Dynamical Calculations of Pseudoscalar and Axial-Vector Meson Parameters Using the Relativistic Lippmann-Schwinger Equation*

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A multichannel dynamical calculation of pseudoscalar and axial-vector meson parameters is performed using the relativistic Lippmann-Schwinger equation. The channel considered is $PV \to PV$. The potential is computed using the usual Feynman rules and the resulting equations are solved on a computer. The cutoff is fixed by adjusting the output mass of one resonance or bound state or the output width to the experimental value. For $J^P=0^-$ we fix $m_K=496$ MeV and obtain as output $m_\pi=135$ MeV, $m_\eta=587$ MeV. For $J^P=1^+$, we fix $\Gamma_B=114$ MeV and obtain as output $m_B=1060$ MeV, $m_{K_c^*}=1310$ MeV, $\Gamma_{K_c^*}=87$ MeV, $m_H=975$ MeV, $\Gamma_H=50$ MeV, $m_{H'}=1400$ MeV, $\Gamma_{H'}=43$ MeV. Furthermore, if we fix $m_{A_1}=1080$ MeV, we obtain $\Gamma_{A_1}=110$ MeV, $m_D=1370$ MeV as a bound state.

I. INTRODUCTION

PREVIOUSLY[1] we have reported a calculation for the parameters of the vector mesons using the relativistic Lippmann-Schwinger equation. The most satisfying result of that calculation was that it was possible to produce whole octets with one adjustable parameter (the cutoff for the integral equations) and an assumption about the behavior of the off-shell potential. With this and with input broken $SU(3)$ symmetry the octet emerges as a necessary consequence of the equations with reasonable values for the masses and widths.

The problem involving the pseudoscalar-vector meson channels has received some attention in the past.[2] We have looked at this problem again from the point of view of the multichannel Lippmann-Schwinger equation. The potential is computed using the usual Feynman rules and the resulting coupled integral equations are solved numerically on a computer. In cases where S and D waves contribute to the scattering, we keep both parts and thus double the dimensionality of the channel space.

We have sought to answer two questions which appear to be relevant to a practical realization of the bootstrap philosophy. First, for a given spin and parity state, is it possible to produce whole octets with at most one adjustable parameter? We have previously found that this is indeed the case for vector mesons (1^- state).[1] We find that this is also the case for the pseudoscalars (0^- state) and axial vectors (1^+ state). This would imply that within the context of the present model, no member of an octet is more fundamental than another. It also indicates that broken $SU(3)$ reproduces itself. The other question is whether it is possible to climb up the bootstrap ladder gradually. There exists a large number of known mesons and a true bootstrapper should write down a horrendous number of integral equations of still unknown properties and upon solving them would explain all mesons in one grand fashion. But for those interested in a more modest proposal, it is interesting to see whether within the context of a simple and manageable model one can ascend in successive steps. For example, it is reasonable to start with $PP \to PP$ and try to produce the vector mesons as resonances or bound states. Now assuming some success here, the next step would be to put these vector mesons back as input and consider $PV \to PV$ and see whether it is possible to produce the pseudoscalars, the axial vectors, and others, if possible. This is the program that we follow here.

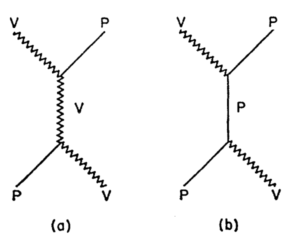

FIG. 1. The Feynman diagrams giving rise to the potentials.

II. CALCULATIONS

The potential is computed from the graphs in Fig. 1. At the PPV vertex the $SU(3)$ invariant interaction is

$$\frac{i}{\sqrt{2}} g_{\rho^0\pi^+\pi^-} V_{\mu j}{}^{i} \left[P_k{}^j \frac{\partial P_i{}^k}{\partial x_\mu} - P_i{}^k \frac{\partial P_k{}^j}{\partial x_\mu} \right], \qquad (1)$$

and at the VVP vertex the interaction is

$$\frac{if}{2\sqrt{2}} \epsilon_{\alpha\beta\mu\nu} \left[\frac{\partial V_\mu{}^{ki}}{\partial x_\alpha} \frac{\partial V_\nu{}^{jk}}{\partial x_\beta} + \frac{\partial V_\mu{}^{jk}}{\partial x_\alpha} \frac{\partial V_\nu{}^{ki}}{\partial x_\beta} \right] P_i{}^j. \qquad (2)$$

From the width of the ρ meson one obtains

$$g^2/4\pi = 2.5\,, \qquad (3)$$

and from the width of ω using the Gell-Mann, Sharp, and Wagner model one obtains[3]

$$m_\rho{}^2 f^2/4\pi = 13.0\,. \qquad (4)$$

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[1] J. Boguta and H. W. Wyld, Jr., Phys. Rev. **164**, 1996 (1967).
[2] R. F. Peierls, Phys. Rev. Letters **12**, 50 (1964); **12**, 119(E) (1964); E. Abers, *ibid.* **12**, 55 (1964); T. K. Kuo, *ibid.* **12**, 465 (1964).
[3] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962).

These values depend on the choice of masses and widths for the appropriate mesons. We take the most recent data, though there still exists some ambiguity.[4] The $SU(3)$ symmetry is broken in the sense that we use the physical masses of the external particles. The masses of the exchanged particles will be taken to be equal, since symmetry breaking for the exchanged particles will produce only very small effects in the final answer.

