$W(q',q)$ is obtained from $W(q,q')$ by interchange of the initial vector meson with the final one.

Now define

$$\tau=-\frac{1}{2}\frac{g^2}{4\pi}\frac{q^{1/2}q'^{\,1/2}}{[(E+\omega)(E'+\omega')]^{1/2}}. \tag{B17}$$

The pseudoscalar-meson contribution is

$$(X+Y)_P=-\tfrac{1}{4}\tau I_6, \tag{B18}$$

$$Z_P=[\tau/(2mm'qq')][qq'\omega\omega' I_2 + (q^2\omega' E'+q'^2\omega E)I_3+qq'EE'I_4], \tag{B19}$$

$$W(q,q')_P=(-\tau/4q'm)[q\omega' I_5+q'EI_6]. \tag{B20}$$

All these elements of the potential must be multiplied by the $SU(3)$ crossing coefficients tabulated in the main part of the paper.

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Hadronic Corrections to the Goldberger-Treiman Relation*

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Using unsubtracted dispersion relations in momentum transfer for the matrix element of the divergence of the axial-vector current between nucleon states, we have examined the hadronic continuum corrections to the Goldberger-Treiman relation for π^+ decay. These are observed to be about +10%. From a rigorous unitarity bound and the assumption that the pion propagator $\Delta_\pi(0)$ is dominated by the pion pole we show that the continuum states of energy greater than two nucleon masses contribute less than $\frac{1}{2}$%. The $\pi\rho$ and $\pi\sigma$ states contributed negligibly. Using Weinberg's extrapolation for the $\pi\pi$ scattering amplitude and chiral dynamics, we find that the presumably dominant 3π state contributes with opposite sign and is more than an order of magnitude too small. In the absence of any simple explanation for the 10% correction, we conjecture that what is required is a 3π threshold enhancement or possible resonance, the tripion, with the quantum numbers of the pion and mass near threshold at 4.2 BeV/c^2 or a possible subtraction in the dispersion relation.

I. INTRODUCTION

As a consequence of the precision measurements of the π^+ lifetime, the rate of Gamow-Teller transitions in neutron β decay, and the π^+ nucleon coupling constant, one may establish in both magnitude and sign the correction to the Goldberger-Treiman relation (GTR)[1]

$$\Delta=1-\frac{(m_p+m_n)g_A}{\sqrt{2}gf_\pi}=+0.105\pm0.026.$$

This number represents the small 10% continuum correction to the single-pion-pole term, and it is this number we will endeavor to understand. We will approach this problem in the conventional way by assuming an unsubtracted dispersion relation in the momentum transfer for the matrix elements of the divergence of the axial-vector current taken between nucleon states. Then Δ is simply related to the continuum integral over the timelike region with the threshold at three pion masses.

As we discover by proceeding in this way, the problem is not to understand why the correction Δ is so small but rather why it is so large. Everything one can estimate from the known low-lying meson spectrum gives a value for Δ more than an order of magnitude too small. The 3π state contribution we estimate gives a number with the wrong sign and an order of magnitude too small. This is primarily because of the very small three-body phase space. Electromagnetic corrections (with a uv cutoff at 1–2 BeV) give at most 1%.

The $\pi\rho$ and $\pi\sigma$ continuum states are negligible also. Using a rigorous unitarity bound and the assumption the pion propagator at $q^2=0$ is dominated by the pion pole, we can argue that high-energy contributions from the region of energy greater than two nucleon masses are less than $\frac{1}{2}$%.

Confronted with the absence of any evident explanation for the observed 10% correction in terms of the known meson spectrum, we conjecture the existence of large forces in the three-body pion system giving rise to an enhancement with the quantum numbers of the pion near the 3π threshold. The tripion, if a genuine resonant state near $m_{\pi'}\approx 3m_\pi$, should be seen in $\pi^+\pi^+\pi^-$ invariant mass distributions near threshold at 4.2 BeV/c^2 in 3- and 4-prong π^-p collisions. No peak is in evidence from the available data but the statistics is poor in the threshold region and a small amplitude peak might not have shown up. Should no tripionic

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[1] M. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958).

enhancement be evident in this region, we are left with an uncomfortably inadequate understanding of the 10% correction on the basis of unsubtracted dispersion relations.

II. ESTIMATION OF DISCREPANCY

A. Dispersion Relation

First we will derive the GTR taking into account the precise values of the coupling constants. The matrix elements of the charged axial-vector current $A_\mu^{(+)}(x)$ between neutron and proton states is specified by

$$\langle N(p')|A_\mu^{(+)}(0)|N(p)\rangle=\left(\frac{m^2}{p_0'p_0}\right)^{1/2}\bar{u}(p')\times[\gamma_\mu i\gamma_5 F_1(q^2)+q_\mu i\gamma_5 F_2(q^2)]u(p), \quad (2.1)$$

$$q_\mu=(p'-p)_\mu,$$

where the axial-vector form factors are presumed free of kinematic singularities. The matrix elements of the divergence of the axial-vector current $-i\partial_\mu A_\mu^{(+)}(x)$ (which has the quantum numbers of the π^+, $G=-1$, $I=1$, $J^P=0^-$) is specified by the combination

$$D(q^2)=(m_p+m_n)F_1(q^2)+q^2F_2(q^2). \quad (2.2)$$

The major assumption in the derivation of the GTR is that $D(q^2)$ is an analytic function of q^2 with a pole at $q^2=\mu^2$, $\mu=$mass of the π^+, and a cut beginning at $q^2=(3\mu)^2$,[2] and satisfies $D(q^2)\to 0$, $q^2\to\infty$. Then we can write the unsubtracted dispersion relation in q^2

$$D(q^2)=\frac{C}{\mu^2-q^2}+\frac{1}{\pi}\int_{(3\mu)^2}^{\infty}\frac{\mathrm{Im}D(q'^2)dq'^2}{q'^2-q^2}. \quad (2.3)$$

Only states with the quantum numbers of the π^+ can contribute to the absorptive part, $\mathrm{Im}D(q^2)$.

