot p+\kappa}-\frac{\frac{1}{2}\rho_2(\kappa^2)}{\gamma\cdot p-\kappa}\right],$$

it follows from $\|A+B\|\leq\|A\|+\|B\|$ that if (13) is obeyed, then the propagators can be majorized according to (since $\kappa\geqslant m$)

$$\|G(k\pm\tfrac{1}{2}K)\|\leq(2\pi/g_0)L/(k^2+\mu^2)^{1/2},\quad 0<\mu<m, \quad (14)$$

where m is the fermion mass and the integral $2\pi/g_0$ over the spectral functions is defined in Eq. (4). Since[13] $2\rho_1\geq\rho_2>0$, the convergence of the integral depends upon the convergence of $\int\rho_1 d\kappa^2$, which usually occurs, as $(\gamma\cdot p)G(p)$ usually has a high-p limit.

The proof of Eq. (13) starts with the following observations. If the vector K is written in its real and imaginary parts as $K=K_R+iK_I=(\mathbf{K}_R+i\mathbf{K}_I,\,E_R+iE_I)$, and if real Euclidean vectors a, b are defined according to

$$\mathbf{a}=\mathbf{K}_R,\quad a_4=E_I, \quad (15)$$
$$\mathbf{b}=\mathbf{K}_I,\quad b_4=-E_R,$$

then

$$\gamma\cdot(k+\tfrac{1}{2}K)+m=\gamma\cdot(k+\tfrac{1}{2}a)+m+\tfrac{1}{2}i\gamma\cdot b. \quad (16)$$

Now imagine two operators A,B. If $\alpha^{-1}=\|A^{-1}\|$, $\beta=\|B\|$, and $\alpha\geq m>\beta$, then it is evident from the convergent formal expansion of $(A+B)^{-1}$ in powers of $A^{-1}B$ that $\|(A+B)^{-1}\|<(\alpha-\beta)^{-1}$. Then it follows that

$$\|(A+B)^{-1}\|<(1-\beta/m)^{-1}\alpha^{-1}.$$

If

$$b^2<4m^2, \quad (17)$$

this result can be applied to the case of Eq. (16) with $A=\gamma\cdot(k+\frac{1}{2}a)+m$ and $B=\frac{1}{2}i\gamma\cdot b$. Using Eq. (10), it is apparent that for all k,

$$\|[\gamma\cdot(k+\tfrac{1}{2}K)+m]^{-1}\|<N^{-1}[(k+\tfrac{1}{2}a)^2+m^2]^{-1/2}, \quad (18)$$

$$N\equiv 1-(b^2)^{1/2}/2m. \quad (19)$$

Then to prove Eq. (13), it is sufficient to find when the right-hand side of Eq. (18) is always less than or equal to the right-hand side of Eq. (13). That condition is equivalent to

$$(k+\tfrac{1}{2}a)^2+m^2\geq(k^2+\mu^2)/L^2N^2. \quad (20)$$

Inequality (20) is quadratic in k. Completion of the square shows that Eq. (20) is satisfied for all k under the two conditions

$$\eta\equiv(L^2N^2)^{-1}<1, \quad (21)$$

$$\eta(1-\eta)^{-1}a^2+4\eta\mu^2\leq 4m^2. \quad (22)$$

Equation (21) contains condition (17) which need not be imposed separately.

Equations (21) and (22) are the conditions on K needed so that the majorization (13) holds. However, the definitions (15) of a and b show that these conditions are not Lorentz invariant. The reason for this is that Wick rotation specifies the time component of a four-vector, so it is not an operation independent of the Lorentz frame. On the other hand, this fact can be turned to advantage, because it means that it is possible to consider all Lorentz frames and see whether there is any in which conditions (21) and (22) are satisfied. If there is one, then all the steps of the proof of the existence and analyticity in s of the iteration solution to the BS equation (1) may be carried out in that frame. (These steps are Wick rotation, the majorizations, the proof of the convergence of the majorizing series, the proof of term-by-term analytcity of the original series, and the Wick rotation's reversal, which is assumed to be possible in the final solution.) It will be shown that there is indeed a Lorentz frame which is "most favorable" in that the left-hand sides of Eqs. (21) and (22) are simultaneously the least possible, and that in this frame the conditions take a covariant form as a function of the single variable s.

Inspection, with the definition (19), shows that when L is constant the left-hand sides of Eqs. (21) and (22) both decrease as a^2 and b^2 decrease. The definitions (15) show that $a^2=K_R^2+(E_R^2+E_I^2)$ and $b^2=K_I^2+(E_R^2+E_I^2)$. The first term in each is Lorentz invariant, so a^2 and b^2 simultaneously reach their minimum when $E_R^2+E_I^2$ does. The minimum of $E_R^2+E_I^2$ under all real Lorentz transformations can most easily be found by minimizing $|K\cdot X|^2$ with respect to variations of a real vector X which satisfies $X^2=-1$. The minimum found is $\frac{1}{2}(|K^2|-K^*\cdot K)$, which gives the result (recalling $s=-K^2$) that

$$\min a^2=\tfrac{1}{2}(|s|-\mathrm{Re}s)=(\mathrm{Im}s^{1/2})^2, \quad (23)$$

$$\min b^2=\tfrac{1}{2}(|s|+\mathrm{Re}s)=(\mathrm{Re}s^{1/2})^2. \quad (24)$$

It is particularly surprising and satisfactory that the values (23) and (24) depend only on s, and on none of the other available covariant quantities defined by K, such as $K^*\cdot K$.