The potential is a product of two factors, the $SU(3)$ crossing coefficients which are matrices and the momentum dependence. The latter will also assume a matrix form for the $J^P=1^+$ states since we have two possible orbital angular momentum states. Thus, each member of the momentum matrix must be multiplied by the corresponding matrix for the $SU(3)$ crossing coefficients. We use the helicity representation of Jacob and Wick[5] and the partial-wave projection defined by

$$V_{\mu\lambda}{}^{(J)}=\frac{1}{2}\int d\theta\,\sin\theta\, V_{\mu\lambda}d_{-\lambda-\mu}{}^{J}(\theta)\,, \tag{5}$$

where $d_{\mu\lambda}{}^{J}(\theta)$ is the usual d function. For $J=1$ states the potential matrix has the form

$$V_{\mu\lambda}{}^{(1)}=\begin{bmatrix} X & W(q,q') & Y \\ W(q',q) & Z & W(q',q) \\ Y & W(q,q') & X \end{bmatrix}. \tag{6}$$

We carry out a unitary transformation to parity eigenstates with the aid of the matrix

$$\begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \end{bmatrix} \tag{7}$$

to obtain

$$\begin{bmatrix} X+Y & \sqrt{2}W(q,q') & 0 \\ \sqrt{2}W(q',q) & Z & 0 \\ 0 & 0 & X-Y \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} 1^+ \text{ state} \\ \\ 1^- \text{ state} \end{array}. \tag{8}$$

Thus, we use

$$\begin{pmatrix} X+Y & \sqrt{2}W(q,q') \\ \sqrt{2}W(q',q) & Z \end{pmatrix} \tag{9}$$

and $X-Y$ as the 1^+ and 1^- potentials, respectively. The quantities X, Y, W, and Z will receive contributions from both pseudoscalar and vector exchanges. These and other relevant calculations are given in the appendices.

[4] A. H. Rosenfeld, N. Barash-Schmidt, A. Barbaro-Galtieri, M. Ross, W. J. Willis, L. R. Price, P. Söding, and C. G. Wohl, Rev. Mod. Phys. **40**, 77 (1968).

[5] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

The $SU(3)$ crossing coefficients are as follows:

For V exchange,

$I=1$ state:

$$\begin{bmatrix} 2 & 0 & \frac{2}{3}\sqrt{3} & -1 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2} & 0 & 0 \\ \frac{2}{3}\sqrt{3} & 0 & \frac{2}{3} & \frac{1}{3}\sqrt{3} & 0 & 0 \\ -1 & -\sqrt{2} & \frac{1}{3}\sqrt{3} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & \sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & -1 \end{bmatrix} \begin{array}{l} \pi\omega \\ \pi\varphi \\ \eta\rho \\ (\bar{K}K^*+K\bar{K}^*)/\sqrt{2} \\ \pi\rho \\ (\bar{K}K^*-K\bar{K}^*)/\sqrt{2}. \end{array} \tag{10}$$

The ω and φ are physical particles determined in terms of singlet and octet states by the usual simplified mixing formulas

$$\omega=(\sqrt{\tfrac{2}{3}})\varphi_1+(\sqrt{\tfrac{1}{3}})\varphi_8\,,$$
$$\varphi=(\sqrt{\tfrac{1}{3}})\varphi_1-(\sqrt{\tfrac{2}{3}})\varphi_8\,.$$

$I=0$ state:

$$\begin{bmatrix} 2 & -2 & -\sqrt{3} & 0 & 0 \\ -2 & \frac{2}{3} & -\frac{1}{3}\sqrt{3} & 0 & 0 \\ -\sqrt{3} & -\frac{1}{3}\sqrt{3} & 3 & -\frac{1}{3}\sqrt{6} & 0 \\ 0 & 0 & -\frac{1}{3}\sqrt{6} & 8/3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \begin{array}{l} \pi\rho \\ \eta\omega \\ (\bar{K}K^*-K\bar{K}^*)/\sqrt{2} \\ \eta\varphi \\ (\bar{K}K^*+K\bar{K}^*)/\sqrt{2} \end{array} \tag{11}$$

$I=\frac{1}{2}$ state:

$$\begin{bmatrix} -\frac{1}{2} & -1 & -\sqrt{3} & \frac{1}{2} & 0 \\ -1 & -\frac{1}{2} & \frac{1}{2}\sqrt{3} & 1 & \frac{1}{2}\sqrt{6} \\ -\sqrt{3} & \frac{1}{2}\sqrt{3} & \frac{1}{2} & \frac{1}{3}\sqrt{3} & \frac{1}{2}\sqrt{2} \\ \frac{1}{2} & 1 & \frac{1}{3}\sqrt{3} & \frac{1}{6} & -\frac{2}{3}\sqrt{6} \\ 0 & \frac{1}{2}\sqrt{6} & \frac{1}{2}\sqrt{2} & -\frac{2}{3}\sqrt{6} & 1 \end{bmatrix} \begin{array}{l} \pi K^* \\ K\rho \\ K\omega \\ \eta K^* \\ K\varphi. \end{array} \tag{12}$$

For P exchange,

$I=1$ state:

$$\begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \\ -1 & \sqrt{2} & -\sqrt{3} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & \sqrt{2} \\ 0 & 0 & 0 & 0 & \sqrt{2} & -1 \end{bmatrix} \begin{array}{l} \pi\omega \\ \pi\varphi \\ \eta\rho \\ (\bar{K}K^*+K\bar{K}^*)/\sqrt{2} \\ \pi\rho \\ (\bar{K}K^*-K\bar{K}^*)/\sqrt{2} \end{array} \tag{13}$$

$I=0$ state:

$$\begin{bmatrix} 4 & 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & \sqrt{3} & 3 & -\sqrt{6} & 0 \\ 0 & 0 & -\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix} \begin{array}{l} \pi\rho \\ \eta\omega \\ (\bar{K}K^*-K\bar{K}^*)/\sqrt{2} \\ \eta\varphi \\ (\bar{K}K^*+K\bar{K}^*)/\sqrt{2} \end{array} \tag{14}$$