The residue of the π^+ pole, $C=\sqrt{2}gf_\pi\mu^2$, is specified by the $\pi^+\to l+\nu$ decay amplitude f_π and the π^+pn coupling constant $\sqrt{2}g$. From the precision measurement of the π^+ lifetime one obtains

$$|f_\pi|=(0.9320\pm0.0005)\mu.$$

Electromagnetic corrections are at most 1%. Extrapolation of the π^-p scattering data to the neutron pole yields $f^2=0.0822\pm0.0018$ with $f^2=(g^2/4\pi)(\mu/2m)^2$ or

$$|g|=13.66\pm0.15.$$

This then fixes the π^+ pole residue C to about 1% accuracy.

From Eq. (2.2), one has

$$D(0)=(m_p+m_n)F_1(0)=(m_p+m_n)g_A.$$

From the observed rate of Gamow-Teller transitions in nucleon β decay,

$$g_A=1.198\pm0.022.$$

It is a consequence of *two* precision experimental determinations of weak decay amplitudes and the π^+pn coupling constant that we are given the opportunity of discussing the correction to the GTR.

Evaluating the dispersion relation Eq. (2.3) at $q^2=0$, we have a sum rule,

$$(m_p+m_n)g_A=\sqrt{2}gf_\pi+\frac{1}{\pi}\int_{(3\mu)^2}^{\infty}\frac{\mathrm{Im}D(q^2)dq^2}{q^2}. \quad (2.4)$$

Were we to neglect the continuum integral we would have the GTR

$$(m_p+m_n)g_A\approx\sqrt{2}gf_\pi, \quad (2.5)$$

which, assuming the signs of the couplings are correct, is satisfied to about 10%. Evidently the continuum states with $q^2>(3\mu)^2$ comprise only $\frac{1}{10}$ of the single-pion-pole term. To study the origin of this correction, we introduce the discrepancy Δ defined by

$$\Delta=1-\frac{(m_p+m_n)g_A}{\sqrt{2}gf_\pi}=\frac{-1}{\sqrt{2}gf_\pi\pi}\int_{(3\mu)^2}^{\infty}\frac{\mathrm{Im}D(q^2)dq^2}{q^2}. \quad (2.6)$$

If $\Delta=0$ the GTR is exact. From the measured values of the couplings we obtain

$$\Delta^{\mathrm{expt}}=+0.105\pm0.026, \quad (2.7)$$

which is the number we want to understand.

B. Partially Conserved Axial-Vector Current (PCAC)

It is worthwhile to make contact with another approach in which the charged pion field $\pi^+(x)$ is *defined* through the weak interactions according to PCAC[3,4]

$$\mu^2f_\pi\pi^{(+)}(x)=i\partial_\mu A_\mu^{(+)}(x). \quad (2.8)$$

The proportionality constant μ^2f_π has been chosen so the definition agrees with the π^+ lifetime. The pion-nucleon form factor $K(q^2)$ is then defined from the weak interactions by

$$\langle N(p')|j_\pi^{(+)}(0)|N(p)\rangle=\left(\frac{m^2}{p_0'p_0}\right)^{1/2}\bar{u}(p')\sqrt{2}i\gamma_5K(q^2)u(p), \quad (2.9)$$

where $j_\pi^{(+)}(x)=(\square+\mu^2)\pi^+(x)$ and we have normalized $K(\mu^2)=g$. The connection between $K(q^2)$ and $D(q^2)$ directly follows:

$$D(q^2)=\frac{\sqrt{2}\mu^2f_\pi K(q^2)}{\mu^2-q^2}. \quad (2.10)$$

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[2] There is a cut beginning at $q^2=\mu^2$ corresponding to the $\gamma\pi^+$ threshold. We will not be considering electromagnetic contributions in detail since they can be shown to contribute less than 1%. The $\gamma\pi^+$ state as calculated in cutoff perturbation theory is found to be small, $\sim\alpha/2\pi$ relative to $\Delta\sim+0.1$.

[3] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).

[4] Chou Kuang-Chao, Zh. Eksperim. i Teor. Fiz. **39**, 703 (1960) [English transl.: Soviet Phys.—JETP **12**, 492 (1961)].

From this equation and the assumption that $D(q^2)$ satisfies an unsubtracted dispersion relation, it follows that $K(q^2)$ will satisfy a once subtracted dispersion of the form

$$K(q^2)=g+\frac{q^2-\mu^2}{\pi}\int_{(3\mu)^2}^{\infty}\frac{\mathrm{Im}K(q'^2)dq'^2}{(q'^2-\mu^2)(q'^2-q^2)}, \quad (2.11)$$

where we have performed the subtraction at $q^2=\mu^2$. The correction Δ is then simply related to the continuum contribution to $K(q^2)$. From Eqs. (2.6), (2.10), and (2.11),

$$g\Delta=\frac{\mu^2}{\pi}\int_{(3\mu)^2}^{\infty}\frac{\mathrm{Im}K(q^2)dq^2}{(q^2-\mu^2)q^2}. \quad (2.12)$$

These dispersion relations suggest that the correction Δ be looked upon as a mean square radius $\langle r^2\rangle_{\pi N}$ of the πN interaction defined by the weak interaction

$$g\Delta\approx\mu^2\frac{dK(q^2)}{dq^2}\bigg|_{q^2=0}=\tfrac{1}{6}\mu^2\langle r^2\rangle_{\pi N}. \quad (2.13)$$

From the 3π state we have a long-range force so $\langle r^2\rangle_{\pi N}\sim 1/(3\mu)^2$ and should be the most important. The $\pi\rho$ state is characterized by $\langle r^2\rangle_{\pi\rho}\sim 1/(m_\rho+\mu)^2$. We now turn to making these statements more precise.