Therefore, there is a Lorentz frame in which conditions (21) and (22) can be satisfied, and hence in which all the steps in the proof of the existence of an iteration solution to the BS equation (1) can be carried out, if

$$N\equiv 1-|\mathrm{Re}s^{1/2}|/2m, \tag{25}$$

$$\eta\equiv(L^2N^2)^{-1}<1, \tag{26}$$

$$\eta(1-\eta)^{-1}(\mathrm{Im}s^{1/2})^2+4\eta\mu^2\leq 4m^2. \tag{27}$$

It is assumed that after $T(p,q;K)$ has been found in this most favorable frame, the Wick rotation of p^0, q^0 can be reversed without destroying the existence of T or changing its domain of analyticity in s. Then, an arbitrary Lorentz transformation can be performed, showing that these properties of T hold in any frame, that is, always.

Conditions (26) and (27) still contain the majorization constant L, introduced in Eq. (13). L has not been specified except that Eqs. (10) and (13) show that it must obey $L>1$. It will now be shown that there is an upper bound to L, determined by the requirement that the iteration solutions to the majorizing equation converge.

V. CONVERGENCE OF THE MAJORIZING EQUATION

The majorizations of the components I and G of the BS equation (1), given in Eqs. (8) and (14) of the last two sections, show that the iteration solution $T(p,q;K)$ to Eq. (1) is majorized by the iteration solution to the equation

$$F(p,q)=\frac{g^2}{(p-q)^2}+g^2\left(\frac{2\pi}{g_0}\right)^2L^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(p-k)^2}\frac{1}{k^2+\mu^2}F(k,q), \tag{28}$$

the convergence of the latter therefore implying that of the former.

The convergence of the iteration solution of Eq. (28) depends only upon the coefficient of the integral in (28), as will now be proven. If the substitution $F(p,q)=\mu^{-2}F_1(p/\mu,q/\mu)$ is made in (28), it is evident that the resultant equation for the function $F_1(x,y)$ is independent of μ. [This scale invariance of the integral operator in Eq. (28) is an example of a characteristic of BS equations for the interaction of fermions which has been exploited by Baker, Johnson, and Willey[14] in QED.] Then the convergence, or otherwise, of the interation solution of the equation for $F_1(x,y)$ is independent of μ, and hence that of Eq. (28) is also. Therefore, the convergence of the iteration solution of Eq. (28) can equally well be tested for the special case $\mu=0$. [The reader may verify this by looking at the explicit solution of Eq. (28) with $\mu\neq 0$ which has been found by Willey.[10]]

Let $C_n(\cos\theta)$ be the usual surface harmonic of Euclidean four-space,[15] such that

$$\frac{1}{(p-q)^2}=\sum_{n=0}^{\infty}\frac{1}{PQ}\left(\frac{P_<}{Q_>}\right)^{n+1}C_n(\hat{p}\cdot\hat{q}), \tag{29}$$

$$P\equiv(p^2)^{1/2},\quad \frac{P_<}{Q_>}\equiv\min\left(\frac{P}{Q},\frac{Q}{P}\right),$$

$$C_n(\cos\theta)=\frac{\sin(n+1)\theta}{\sin\theta}. \tag{30}$$

Then the iteration solution to Eq. (28), with $\mu=0$, may be explicitly verified by induction to be

$$[F(p,q)]_{\mu=0}=g^2\sum_{n=0}^{\infty}\frac{n+1}{\epsilon_n}\frac{1}{PQ}\left(\frac{P_<}{Q_>}\right)^{\epsilon_n}C_n(\hat{p}\cdot\hat{q}), \tag{31}$$

$$\epsilon_n\equiv[(n+1)^2-(Lg/g_0)^2]^{1/2}, \tag{32}$$

if Lg/g_0 is small enough so that the series converges. [The induction is most easily carried out on powers of the variable $(n+1-\epsilon_n)$, instead of Lg/g_0.] It is evident from Eq. (31) that the sum over n converges to a finite number unless $p=q$, in which case we expect divergence since the inhomogeneous term of Eq. (28) is singular there. The single exception to the finiteness of Eq. (31) (except when $p=q$) occurs in the zeroth partial wave ($n=0$) which becomes infinite when either p or q is zero, since $\epsilon_0<1$. Inspection of Willey's solution[10] to Eq. (28) shows that when μ is not zero, the $n=0$ component is finite when p or q is zero. Thus the convergence of the iteration solution to Eq. (28) is established everywhere except at the singularity $p=q$.