$I=\frac{1}{2}$ state:

$$\begin{bmatrix} \frac{1}{2} & -2 & 0 & \frac{3}{2} & 0 \\ -2 & \frac{1}{2} & -\frac{1}{2}\sqrt{3} & 0 & \frac{1}{2}\sqrt{6} \\ 0 & -\frac{1}{2}\sqrt{3} & -\frac{1}{2} & 0 & \frac{1}{2}\sqrt{2} \\ \frac{3}{2} & 0 & 0 & -\frac{3}{2} & 0 \\ 0 & \frac{1}{2}\sqrt{6} & \frac{1}{2}\sqrt{2} & 0 & -1 \end{bmatrix} \begin{array}{l} \pi K^* \\ K\rho \\ K\omega \\ \eta K^* \\ K\varphi. \end{array} \tag{15}$$

There is one more point to be considered. The ρ-meson decay into two pions reflects itself in the fact that the propagator for the diagram in Fig. 1(b) can have a pole for physical values of the external momentum. This in

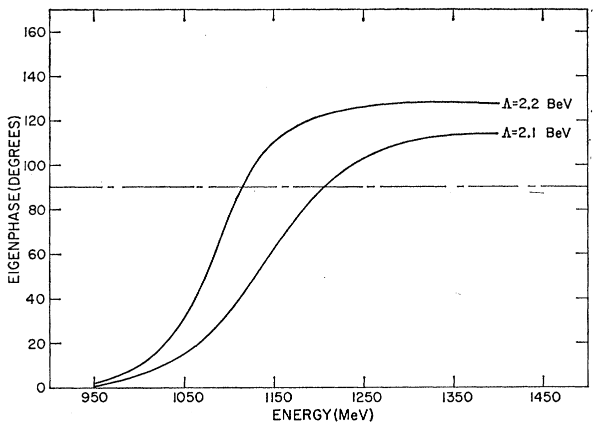

FIG. 2. Eigenphase for $J^{PG}=1^{++}$ state with $I=1$, $g^2/4\pi=2.5$, $m_\rho^2 f^2/4\pi=13$, and $\Lambda=2.1$ BeV, $\Lambda=2.2$ BeV.

turn appears as a logarithmic singularity in the partial-wave projection. Though this is an integrable singularity, the usual matrix inversion method with Gaussian quadrature mesh points fails to solve the integral equations. Either one has to use logarithmic Gaussian quadratures or be careful in the choice of mesh points about the singularities.[6] We have done neither. Fundamentally, PV scattering is a three-body problem beyond the scope of the present investigation. Thus, we resort to the following device: The mass of the *exchanged* pseudoscalar meson is increased to a value such that no singularities occur. In short, we take the mass of the exchanged pseudoscalar octet to be 775 MeV. This value is chosen for convenience and the answers do not depend on it very much once we are above the critical value to prevent the singularity from occurring.

III. RESULTS OF CALCULATION

A. $J^P=1^+$ States

Study of the $SU(3)$ crossing coefficients reveals that there are two distinct combinations for different values of G parity. The $I=\frac{1}{2}$ state of course is an exception. Thus, we present the results for the $J^P=1^+$ states in two distinct subsections. This is because we obtain two $J^P=1^+$ results which are produced with different cutoffs. The first set of particles will be denoted by B. This is the one into which the $B(1208)$ is assigned. The other set will be designated by A. This is the one to which the $A_1(1080)$ is assigned.

Set B

$I=1$, $Y=0$, $G=+1$. Experimentally it is known that there exists an $I=1$ state at about 1208 MeV. Its spin and parity are not well known, but 1^+ is favored, 2^+ is not yet ruled out, while 2^- is not favored.[7] Theoretical calculations of this state have yielded conflicting results. Some have favored a 1^+ state, while others favor a 2^- state.[2] These investigations are based upon inspection of the Born term or an approximate N/D calculation. We have performed a dynamical calculation based on the relativistic Schrödinger equation.

We consider the following coupled channels: $\pi\omega$, $\pi\varphi$, $\eta\rho$, $(\bar{K}K^*+K\bar{K}^*)/\sqrt{2}$. There are two orbital angular momentum values S and D, which can combine with the spin to yield a $J^P=1^+$ state. Likewise, there are two orbital angular momentum values P and F that can combine with the spin to produce a $J^P=2^-$ state. In the present calculation we keep both angular momentum values, which makes the momentum part of the potential a 2×2 matrix.

It proved impossible to produce a resonance in the $J^P=2^-$, $I=1$, $G=+1$ state. The phase shifts are completely insensitive to cutoff and only slightly sensitive to the coupling constants. The independence from the cutoff can be easily understood by noting that for higher partial waves the kernel converges rapidly and the cutoff is superfluous. On the other hand, a $J^P=1^+$ state is extremely easy to produce.

The results are shown in Fig. 2. We see that the numerical results are sensitive to the cutoff. We adjust the cutoff so that the width of the B meson is 114 MeV. This corresponds to $\Lambda=2.2$ BeV. Then the mass of the B meson is 1060 MeV. If the mass is adjusted to the experimental value of 1208 MeV, then the width increases to about 200 MeV and the cutoff becomes 2.1 BeV. We shall work with 2.2 BeV, though 2.1 BeV is just as acceptable. The resonance positions are just shifted by about 150 MeV in this case, and about 70 MeV in other cases.