C. Bound on High-Energy Contribution

Our program is now to estimate the contribution to Δ from the dispersion integral (2.12). In searching for the major contribution to the integral we will consider separately the contribution from continuum states of invariant mass squared $q^2<(2m)^2$ and $q^2>(2m)^2$, where m is the nucleon mass,

$$g\Delta=g\Delta_L+g\Delta_H$$

with

$$g\Delta_L=\frac{\mu^2}{\pi}\int_{(3\mu)^2}^{(2m)^2}\frac{\mathrm{Im}K(q^2)dq^2}{(q^2-\mu^2)q^2},$$
$$g\Delta_H=\frac{\mu^2}{\pi}\int_{(2m)^2}^{\infty}\frac{\mathrm{Im}K(q^2)dq^2}{(q^2-\mu^2)q^2}. \quad (2.14)$$

The point in doing this is that for the high-energy piece Δ_H we may establish a bound which we now derive.

Using the unitarity condition to calculate $\mathrm{Im}K(q^2)$ and application of the Schwarz inequality to the sum on states implies[5,6]

$$|\mathrm{Im}K(q^2)|^2\leqslant\pi(q^2-\mu^2)^2\left(\frac{q^2-4m^2}{q^2}\right)^{1/2}\sigma_T^0(q)\rho_\pi(q^2). \quad (2.15)$$

In this expression there appears the pion spectral

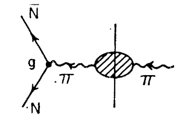

FIG. 1. Propagator corrections.

function $\rho_\pi(q^2)$ defined in the propagator $\Delta_\pi(q^2)$

$$\Delta_\pi(q^2)=\frac{1}{q^2-\mu^2}+\int_{(3\mu)^2}^{\infty}\frac{dq'^2\rho_\pi(q'^2)}{q^2-q'^2}, \quad (2.16)$$

where we assume the integral over $\rho_\pi(q^2)\geqslant 0$ exists.[7]

Here $\sigma_T^0(q)$ for $q=\sqrt{(q^2)}\geqslant 2m$ is the total nucleon-antinucleon annihilation cross section in the $^1S_0$ state of the $N\bar{N}$ system, with barycentric energy q. For $q\geqslant 2m$ it is bounded by unitarity:

$$\sigma_T^0(q)\leqslant 16\pi/q^2-4m^2,\quad q^2\geqslant(2m)^2. \quad (2.17)$$

For $q<2m$ the unitarity condition can be analytically continued to the unphysical region to yield bounds but these restrictions are not too useful.

Combining Eqs. (2.14), (2.15), and (2.17), we have

$$|g\Delta_H|\leqslant 4\mu^2\int_{(2m)^2}^{\infty}\frac{dq^2}{q^2}[\sqrt{\rho_\pi(q^2)}][q^2(q^2-4m^2)]^{-1/4}$$

or by an application of the Schwarz integral inequality

$$(g\Delta_H)^2\leqslant\frac{2\mu^4}{m^2}\int_{(2m)^2}^{\infty}\frac{dq^2}{q^2}\rho_\pi(q^2). \quad (2.18)$$

This rigorous bound on Δ_H is useless unless we can estimate the integral over the spectral function.

To establish an estimate we turn to the connection between the dispersion relation vertex $K(q^2)$ containing all the self-mass insertions and the irreducible vertex $k(q^2)$,

$$K(q^2)=k(q^2)\Delta_\pi(q^2)(q^2-\mu^2)$$

or from Eq. (2.16)

$$K(q^2)=k(q^2)\left[1+(q^2-\mu^2)\int_{(3\mu)^2}^{\infty}\frac{dq'^2\rho_\pi(q'^2)}{q^2-q'^2}\right]. \quad (2.19)$$

There is thus a well-defined set of contributions to $\mathrm{Im}K(q^2)$ arising from modification of the pion propagator through $\rho_\pi(q^2)$. Were we to set $k(q^2)=k(\mu^2)=g$ corresponding to the diagram of Fig. 1 we see from Eq. (2.19) this class of diagrams contributes to $\mathrm{Im}K(q^2)$ a term $-g\pi(q^2-\mu^2)\rho_\pi(q^2)$. Let us then write

$$\mathrm{Im}K(q^2)=-g\pi(q^2-\mu^2)\rho_\pi(q^2)+\mathrm{Im}\tilde{K}(q^2),$$

where $\mathrm{Im}\tilde{K}(q^2)$ represents the contribution of all intermediate states not represented by Fig. 1. From

[5] S. D. Drell and F. Zachariasen, Phys. Rev. **119**, 463 (1960).
[6] S. D. Drell, A. C. Finn, and A. C. Hearn, Phys. Rev. **136**, B1439 (1964).

[7] This is consistent with, but by no means implied by, the assumption that $D(q^2)$ satisfies an unsubtracted dispersion relation.

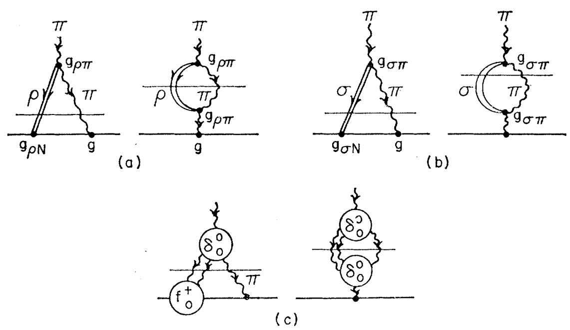

FIG. 2. $\rho\pi$ and $\sigma\pi$ contributions.

the dispersion integral Eq. (2.12), Δ may be expressed as

$$g\Delta=-g\mu^2\int_{(3\mu)^2}^{\infty}\frac{dq^2}{q^2}\rho_\pi(q^2)+\frac{\mu^2}{\pi}\int_{(3\mu)^2}^{\infty}\frac{\mathrm{Im}\tilde{K}(q^2)dq^2}{(q^2-\mu^2)q^2}. \quad (2.20)$$