Since every function in Eq. (28) is real, the iteration solution of (28) must be real if it exists. But the zeroth partial-wave component of Eq. (31) is shown by Eq. (32) to become first infinite, then complex, as Lg/g_0 increases through the value 1. Thus, the iteration solutions to Eq. (28) will converge when, and only when

$$L<g_0/g. \tag{33}$$

This is the upper bound on L imposed by the convergence requirement.

In the particular case of the illustrative QED equation (2), it has been shown that $g_0=2\pi$ and $g=2e$. Therefore Eq. (33) gives the inequality

$$L<\pi/e \tag{34}$$

as the condition for the convergence of the iteration solution of the majorizing equation of Eq. (2).

VI. ANALYTICITY OF EACH TERM OF THE ITERATION

The last section has shown the iteration solution of the BS equation (1) to be a convergent series, under

[15] G. Domokos and P. Suranyi, Nucl. Phys. 54, 529 (1964).

the conditions (26) and (27) for s and (33) for L. Therefore if each term of the iteration solution of (1) is analytic in s, then the iteration solution itself will be analytic in s,[16] when (26), (27), and (33) are true.

The product GG of the propagators in Eq. (1) is analytic in the s plane cut along the real axis from $s=4m^2$ to ∞.[17] Therefore GG is analytic in the domain defined by (26) and (27).

It should be remarked that Wick rotation may indeed be carried out on each term of the iteration solution in this domain. In fact, the condition defining the domain of analyticity, Eq. (35) below, implies that $|\mathrm{Re}s^{1/2}|<2m$ which is precisely the condition for Wick rotation to be possible.[1]

For simplicity, it is assumed in this paper that the kernel is analytic in s within the domain defined by all the other conditions. As mentioned earlier, this excludes low-mass annihilation diagrams from it.

VII. RESULTANT CONDITIONS

The conditions (26) and (27) on s and (33) on L are now combined to give the final result of this paper.

The discussion of μ in Sec. V shows that μ may be as small as wished, as long as $\mu>0$, without changing the finiteness of the majorizing function $F(p,q)$ at $p=0$ or $q=0$. Therefore in the inequalities (26) and (27), μ can be left out if the $\leq$ sign is changed to $<$.

The left-hand sides of (26) and (27) are smallest when L is largest. Therefore by Eq. (33), we can replace L by g_0/g. In this case, Eqs. (26) and (27) finally become the single condition

$$\frac{(\mathrm{Im}s^{1/2})^2}{4m^2}+1<\left[\frac{g_0}{g}\left(1-\frac{|\mathrm{Re}s^{1/2}|}{2m}\right)\right]^2. \tag{35}$$

As has been explained in Sec. IID, Eq. (35) defines a domain $D(g/g_0)$ of the s plane in which the iteration solution $T(p,q;K)$ to the BS equation (1) exists and is analytic in s.

The domain $D(g/g_0)$ is sketched for a few values of g/g_0 in Fig. 4.

The consequences of the limits on $D(g/g_0)$ for real s were discussed in Sec. IIC.

As described in Sec. IIC, the result (35) is a proof that Goldstein's pseudoscalar homogeneous solution to the QED equation (2) at $K=0$ does not correspond to a bound state when $\alpha=e^2/4\pi$ is small [since $g^2/g_0{}^2=4\alpha/\pi$ for Eq. (2)]. It should be mentioned that Willey has almost proven this already.[10] He showed that the pseudoscalar additive component of Eq. (2) at $K=0$ has a finite particular solution. Equation (35) completes the proof[18] by showing that the solution to (2) is actually analytic in s at $s=0$.

VIII. CONCLUSION

The details of the assumptions and results have already been given in Sec. II.

The approach used in this paper to prove existence and analyticity arose from the impossibility of using any of the standard Hilbert-space techniques on the usual kinds of BS equations for the interaction of spin-$\frac{1}{2}$ particles.

The majorization technique used here cannot give any formal method for calculating any of the interesting properties of the solution of Eq. (1), such as the actual bound-state energies and vertex functions, the Regge trajectories, and so on. Nevertheless, at the present stage of knowledge about the solutions of equations such as (1) and (2), it is reassuring that solutions so easily can be proven to exist at all. It is also interesting that the general condition $B<2mg/g_0$ should emerge, for this is a limit on the binding energy B of a bound state in terms of general characteristics of the BS equation. The value of g depends upon the coupling constant, and the value of g_0 depends only upon the spectral representation of the propagators.

There is no reason that the same technique cannot also be used on the BS equations for the interaction of scalar particles, since majorizations of the type given in Sec. IV are even easier to carry out on scalar-particle propagators. However, since the properties of scalar equations are well understood anyway, there is not much need for the information which can be obtained by the majorization methods given here.

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[16] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, England, 1962), Sec. 5.3.

[17] Reference 7, Sec. 2.9.

[18] As a matter of fact Willey worked in the Landau gauge $D_{\mu\nu}(k)=(g_{\mu\nu}-k_\mu k_\nu)/k^2$, because of certain gauge requirements in the treatment of QED by Johnson, Baker, and Willey. The conclusion is not changed because g^2 just becomes $5e^2$ instead of $4e^2$.