Of course, if the mass of the B meson is chosen to be 1060 MeV, then the only decay mode available is $\pi\omega$. But even if we adjust the mass to be 1208 BeV, thus opening the new decay channel $\pi\varphi$, the B still prefers to

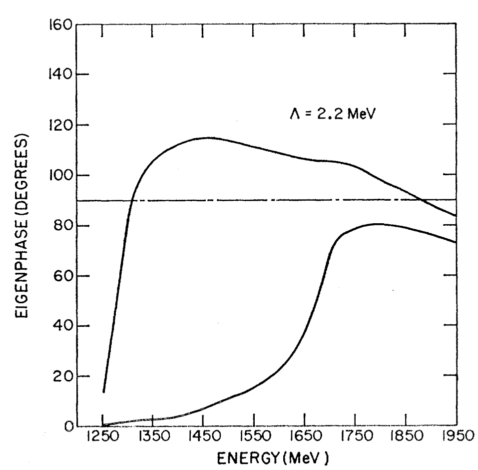

FIG. 3. Eigenphase for $J^P=1^+$ state with $I=\frac{1}{2}$, $g^2/4\pi=2.5$, $m_\rho^2 f^2/4\pi=13$, and $\Lambda=2.2$ BeV.

[6] C. E. Jones and G. Tiktopoulos, J. Math. Phys. 7, 311 (1966).
[7] G. Ascoli, H. B. Crawley, D. W. Mortara, and A. Shapiro, Phys. Rev. Letters 20, 1411 (1968).

decay into $\pi\omega$. The branching ratio is

$$\left.\frac{B\to\pi\varphi}{B\to\pi\omega}\right|_{1208}=7.7\times10^{-4}.$$

The smallness of this result can be understood from the crossing coefficients. Equations (10) and (13) reveal that the direct coupling does not occur and thus the decay of this mode should be small. This is consistent with the experimental result that $\pi\omega$ is the predominant mode in decay for the B meson.

The B meson is produced mainly by vector exchange. If we look at the $SU(3)$ crossing coefficients for pseudoscalar exchange, Eq. (13), we see that the low-mass channels do not interact. This is explicitly confirmed by actual calculation.

$I=\frac{1}{2}$, $Y=1$ state. The experimental situation for the $I=\frac{1}{2}$ states is not clear at the present. There exists evidence for a number of bumps in the energy range 1300–1450 MeV.[8] In our model, there appear to be two $J^P=1^+$ octets. If this is indeed the case, the $I=\frac{1}{2}$ members could be badly mixed. We consider the five coupled channels πK^*, $K\rho$, $K\omega$, ηK^*, and $K\varphi$. There is one resonance at 1310 MeV with a width of 87 MeV. The other eigenphase just misses 90°. Our results are shown in Fig. 3. It usually turns out that if one includes more higher-mass channels, the lower-mass eigenphases rise more steeply. Thus it is hoped that the addition of more channels would produce another $I=\frac{1}{2}$ resonance.

$I=0$, $Y=0$, $G=-1$. Keeping the parameters fixed we look at the $I=0$ state. Here we couple $\pi\rho$, $\eta\omega$, $(\bar{K}K^*-K\bar{K}^*)/\sqrt{2}$, and $\eta\varphi$. The results are shown in Fig. 4. There are two resonances: one at about 975 MeV, which we identify with the $H(990)$, and the other one at 1400 MeV which has not been observed. We call it H'. The widths of the two mesons are 50 and 43 MeV, respectively. The decay mode of H is pure $\pi\rho$, while the predominant decay modes of H' is $\eta\omega$ with the predicted

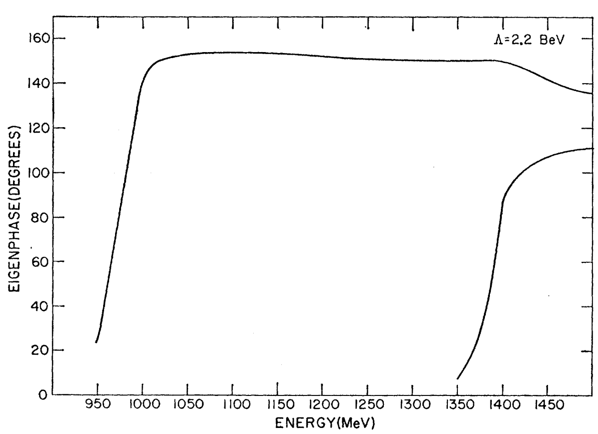

FIG. 4. Eigenphase for $J^{PG}=1^{+-}$ state with $I=0$, $g^2/4\pi=2.5$, $m_\rho^2f^2/4\pi=13$, and $\Lambda=2.2$ BeV.

[8] G. Goldhaber, A. Firestone, and B. C. Shen, Phys. Rev. Letters **19**, 972 (1967).

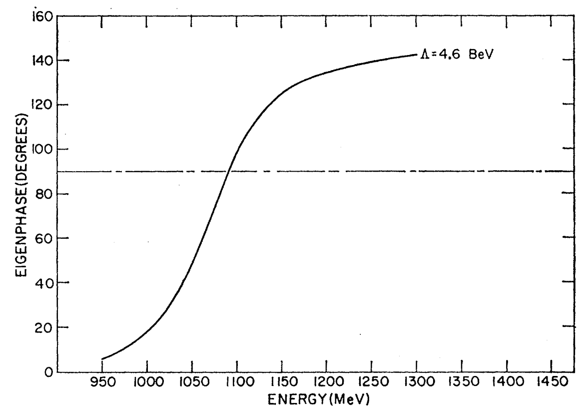

FIG. 5. Eigenphase for $J^{PG}=1^{+-}$ state with $I=1$, $g^2/4\pi=2.5$, $m_\rho^2f^2/4\pi=13$, and $\Lambda=4.6$ BeV.

branching ratios

$$\frac{H'\to(\pi\rho)}{H'\to(\eta\omega)}=0.65\,,\qquad\frac{H'\to(\pi\rho)}{H'\to((\bar{K}K^*-K\bar{K}^*)\ \sqrt{2})}=0.8\,.$$

Set A

$I=1$, $G=-1$ state. In cases when the momentum part of the potential is just a pure number, the reversal of the signs of the $SU(3)$ crossing coefficients usually means that the force has been changed from attraction to repulsion, or vice versa. And if one obtains a resonance in the first case, a resonance is not expected in the latter case.