We note from this expression that [as a consequence of the positivity condition $\rho_\pi(q^2)\geqslant 0$] the first term necessarily contributes a negative amount to Δ and hence in the *opposite* direction to the observed $\Delta\sim+0.1$. We conclude that any modification of the pion propagator, if considered alone, necessarily leads to the wrong sign for Δ.[8]

We separate the propagator term in Eq. (2.20) into a high- and low-energy piece according to

$$g\Delta=-g\mu^2\int_{(3\mu)^2}^{(2m)^2}\frac{dq^2\rho_\pi(q^2)}{q^2}-g\mu^2\int_{(2m)^2}^{\infty}\frac{dq^2\rho_\pi(q^2)}{q^2}+\frac{\mu^2}{\pi}\int_{(3\mu)^2}^{\infty}\frac{\mathrm{Im}\tilde{K}(q^2)dq^2}{(q^2-\mu^2)q^2},$$

each term of which represents a definite set of contributions from continuum states. If we now assume that the numerical success of the GTR based on single-pion-pole dominance is not the result of accidental cancellations between large terms in the continuum integral (each of the magnitude of the pion-pole term), then each of the terms in the above expression should not be much larger than $g\Delta\sim g(0.1)$. In particular this assumption implies for the second term

$$\mu^2\int_{(2m)^2}^{\infty}\frac{dq^2}{q^2}\rho_\pi(q^2)\leqslant 0.1. \quad (2.21)$$

This assumption is equivalent to assuming the pion propagator $\Delta_\pi(0)$, Eq. (2.16), is dominated by the pole, consistent with the PCAC philosophy.

From the very conservative bound represented by Eq. (2.12), Eq. (2.18) implies

$$|\Delta_H|\leqslant 0.004 \quad (2.22)$$

or maximum of 0.4% out of an observed 10%. Even if we assumed the spectral integral Eq. (2.21) were $\leqslant 1.0$ so the GTR would be an accident, then $|\Delta_H|\leqslant 0.012$ or 1.2%. Hence we assert with some confidence that the high-energy contribution to Δ from states with $q^2>(2m)^2$ is completely negligible and we need consider only the contribution from states with $(3\mu)^2\leqslant q^2\leqslant(2m)^2$

$$g\Delta\cong\frac{\mu^2}{\pi}\int_{(3\mu)^2}^{(2m)^2}\frac{\mathrm{Im}K(q^2)dq^2}{(q^2-\mu^2)q^2}. \quad (2.23)$$

This restricts our search for the origin of the correction to the relatively low-energy meson spectrum.

D. ρπ and σπ Contributions

Besides the nonresonant 3π state there are possible states for which a 2π system resonates in the P or S wave as a ρ or σ. If these states are reasonably narrow we may approximate the three-body state as two bodies. This is the basis for the present estimate.

The $\rho\pi$ contribution in the Born approximation indicated in Fig. 2(a) is directly calculated to be

$$\mathrm{Im}K_{\pi\rho}(q^2)=-g\left(\frac{g_{\rho\pi}^2}{4\pi}\right)\frac{(q^2-m_\rho^2)^3}{2q^2m_\rho^2}\left[\frac{1}{q^2}+\frac{q^2}{(4m^2-q^2)(q^2-m_\rho^2)}\ln\left|\frac{\sqrt{(q^2)}-(4m^2-q^2)^{1/2}}{\sqrt{(q^2)}+(4m^2-q^2)^{1/2}}\right|\right]\times\theta(q^2-m_\rho^2). \quad (2.24)$$

We have neglected terms of order μ^2/m_ρ^2, assumed $g_{\rho\pi}=g_{\rho N}$ in accord with universality of ρ couplings, and dropped a possible magnetic coupling of the ρ to the nucleon. As is necessary in all estimates of the present type, we assumed that the coupling of the external pion of mass $>3\mu$ to the $\pi\rho$ system has not changed much from its physical value $g_{\rho\pi}$ at $q^2=\mu^2$. The PCAC assumption is that pion amplitudes vary little in the range $0\leqslant q^2\leqslant\mu^2$, but here we are assuming this variation is small out to $q^2=9\mu^2$. Later we will critize this assumption but for our present purposes even a change of the amplitude by a factor of 5 from $q^2=\mu^2$ to $q^2=9\mu^2$ does not influence our conclusion. With $g_{\rho\pi}^2/4\pi\approx 1.8$ one obtains from the dispersion integral Eq. (2.23)

$$\Delta_{\pi\rho}\approx 0.004,$$

which is completely negligible. This smallness is easily understood from Eq. (2.23) since $\Delta_{\pi\rho}$ is proportional to $\mu^2/m_\rho^2\approx 0.04$, and also because the P-wave coupling of the ρ to 2π introduces an additional factor of the momentum beyond the two-body phase space which is small in the region of integration.

The σ contribution is also estimated from the Born approximation and the unitarity condition [Fig. 2(b)].

[8] We expect such propagator contributions to be very small.

$$\mathrm{Im}K_{\pi\sigma}(q^2)=\frac{2g(q^2-m_\sigma^2)}{q^2}\left[\frac{g_{\sigma N}g_{\sigma\pi}}{4\pi m}-\frac{g_{\sigma\pi}^2}{4\pi}\frac{1}{q^2}\right]\times\theta(q^2-m_\sigma^2) \quad (2.25)$$

again neglecting terms of $O(\mu^2/m_\sigma^2)$. The combination of coupling constants $g_{\sigma N}g_{\sigma\pi}$ that appears in this expression can be bounded by the observed threshold behavior of πN scattering. Assuming that all the $I=0$ t-channel $NN\to 2\pi$ transition comes from σ exchange, one finds for the scattering lengths

$$\tfrac{1}{3}(a_1+2a_2)\left(1+\frac{\mu}{m}\right)=-\frac{g_{\sigma N}g_{\sigma\pi}}{4\pi m_\sigma^2}.$$

This is just the combination the Adler consistency condition,[9] which assumes the absence of σ terms in the current commutators, implies should vanish. From the observed scattering lengths we have

$$g_{\sigma N}g_{\sigma\pi}/4\pi m_\sigma^2=0.014/\mu.$$

Assuming $m_\sigma\approx 5\mu$ and $\Gamma_\sigma\approx 2\mu$, one has $g_{\sigma\pi}^2/4\pi\approx 7\mu^2$ and from the dispersion integral one finds

$$\Delta_{\pi\sigma}\approx -0.025$$

which is too small and has the wrong sign because it comes mostly from the propagator term.

One may question the validity of the narrow-resonance approximation for the σ which, if it exists, probably a broad resonance or S-wave enhancement. To answer this question we have retained the full $NN\to 2\pi$, $I=0$, $J=0$ amplitude $f_0^+(E)$ analyzed by Hamilton *et al.*[10] and parametrized the $\pi\pi\to\pi\pi$ $I=0$, $J=0$ amplitude in terms of the phase shift $\delta_0^0(E)$, where E is barycentric energy of the 2π system. This is indicated in Fig. 2(c). For a range of S-wave enhancements corresponding to $3\mu\leqslant m_\sigma\leqslant m$, $2\mu\leqslant\Gamma_\sigma\leqslant 8\mu$, we find $|\Delta_{\pi\sigma}|\leqslant 0.01$, completely negligible. We conclude that the $\pi\rho$ and $\pi\sigma$ states cannot account for the observed discrepancy.

E. Nonresonant 3π Continuum

Next we consider the 3π contribution as represented in Fig. 3(a). What is required to establish an estimate is the $\pi\to 3\pi$ transition amplitude for a virtual pion with $q^2\geqslant 9\mu^2$ in conjunction with the $N\bar{N}\to 3\pi$ annihilation amplitude continued deep into the unphysical region $9\mu^2\leqslant q^2\leqslant 4m^2$. Besides information on these unphysical processes, a further complication is introduced by the necessity of considering full three-body kinematics in the angular integration of the three-pion system. These features inhibit the possibility of a highly reliable quantitative calculation; however, they should not dissuade us from examining the qualitative aspects of the 3π contribution.

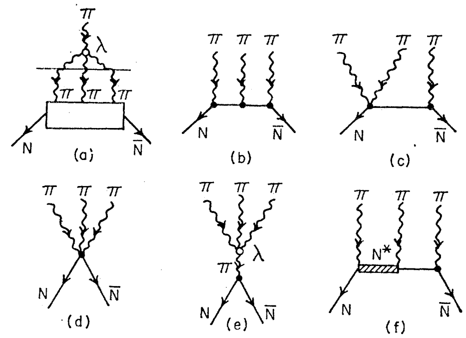

FIG. 3. 3π contribution.

If we evaluate the transition matrix elements at the threshold $q^2=9\mu^2$ at which the three pions have zero momentum and an isotropic distribution in the barycentric system and assume they remain approximately constant away from threshold (as would be expected if there are no resonances near threshold), then the troublesome angular integrals are trivial and reduce to three-body phase space. Hence in this threshold approximation $\mathrm{Im}K_{3\pi}(q^2)$ will be proportional to three-body phase space. The proportionality constant is to be abstracted from the dynamics. In this way we incorporate the correct threshold behavior into the calculation and this should be the most important region for the calculation of the πN interaction radius.

The $\pi\to 3\pi$ transition amplitude is specified by

$$\langle\pi_{\alpha_1}(q_1),\pi_{\alpha_2}(q_2),\pi_{\alpha_3}(q_3)|j_\pi{}^a(0)|0\rangle = -i\left(\frac{1}{8q_1{}^0q_2{}^0q_3{}^0}\right)^{1/2}\phi_b{}^{*\alpha_1}\phi_c{}^{*\alpha_2}\phi_d{}^{*\alpha_3}M_{bc;da},$$

where $q_{1,2,3}$ are the momenta of the three π's of isospin $\alpha_{1,2,3}$. The matrix element has the general form

$$M_{bc;da}=\delta_{bc}\delta_{da}A(s,t,u,q_1^2,q_2^2,q_3^2,q^2)+\delta_{bd}\delta_{ca}B(s,t,u,q_1^2,q_2^2,q_3^2,q^2)+\delta_{ba}\delta_{cd}C(s,t,u,q_1^2,q_2^2,q_3^2,q^2),$$

with $s=(q_1+q_2)^2$, $t=(q_1+q_3)^2$, $u=(q_1-q)^2$, $q_1^2=q_2^2=q_3^2=\mu^2$ and $s+t+u=3\mu^2+q^2$. At the threshold for $\pi\to 3\pi$, $q^2=9\mu^2$, $s=t=u=4\mu^2$, and crossing symmetry and Bose statistics informs us that here

$$\lambda=A(4\mu^2,4\mu^2,4\mu^2,\mu^2,\mu^2,\mu^2,9\mu^2)=B(4\mu^2,4\mu^2,4\mu^2,\mu^2,\mu^2,\mu^2,9\mu^2)=C(4\mu^2,4\mu^2,4\mu^2,\mu^2,\mu^2,\mu^2,9\mu^2),$$

so

$$M_{bc;da}=\lambda(\delta_{bc}\delta_{da}+\delta_{bd}\delta_{ca}+\delta_{ba}\delta_{cd}). \quad (2.26)$$

The number λ which measures the strength of the $\pi\to 3\pi$ transition at threshold we will leave unspecified

[9] S. L. Adler, Phys. Rev. **137**, B1022 (1965).
[10] J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. **128**, 1881 (1962).

for the moment; later we appeal to Weinberg's[11] and Khuri's[12] analysis of $\pi\pi$ scattering and smoothness assumptions to establish an estimate for λ.