For the $J^P=1^+$, $G=-1$ state the potential is a 2×2 matrix and this opens the possibility that even if we reverse the signs of all the $SU(3)$ crossing coefficients, the phase shifts still can resonate. And this indeed does happen. If we look at the crossing coefficients for $I=1$, $G=-1$, we see that the diagonal elements are negative. But if one increases the cutoff to $\Lambda=4.6$ BeV then we obtain another resonance.

We consider the following coupled channel: $\rho\pi$, $(K\bar{K}^*-\bar{K}K^*)/\sqrt{2}$. The result is illustrated in Fig. 5. There exists experimental uncertainties about the state A_1. Some do not favor an A_1 at 1080 MeV and take it to be just the Deck effect[9] and there appears to be evidence for an object called $A_{1.5}$.[10] Since the separation in energy between A_1 and $A_{1.5}$ is small, we fix the cutoff so that there be a resonance at 1080 MeV and investigate the consequences. The output width is 110 MeV.

$I=0$, $Y=0$, $G=+1$ state. With the cutoff $\Lambda=4.6$ BeV, we look at the $I=0$, $G=+1$ state. Here we have a single channel $(K\bar{K}^*+\bar{K}K^*)/\sqrt{2}$. The result is that there is a bound state at a mass of 1370 MeV. This can be identified with the $D(1285)$. In Fig. 6, we plot the

[9] R. T. Deck, Phys. Rev. Letters **13**, 169 (1964); U. Maor and T. A. O'Halloran, Jr., Phys. Letters **15**, 281 (1965); M. Ross and Y. Y. Yam, Phys. Rev. Letters **19**, 546 (1967).

[10] G. Ascoli, H. B. Crawley, U. Kruse, D. W. Mortara, E. Schafer, A. Shapiro, and B. Terreault, Phys. Rev. Letters **21**, 113 (1968).

determinant of the homogeneous integral equation. The bound-state problem is an eigenvalue problem, and the energy is determined by requiring the determinant to vanish. The output mass of the D meson can be reduced if we reduce the mass of the A_1.

B. $J^P=0^-$ States

The pseudoscalar mesons are well-established particles. Thus, it is interesting to see whether their octet can be reproduced by our model. If so, this would complete a bootstrap cycle. We start with $PP \rightarrow PP$ and produce the vector mesons. Then we put back the vector mesons and consider $PV \rightarrow PV$ and reproduce the pseudoscalars themselves.

$I=\frac{1}{2}$, $Y=1$ state. We consider the five channels represented by states πK^*, $K\rho$, $K\omega$, ηK^*, and $K\varphi$. We solve the eigenvalue problem, varying the cutoff so that the determinant vanishes at 496 MeV. This corresponds to the cutoff $\Lambda=3.33$ BeV. The determinant is plotted in Fig. 7.

$I=0$, $Y=0$, $G=+1$ state. With the cutoff determined by the K meson, we look at the $I=0$, $G=+1$ state. Here we have a single channel $(K\bar{K}^*+\bar{K}K^*)/\sqrt{2}$. With the cutoff $\Lambda=3.33$ BeV, we produce a bound state at 587 MeV.

We plot the determinant in Fig. 8. The model predicts only one state. The good agreement with experiment here (and for the π) should not be taken too seriously because of the crude treatment of the pseudoscalar exchange as discussed in the Introduction. But, nevertheless, there are several interesting points. First, the mass of the K comes out to be smaller than that of the η. This is a general result. For if we decrease the mass of η by increasing the cutoff, the mass of K will be correspondingly decreased. Furthermore, if we truncate the K meson problem to two or three channels, then the mass of the K turns out to be larger than that of the η.

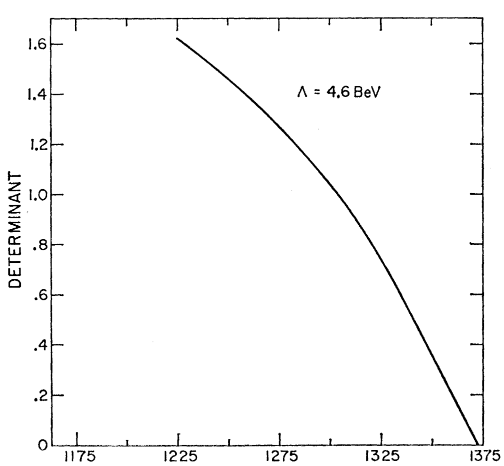

FIG. 6. Determinant for $J^{PG}=1^{++}$ state with $I=0$, $g^2/4\pi=2.5$, $m_\rho^2 f^2/4\pi=13$, and $\Lambda=4.6$ BeV.

So it appears that broken $SU(3)$ symmetry gets propagated in a nontrivial way, in which the number of coupled channels considered is important.

$I=1$, $Y=0$, $G=-1$ state. The two coupled channels are $\pi\rho$ and $(K\bar{K}^*-\bar{K}K^*)/\sqrt{2}$. With the cutoff fixed at $\Lambda=3.33$ BeV, the bound-state energy of this state turns out to be 135 MeV, which is close to the experimental value for the π. We plot the determinant in Fig. 9. This is a satisfying result for the following reasons. The smallness of the π-meson mass in relation to the other members of the pseudoscalar octet raises the question whether it is not somehow more fundamental.[11] If this were so, nuclear democracy would develop regal trimmings. Of course, our calculation is not a true bootstrap, and in no sense is the mass of the π determined in a fundamental way. But within the context of the model, the π does not seem to have any special status and the smallness of the mass is just a consequence of the dynamical equations plus the input.