Similarly we may treat the $N\bar{N}\to 3\pi$ amplitude at the unphysical threshold. Crossing symmetry informs us that at this threshold

$$\langle\bar{N}(p_+)N(p_-)|T|\pi^{\alpha_1}(q_1),\pi^{\alpha_2}(q^2),\pi^{\alpha_3}(q_3)\rangle = (m^2/8p_+{}^0p_-{}^0q_1{}^0q_2{}^0q_3{}^0)^{1/2}i\bar{u}(p_-) \times[Mi\gamma_5(\not{q}/2m)(\delta_{bc}\tau_d+\delta_{bd}\tau_c+\delta_{dc}\tau_b)] \times v(p_+)\phi_b{}^{\alpha_1}\phi_c{}^{\alpha_2}\phi_d{}^{\alpha_3}. \quad (2.27)$$

There are also possible additional terms antisymmetric in the isospin indices b, c, and d; however in our calculation we will not include them since they vanish when contracted with the purely symmetric $\pi\to 3\pi$ amplitude Eq. (2.26). The constant amplitude M appearing in Eq. (2.27) must be computed from dynamics. In particular we will consider the contribution from Figs. 3(b)–3(f).

Since the three external pions appearing in these diagrams are all soft in the threshold limit, we can appeal to the technology of low-energy theorems for pion processes either as implemented by current algebra[13] or phenomenological Lagrangians exhibiting chiral symmetry.[14] We will use pseudovector coupling of single pions to the nucleons as required by PCAC and the chiral symmetric Lagrangian of Ref. (14) to compute the tree graphs 3(c) and 3(d).

The threshold amplitude $M_{(b)}$ corresponding to Fig. 3(b) is calculated as

$$M_{(b)}=g^3/6\mu^2. \quad (2.28)$$

The all important factor of the pion mass appearing in the denominator of this amplitude emerges as a consequence of the virtual nucleons in Fig. 2(b) approaching their mass shell at the threshold developing singularities like $1/\mu$. Two such nucleons contribute the factor $1/\mu^2$ which is just what is required to eliminate the multiplicative factor μ^2 in the dispersion relation for the discrepancy Δ, Eq. (2.23).

The amplitude for Fig. 2(c) with the pion-pair contact interaction is essentially the ρ and σ contributions considered before. Here we see the pion pair with $I=1$ will not contribute at all since the amplitude is antisymmetric in the isospin of the pair and vanishes when contracted into the $\pi\to 3\pi$ transition amplitude, and if $I=0$ the σ term is very small relative to $M_{(b)}$. The amplitude for Fig. 3(d) occurs with the ratio

$$\frac{M_{(d)}}{M_{(b)}}=\frac{3g_{\rho\pi}{}^2}{g^2}\left(\frac{\mu}{m_\rho}\right)^2\approx 0.018. \quad (2.29)$$

Quite independent of what we chose for the $\pi\to 3\pi$ interaction strength λ, the relative contribution of transitions 3(d) to 3(b) to the discrepancy is very small, $\sim 1/50$. We ignore them.

We also consider the contribution of a parity doublet of the nucleon with $J^P=\frac{1}{2}^-$ as illustrated in Fig. 3(f). Such contributions might be important for, as emphasized by Fubini and Furlan,[15] they become singular if the masses are degenerate, $m\to m^*$. We consider the observed state with $J^P=\frac{1}{2}^-$, $m^*=1550$ MeV as a candidate. Using vector coupling of the pion for the $N^*N\pi$ vertex characterized by the constant g^* and letting $\mu^2\ll m^2$, m^{*2}, we have

$$\frac{M_{(f)}}{M_{(b)}}=\frac{8g^{*2}}{g^2}\left(\frac{m}{m^*+m}\right)\left(\frac{\mu}{m^*-m}\right)^2, \quad (2.30)$$

which indeed exhibits the singularity as $m^*\to m$. The coupling g^* is related to the partial width of the N^*:

$$\Gamma_{N^*\to N\pi}=\left(\frac{3}{4}\right)\left(\frac{g^{*2}}{4\pi}\right)\frac{(m^*+m)^2(m^{*2}-m^2)}{m^{*3}},$$

and from the observed full width $\Gamma=130$ MeV and 30% branching ratio into $N\pi$ we calculate $g^{*2}/4\pi\approx 0.021$, a very small number. We obtain from Eq. (2.30)

$$M_{(f)}/M_{(b)}\approx 0.0014,$$

a negligible relative contribution.

Finally, we examine the π propagator term indicated in Fig. 3(e), which is of the class shown in Fig. 1. On general grounds this must contribute with the wrong sign to Δ. Using Eq. (2.26) for the $3\pi\to\pi$ amplitude, we have

$$M_{(e)}=-g\lambda/8\mu^2. \quad (2.31)$$

Here the factor $1/8\mu^2$ arises from the pion propagator $(q^2-\mu^2)^{-1}$ as $q^2\to 9\mu^2$. In summary, only the graphs 3(b) and 3(e) can be expected to contribute significantly.

For the absorptive part in the threshold approximation we have, using Eqs. (2.28), (2.31), and (2.26),

$$\mathrm{Im}K_{3\pi}(q^2)=g\left(\lambda-\frac{3\lambda^2}{4g^2}\right)\left(\frac{g^2}{4\pi}\right)\left(\frac{5}{12}\right)\frac{\Phi_3(q^2)}{\mu^2(2\pi)^4}\times\theta(q^2-9\mu^2). \quad (2.32)$$

Here the three-body phase space is

$$\Phi_3(q^2)=\int\frac{d^3q_1}{2q_1{}^0}\frac{d^3q_2}{2q_2{}^0}\frac{d^3q_3}{2q_3{}^0}\delta^4(q_1+q_2+q_3-q) \cong\frac{\sqrt{3}\pi^3}{32}\frac{[(q-\mu)^2-4\mu^2]^2}{q^2}, \quad (2.33)$$

where the approximation introduces only small errors

[11] S. Weinberg, Phys. Rev. Letters **17**, 616 (1966).
[12] N. N. Khuri, Phys. Rev. **153**, 1477 (1967).
[13] Lay-Nam Chang, Phys. Rev. **162**, 1497 (1967).
[14] J. Wess and Bruno Zumino, Phys. Rev. **163**, 1727 (1967).
[15] S. Fubini and G. Furlan, Ann. Phys. (N. Y.) **48**, 322 (1968).

for all q^2. The two terms in the factor $(\lambda-3\lambda^2/4g^2)$ appearing in Eq. (2.32) come from the Figs. 3(b) and (e), respectively. We expect that the dominant contribution to $\mathrm{Im}K(q^2)$ in the relatively low-energy region $3\mu\leqslant q\leqslant 7\mu$ is approximately specified by Eq. (2.32). For large q^2 the approximation represented by Eq. (2.32) must fail since the three-body phase space $\Phi_3(q^2)$ grows linearly with q^2 as $q^2\to\infty$ and $\mathrm{Im}K(q^2)\to$ (constant)$\times q^2$, violating the behavior required for a once-subtracted dispersion relation for $K(q^2)$. Evidently the approximation of the matrix elements as constants fails for large q^2 and the required convergence factor must be supplied by a more precise treatment of the dynamics. However, only the low-q^2 region can be important as we have argued, so we may introduce a low cutoff.

To consider the contribution to Δ from the 3π states of energy q with $3\mu\leqslant q\leqslant\Lambda\mu$, we have the dispersion integral

$$g\Delta_{3\pi}=\frac{\mu^2}{\pi}\int_{(3\mu)^2}^{(\Lambda\mu)^2}\frac{\mathrm{Im}K_{(3\pi)}(q^2)dq^2}{(q^2-\mu^2)q^2}$$

or, using Eq. (2.32),

$$\begin{aligned}\Delta_{3\pi}=&+0.004[\lambda-(3\lambda^2/4g^2)][12((\Lambda)^{-1}-\tfrac{1}{3})\\&+\tfrac{9}{2}((\Lambda^2)^{-1}-\tfrac{1}{9})-7\ln\tfrac{1}{3}\Lambda+8\ln(\tfrac{1}{2}(\Lambda-1))]\\\approx&+0.001[(\lambda-(3\lambda^2/4g^2))]\quad\Lambda\approx 7.\end{aligned}\tag{2.34}$$

The primary reason for the smallness of the integral is the fact that the three-body phase space acts as a very strong suppression. Variations of the cutoff $\Lambda\leqslant 14$ will not alter the conclusions we will make regarding the 3π state.

To estimate λ we examine the analysis of Weinberg,[11] who has pointed out that if the $\pi\pi$ amplitude does not vary much from the region $0\leqslant s,t,u,\ \leqslant\mu^2, 0\leqslant q_1^2, q_2^2, q_3^2, q^2\leqslant\mu^2$ where it can be calculated to the physical threshold $s=4\mu^2$, $u=t=0$, $q_1^2=q_2^2=q_3^2=q^2=\mu^2$, then we can calculate the scattering lengths. The primary assumption is the smoothness of the amplitude in the variables. Assuming that Weinberg's crossing-symmetric expansion of the amplitude retaining just linear terms in s, t, and u (there is no dependence on the external masses to this order) is approximately valid out to the unphysical threshold for $\pi\to 3\pi$ at $s=t=u=4\mu^2$, then we can determine λ by extrapolation with the result

$$\lambda=-6\mu^2(g_V/f_\pi)^2=-7.0.\tag{2.35}$$

This is, of course, a large extrapolation to an unphysical point. If we examine the analysis of Khuri[12] who also assumes that a self-consistent perturbative approach to $\pi\pi$ scattering is valid and retains quadratic terms in the expansion of the amplitude so the mass variables q_1^2, q_2^2, q_3^2, q^2 explicitly enter, we find the extrapolation to the point $s=t=u=4\mu^2$, $q_1^2=q_2^2=q_3^2=\frac{1}{9}q^2=\mu^2$ yields $\lambda=-6\mu^2(g_V/f_\pi)^2-18h\mu^4$. The correction factor $-18h\mu^4$ is estimated to be small, $-18h\mu^4\approx 6a_0^2\mu^2\approx 0.24$, and does not significantly change the estimate Eq. (2.35). If the extrapolation required to calculate the scattering lengths is all right and that required to estimate λ is wrong (by an order of magnitude), then the dependence of the amplitude on cubic and higher-order terms in the expansion would have to be very strong in just such a way to influence one extrapolation and not the other. This seems to us unlikely although probably not impossible.

Using the value of λ, Eq. (2.35), obtained in this way, we find from Eq. (2.34) for the 3π contribution to the discrepancy

$$\Delta_{3\pi}\approx -0.007\,,$$

which is of the wrong sign and more than one order of magnitude smaller than the observed $\Delta\approx +0.1$. Now we discuss the consequences of the complete failure of our assumptions to provide the observed value.

III. 3π THRESHOLD ENHANCEMENT OR TRIPION

We have argued that the high-energy contributions for $q^2>(2m)^2$ and the $\pi\rho$ and $\pi\sigma$ states contribute negligibly to the discrepancy Δ. If the 10% correction to the GTR is to be understood it must be in terms of the dynamics of the 3π $J^P=0^-$ state. Assuming constant matrix elements (or at least that they do not increase by an order of magnitude) for the $\pi\to 3\pi$ and $N\bar{N}\to 3\pi$ transitions, we estimated

$$\Delta_{3\pi}\simeq 0.001(\lambda-3\lambda^2/4g^2)\,.\tag{3.1}$$

Smoothness assumptions on the $\pi\pi$ amplitude implied a value $\lambda\simeq -7.0$ which from Eq. (3.1) yields a correction of the wrong sign and is an order of magnitude too small.