Some remarks about the above results are in order. Since the pseudoscalar contribution to the potential was not handled in a completely satisfactory way, the question is how important are these exchanges. The only state which is almost independent of the pseudoscalar exchange is the B meson, since the $SU(3)$ coupling coefficients are zero for the low-mass channels. A direct

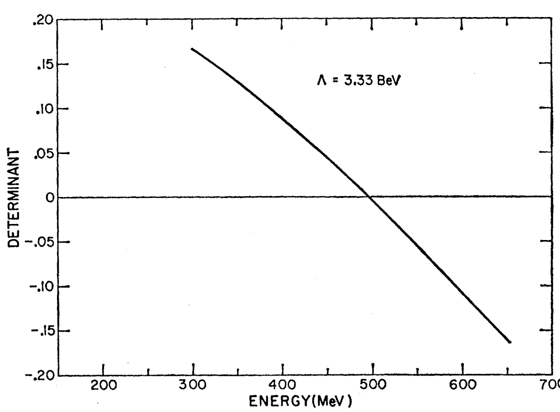

FIG. 7. Determinant for $J^P=0^-$ state with $I=\frac{1}{2}$, $g^2/4\pi=2.5$, $m_\rho^2 f^2/4\pi=13$, and $\Lambda=3.33$ BeV.

calculation also reveals that the $I=\frac{1}{2}$, $J^P=1^+$ state is also almost independent of the pseudoscalar exchange, while the H meson becomes a bound state without the pseudoscalar exchange. The A_1 and the D meson will not appear if we leave off the pseudoscalar contribution. For the $J^P=0^-$ projections, the vector exchange yields attractive forces, but one needs very large cutoffs to produce the bound states. The addition of the pseudoscalar exchange, with the artificially large mass, reduces the cutoff to a more reasonable value.

The other question is, how important a role would one expect the exchanged mass to play? If one deals with a

[11] G. F. Chew, Comments Nucl. Particle Phys. **6**, 187 (1967).

pure Yukawa potential then it is clear that the exchanged mass plays an important role, since it determines the range of the potential. Our potentials are only roughly of the Yukawa type, and the most important contributions to the scattering come from the very singular pieces which are determined by the parameters of the external legs, rather than the exchanged mass.

C. Other States

The exchange of a pseudoscalar or vector meson octet in the t channel will give rise to forces in the singlet,

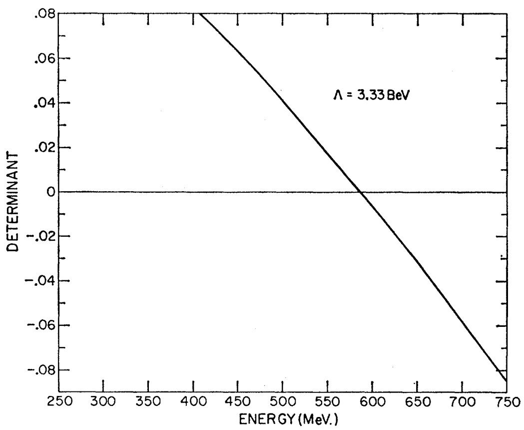

FIG. 8. Determinant for $J^{PG}=0^{-+}$ state with $I=0$, $g^2/4\pi=2.5$, $m_\rho^2f^2/4\pi=13$, and $\Lambda=3.33$ BeV.

symmetric and antisymmetric octets and the **27** representation in the s channel. There is no contribution to the decimet or antidecimet representation. The crossing coefficients for the **27** representation are

Vector exchange

$I=1$, $Y=2$:

$$(2)KK^*\,, \tag{16}$$

$I=2$, $Y=0$:

$$(2)\pi\rho\,, \tag{17}$$

$I=\frac{3}{2}$, $Y=1$:

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{matrix}\pi K^*\\ K\rho\end{matrix}\,. \tag{18}$$

For pseudoscalar exchange, the sign of all coefficients should be reversed.

Our approach has been to fix the cutoff so that the mass of one known member of the representation is the experimental value and then see whether the other members are produced. Thus far there has been no clear cut experimental evidence for existence of the **27** representation of the mesons and hence we cannot make any definite predictions, but there are some considerations which seem worthy of discussion.

We have considered the $J^P=0^-$ and 1^+ projections. For the $J^P=0^-$ state in the **27** representation the vector

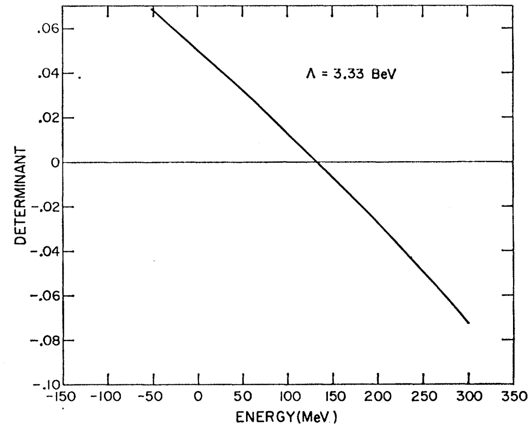

FIG. 9. Determinant for $J^{PG}=0^{--}$ state with $I=1$, $g^2/4\pi=2.5$, $m_\rho^2f^2/4\pi=13$, and $\Lambda=3.33$ BeV.

exchange contributes a repulsive potential and the pseudoscalar yields attractive potential. If the cutoff is chosen to be $\Lambda=3.33$ BeV, the same value as used for the pseudoscalar octet, then there are no extra bound states or resonances. Resonances or bound states can be produced if the cutoff is increased to 5 or 6 BeV.

If we consider the $J^P=1^+$ state and take the cutoff to be the same as for set $B(\Lambda=2.2$ BeV) then we produce an $I=2$, $Y=0$ resonance at about 1200 MeV with a width of approximately 350 MeV. The output width can be reduced considerably if the position of the resonance is reduced slightly. For the other states of the **27** representation, the phase shifts rise rapidly, but then flatten off. The $I=\frac{3}{2}$ eigenphase goes through 90° very slowly at about 1.4 BeV. The $I=1$, $Y=2$ state does not reach 90° at all. One can make all states of the **27** representation resonate by increasing the cutoff. These resonances would then lie between 1 and 1.3 BeV, with the $I=2$ member being lowest in mass. Since there is only slight evidence for an $I=2$, $Y=0$ resonance at about 1325 MeV and no evidence for lower-mass resonances one must conclude that the cutoff cannot be greater than 2.2 BeV. Thus there are two possibilities which are consistent with our model. Either no member of the $J^P=1^+$, **27** representation resonates or only the $I=2$, $Y=0$ member resonates. Whatever the case might be, the model at hand cannot produce resonances of the $J^P=1^+$, **27** representation which have masses greater than about 1.5 BeV.

APPENDIX A

We solve the following Lippmann-Schwinger equation:

$$K_{IL}'^{(J)}(p,q)=V_{IL}'^{(J)}(p,q)-\frac{1}{\pi}\sum_\alpha P \times\int_{\Delta_\alpha}^{\Lambda}\frac{V_{I\alpha}'^{(J)}(p,k)K_{\alpha L}'^{(J)}(k,q)d(\omega_3+\omega_4)_\alpha}{W-(\omega_3+\omega_4)_\alpha}\,. \tag{A1}$$

Here the subscripts label channels, W is the total energy, and

$$(\omega_3+\omega_4)_\alpha=(k^2+m_3^2)^{1/2}+(k^2+m_4^2)^{1/2} \quad \text{(A2)}$$

is the energy in the intermediate state. The quantity

$$\Delta_\alpha=(m_3+m_4)_\alpha \quad \text{(A3)}$$

is the threshold for the α channel and Λ is the cutoff.

The integral equation is solved in the following way: We introduce a quantity

$$U_{IL}^{(J)}(p,q)=\sum_\beta K_{I\beta}'^{(J)}(p,q)\times\left[1-\frac{P}{\pi}\int\frac{K'(k,q)d(\omega_3+\omega_4)_\alpha}{W-(\omega_3+\omega_4)_\alpha}\right]^{-1}_{\beta L} \quad \text{(A4)}$$

and we find that U satisfies the Fredholm equation

$$U_{IL}^{(J)}(p,q)=V_{IL}'(p,q)-\frac{1}{\pi}\sum_\alpha\int\frac{V_{I\alpha}'^{(J)}(p,k)-V_{I\alpha}'^{(J)}(p,q)}{W-(\omega_3+\omega_4)_\alpha}\times U_{\alpha L}(k,q)d(\omega_3+\omega_4)_\alpha. \quad \text{(A5)}$$

Equation (A5) is solved numerically on a computer by matrix inversion using Gaussian quadrature mesh points. Then K' is obtained from the inverse of Eq. (A4), where we again perform a subtraction in order to evaluate the principal-value integral.

For the bound-state problem we solve the usual eigenvalue problem. The equation is

$$K_{IL}'^{(J)}(p,q)=\frac{-1}{\pi}\sum_\alpha P\times\int_{\Delta_\alpha}^{\Lambda}\frac{V_{I\alpha}'^{(J)}(p,k)K_{\alpha L}'^{(J)}(k,q)d(\omega_3+\omega_4)_\alpha}{W-(\omega_3+\omega_4)_\alpha}. \quad \text{(A6)}$$

The integral equations are reduced to a system of homogeneous linear algebraic equations by use of Gaussian quadratures. These equations will have a solution when the determinant vanishes, and this fixes the value of W, which is the bound-state energy.

APPENDIX B

The potential is computed from the diagrams of Figs. 1(a) and 1(b) using the interactions given in Eqs. (1) and (2). The quantity V' is defined by

$$M=\frac{-2\pi W}{(EE'\omega\omega')^{1/2}}\frac{V'}{q^{1/2}q'^{1/2}}, \quad \text{(B1)}$$

$$W=[(E+\omega)(E'+\omega')]^{1/2}, \quad \text{(B2)}$$

where M is the Born term computed from the diagrams using the standard Feynman rules. The normalization is determined by the relation between M and the S matrix

$$S_{fi}=\delta_{fi}-(2\pi)^4 i\delta^4(p_f-p_i)M_{fi}. \quad \text{(B3)}$$

Define the following quantities:

$$I_1=\ln\left|\frac{x+1}{x-1}\right|, \quad \text{(B4)}$$

$$I_2=2-xI_1, \quad \text{(B5)}$$

$$I_3=-xI_2, \quad \text{(B6)}$$

$$I_4=\tfrac{2}{3}-xI_3, \quad \text{(B7)}$$

$$I_5=2x+(1-x^2)I_1, \quad \text{(B8)}$$

$$I_6=\tfrac{4}{3}-xI_5, \quad \text{(B9)}$$

where

$$x=\frac{q^2+q'^2-\frac{1}{4}(E-\omega+E'-\omega')^2+M_{\text{exch}}^2}{2qq'}. \quad \text{(B10)}$$

Here E is the c.m. energy of the incident vector meson and ω is the energy of the pseudoscalar meson; whereas, E' and ω' are the final c.m. energies of the vector and pseudoscalar meson, respectively. M_{exch} is the mass of the exchanged particle.