We now remark that independent of this estimate of λ from the $\pi\pi$ amplitude, Eq. (3.1) possesses a unique positive maximum for $\Delta_{3\pi}(\lambda)$ at $\lambda_m=2g^2/3=117$,

$$\Delta_{3\pi}{}^{\max}\simeq 0.06$$

or 6%. This is the largest positive value for Δ we can obtain in the framework of threshold dominance. But the huge value for λ this requires, suggests that there are strong forces in the 3π 0^- state. This leads us to question not only the assumption of constant behavior of the matrix elements near the 3π threshold but also any smooth extrapolation procedure in the $\pi\pi$ amplitude. We will not comment on the latter question but will examine the consequences of 3π 0^- attractive forces which we urge are required to understand the correction Δ on the basis of unsubtracted dispersion relations.

To do so we consider a simple model of the $\pi\to 3\pi$ amplitude revelant for our calculation, in which we ignore momentum transfers between the final three

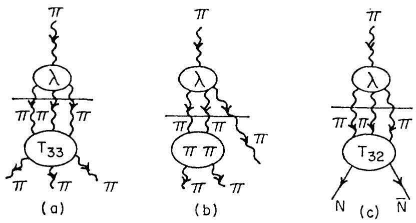

FIG. 4. Unitarity condition.

pions. Then the amplitude depends only on the virtual pion mass $q^2=3(s-\mu^2)$, $s=t=u$ so $\lambda=\lambda(q^2)$. Here $\lambda(\mu^2)$ is the value of the $\pi\pi$ scattering amplitude at the symmetry point $s=t=u=4\mu^2/3$ which we assume given. The dispersion relation for $\lambda(q^2)$ then reads

$$\lambda(q^2)=\lambda(\mu^2)+\frac{q^2-\mu^2}{\pi}\int_{(3\mu)^2}^{\infty}\frac{\mathrm{Im}\lambda(q'^2)dq'^2}{(q'^2-\mu^2)(q'^2-q^2)}, \quad (3.2)$$

where we calculate the absorptive part from unitarity in the 3π approximation as illustrated in Fig. 4(a).

$$\mathrm{Im}\lambda(q^2)=\Phi_3(q^2)\lambda^*(q^2)T_{33}(q^2)\theta(q^2-9\mu^2). \quad (3.3)$$

Here $T_{33}(q^2)$ is the $3\pi\rightarrow 3\pi$ $J^P=0^-$ amplitude. We ignore fly-by diagrams of the type in Fig. 4(b) which also contribute but are small below $q^2=(m_\rho+\mu)^2\simeq 36\mu^2$. The unitarity condition on $K(q^2)$ in the 3π approximation is

$$\mathrm{Im}K(q^2)=\Phi_3(q^2)\lambda^*(q^2)T_{32}(q^2)\theta(q^2-9\mu^2), \quad (3.4)$$

where $T_{32}(q^2)$ is the $\bar{N}N\rightarrow 3\pi$ $^1S_0$ amplitude continued to the unphysical region. It satisfies a unitarity condition

$$\mathrm{Im}T_{32}(q^2)=\Phi_3(q^2)T_{33}{}^*(q^2)T_{32}(q^2)\theta(q^2-9\mu^2). \quad (3.5)$$

Now if we assume there is a strong attractive enhancement in the $3\pi\rightarrow 3\pi$ amplitude $T_{33}(q^2)$, we may simply solve the integral equations with the absorptive parts specified by Eqs. (3.3)–(3.5). The solution is known to imply this enhancement will be manifest in $\lambda(q^2)$ and $K(q^2)$. Very crudely we characterize the enhancement by a pole so

$$\lambda(q^2)=\lambda(\mu^2)\left(\frac{\mu^2-\pi'^2}{q^2-\pi'^2}\right), \quad K(q^2)=K(\mu^2)\left(\frac{\mu^2-\pi'^2}{q^2-\pi'^2}\right), \quad (3.6)$$

where π' is the mass of the enhancement in the 3π 0^- system. We see that $\lambda(q^2)$ cannot be smoothly extrapolated from $q^2=\mu^2$ to q^2 near π'^2; in fact it becomes quite large there. From the very approximate expression for $K(q^2)$, we have for the discrepancy $\Delta\approx\mu^2/\pi'^2$. From $\Delta\approx+0.1$ we obtain $\pi'\approx 3.1\mu$, so the enhancement must be near the 3π threshold.[16]

The purpose of this illustration is only to indicate that attractive 3π forces near threshold provide an alternative to the negligible result for Δ obtained without enhancement. If such a 3π state existed as a resonance, the tripion, it should show up in the invariant mass plots of $\pi^+\pi^+\pi^-$ in 3- and 4-prong production experiments on hydrogen, like $\pi^+p\rightarrow p\pi^+\pi^+\pi^-$. No peak is evident from the data[17] between threshold at 4.2 and 5.5 BeV/c^2 where we would expect it; on the other hand, statistics is poor in this threshold region and a small enhancement might be missed. Should evidence accumulate against the tripion either as a resonance or threshold enhancement, we are left without an adequate picture of the origin of the correction to the GTR on the basis of unsubtracted dispersion relations. Only as a final resort would one appeal to the possibility of a subtraction in the dispersion relation for $D(q^2)$. If this is indeed the case then one loses the GTR unless additional, *ad hoc*, assumptions are made.

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[16] As has also been noted by C. Michael, Phys. Rev. **166**, 1826 (1967).

[17] C. Alff-Steinberger *et al.*, Phys. Rev. **145**, 1072 (1966); P. R. Klein *et al.*, *ibid.* **150**, 1132 (1966); A. W. Key *et al.*, *ibid.* **166**, 1430 (1968).