1. $J^P=0^-$ State

The vector-meson exchange contributes the following potential:

$$V_{00}'^{(0)}(V)=\frac{-f^2}{16\pi}\frac{q^{1/2}q'^{1/2}}{[(E+\omega)(E'+\omega')]^{1/2}}\frac{mm'}{4}I_5, \quad \text{(B11)}$$

where the masses are those of the external vector mesons. The pseudoscalar exchange contributes

$$V_{00}'(P)=-\frac{1}{2}\frac{g^2}{4\pi}\frac{1}{mm'}\frac{1}{2[qq'(E+\omega)(E'+\omega')]^{1/2}}\times\{qq'EE'I_3+(q^2E'\omega'+q'^2E\omega)I_2+qq'\omega\omega'I_1\}. \quad \text{(B12)}$$

2. $J^P=1^+$ State

Define

$$\xi=\frac{-f^2}{16\pi}\frac{1}{4[qq'(E+\omega)(E'+\omega')]^{1/2}}. \quad \text{(B13)}$$

Then the vector-exchange contribution is

$$(X+Y)_V=\xi[-\tfrac{1}{2}EE'qq'I_4+\{q^2q'^2+\tfrac{1}{2}EE'\omega\omega'-\tfrac{1}{2}(q'^2+E'\omega')(q^2+E\omega)\}I_3-qq'(\omega\omega'+\tfrac{1}{2}EE')I_2+\tfrac{1}{2}\{EE'\omega\omega'-(q'^2+E'\omega')(q^2+E\omega)\}I_1], \quad \text{(B14)}$$

$$Z_V=\xi mm'qq'I_6, \quad \text{(B15)}$$

$$W_V(q,q')=\xi[\tfrac{1}{2}mE'qq'I_6-\tfrac{1}{2}\{mE'\omega\omega'-m\omega(q'^2+E'\omega')\}I_5]. \quad \text{(B16)}$$

$W(q',q)$ is obtained from $W(q,q')$ by interchange of the initial vector meson with the final one.

Now define

$$\tau=-\frac{1}{2}\frac{g^2}{4\pi}\frac{q^{1/2}q'^{\,1/2}}{[(E+\omega)(E'+\omega')]^{1/2}}. \qquad \text{(B17)}$$

The pseudoscalar-meson contribution is

$$(X+Y)_P=-\tfrac{1}{4}\tau I_6, \qquad \text{(B18)}$$

$$Z_P=[\tau/(2mm'qq')][qq'\omega\omega' I_2 + (q^2\omega' E'+q'^2\omega E)I_3+qq'EE'I_4], \qquad \text{(B19)}$$

$$W(q,q')_P=(-\tau/4q'm)[q\omega' I_5+q'EI_6]. \qquad \text{(B20)}$$

All these elements of the potential must be multiplied by the $SU(3)$ crossing coefficients tabulated in the main part of the paper.

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Hadronic Corrections to the Goldberger-Treiman Relation*

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Using unsubtracted dispersion relations in momentum transfer for the matrix element of the divergence of the axial-vector current between nucleon states, we have examined the hadronic continuum corrections to the Goldberger-Treiman relation for π^+ decay. These are observed to be about +10%. From a rigorous unitarity bound and the assumption that the pion propagator $\Delta_\pi(0)$ is dominated by the pion pole we show that the continuum states of energy greater than two nucleon masses contribute less than ½%. The $\pi\rho$ and $\pi\sigma$ states contributed negligibly. Using Weinberg's extrapolation for the $\pi\pi$ scattering amplitude and chiral dynamics, we find that the presumably dominant 3π state contributes with opposite sign and is more than an order of magnitude too small. In the absence of any simple explanation for the 10% correction, we conjecture that what is required is a 3π threshold enhancement or possible resonance, the tripion, with the quantum numbers of the pion and mass near threshold at 4.2 BeV/c^2 or a possible subtraction in the dispersion relation.

I. INTRODUCTION

As a consequence of the precision measurements of the π^+ lifetime, the rate of Gamow-Teller transitions in neutron β decay, and the π^+ nucleon coupling constant, one may establish in both magnitude and sign the correction to the Goldberger-Treiman relation (GTR)[1]

$$\Delta=1-\frac{(m_p+m_n)g_A}{\sqrt{2}gf_\pi}=+0.105\pm0.026.$$

This number represents the small 10% continuum correction to the single-pion-pole term, and it is this number we will endeavor to understand. We will approach this problem in the conventional way by assuming an unsubtracted dispersion relation in the momentum transfer for the matrix elements of the divergence of the axial-vector current taken between nucleon states. Then Δ is simply related to the continuum integral over the timelike region with the threshold at three pion masses.

As we discover by proceeding in this way, the problem is not to understand why the correction Δ is so small but rather why it is so large. Everything one can estimate from the known low-lying meson spectrum gives a value for Δ more than an order of magnitude too small. The 3π state contribution we estimate gives a number with the wrong sign and an order of magnitude too small. This is primarily because of the very small three-body phase space. Electromagnetic corrections (with a uv cutoff at 1–2 BeV) give at most 1%.

The $\pi\rho$ and $\pi\sigma$ continuum states are negligible also. Using a rigorous unitarity bound and the assumption the pion propagator at $q^2=0$ is dominated by the pion pole, we can argue that high-energy contributions from the region of energy greater than two nucleon masses are less than ½%.

Confronted with the absence of any evident explanation for the observed 10% correction in terms of the known meson spectrum, we conjecture the existence of large forces in the three-body pion system giving rise to an enhancement with the quantum numbers of the pion near the 3π threshold. The tripion, if a genuine resonant state near $m_{\pi'}\approx 3m_\pi$, should be seen in $\pi^+\pi^+\pi^-$ invariant mass distributions near threshold at 4.2 BeV/c^2 in 3- and 4-prong π^-p collisions. No peak is in evidence from the available data but the statistics is poor in the threshold region and a small amplitude peak might not have shown up. Should no tripionic

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