$A_\mu(x)$ is the electromagnetic vector potential, then Eqs. (68) and (69) are Maxwell's equations for particles described by point singularities

$$\Box A_\mu = 0, \tag{70}$$

$$\partial_\mu A_\mu = 0. \tag{71}$$

Equation (30) becomes the familiar result

$$\partial_\mu \mathcal{V}_\mu{}^i = -e\epsilon_{i3k} A_\mu \mathcal{V}_\mu{}^k, \tag{72}$$

based on the principle of minimal electromagnetic interactions.

The equations for $\Gamma_\mu{}^{(1)i}$ were obtained in the Lorentz gauge equation (69). Let us consider the question of the gauge invariance of our first-order equations. In first order, our field equations are

$$R_{[\mu\nu]}{}^{(1)i} = 0, \tag{73}$$

where $R_{[\mu\nu]}{}^{(1)i}$ is given by Eq. (63) for $i=1,2,3$. If we consider the gauge transformation

$$\Gamma_{\mu\nu}{}^{\alpha(1)i} \to \Gamma_{\mu\nu}{}^{\alpha(1)i} - \tfrac{1}{6}(\delta_\mu{}^\alpha \partial_\nu \Lambda^i - \delta_\nu{}^\alpha \partial_\mu \Lambda^i), \tag{74}$$

then it is easily shown by virtue of Eq. (63) that our field equation (73) remains invariant under this gauge transformation for an arbitrary scalar triplet $\Lambda^i(x)$. Upon contracting (74), we get

$$\Gamma_\mu{}^{(1)i} \to \Gamma_\mu{}^{(1)i} + \partial_\mu \Lambda^i. \tag{75}$$

Thus, Eq. (66) becomes

$$\Gamma_\mu{}^{(1)i} \to (2/\beta)\partial_\nu g_{[\mu\nu]}{}^{(1)i} + \partial_\mu \Lambda^i \tag{76}$$

and Eq. (68) is only valid in the Lorentz gauge corresponding to the choice $\Box\Lambda^i = 0$. The identification $\lambda\Gamma_\mu{}^{(1)i} = e\delta_{i3} A_\mu$ then corresponds to the familiar gauge transformation

$$A_\mu \to A_\mu + (\lambda/e)\partial_\mu \Lambda^3. \tag{77}$$

In principle, we now have a theory to calculate the symmetry violations to all orders in λ, and in this way we can determine the amount of breaking occurring in the hadron symmetries to any order. The solutions of the divergence equations in higher orders will be discussed elsewhere.

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Exact Solution for the Scattering of Electromagnetic Waves from Bodies of Arbitrary Shape. III. Obstacles with Arbitrary Electromagnetic Properties*

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The analytical technique developed earlier for the scattering of electromagnetic wave fields by perfect conductors is generalized to the case where the scattering obstacle is not only of arbitrary geometrical shape, but where both the scattering obstacle and the exterior environment have arbitrary, though homogeneous, electromagnetic properties. As before, the solution obtained is analytically exact and thus equally valid in the near and far zones, as well as over the entire frequency range. The special cases of the first-order solution and of an incident plane wave are considered in detail. The form of the solution is particularly well suited for methodical numerical evaluation.

I. INTRODUCTION

In two previous papers,[1,2] hereinafter referred to as I and II, respectively, we obtained exact analytical series solutions for the problem of the scattering of electromagnetic waves by conductors of irregular geometrical shape. The solution presented in I was restricted to the case of complete cylindrical symmetry with the incident wave field corresponding to that of a plane wave. This solution was generalized in II for the case of scatterers of completely arbitrary shape, as well as an arbitrary incident radiation field. Although a perturbation technique was employed in obtaining these solutions, the final analytic series solutions were valid to all orders in the perturbation, and thus represented the exact solution to the scattering problem within the range of convergence of the series.

However, the solutions presented in both I and II were restricted to the case where the scattering obstacles were perfect conductors in an environment possessing the electromagnetic properties of the vacuum. The aim of the present paper is to generalize our method to the case where the scatterer is not only of arbitrary shape, but where both the scattering obstacle as well as its exterior environment have arbitrary, though homogeneous, electromagnetic structure, as characterized by specified values of the electric permittivity ϵ, the

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[1] V. A. Erma, Phys. Rev. **173**, 1243 (1968).
[2] V. A. Erma, Phys. Rev. **176**, 1544 (1968).

magnetic permeability μ, and the conductivity σ.[3] This case differs physically from those considered earlier in that the boundary conditions to be satisfied at the surface of the scatterer are more complex. Thus, for the present case both the magnetic and electric fields must be considered everywhere, including the region inside the scattering obstacle. As in II, the solution will be developed for the case of an arbitrary incident radiation field, of which a plane wave represents but a special case.

The scattering problem is formulated in Sec. II, and the general solution is developed in Sec. III. For purposes of illustration and comparison with earlier results, the first-order solution and the general solution for the case of an incident radiation field corresponding to a plane wave are developed explicitly in Sec. IV. Finally, Sec. V is devoted to a few concluding remarks.

Inasmuch as the general method of our approach closely parallels that followed in I and II, we shall not dwell on explanations of analytical detail, for which the reader is referred to I and II.[4]

II. FORMULATION OF THE PROBLEM

The problem under consideration concerns the scattering of a given electromagnetic wave field by an obstacle of arbitrary geometrical shape and electromagnetic structure. Both the scattering obstacle and its exterior environment are taken to be homogeneous, isotropic, and linear in their electromagnetic properties, which are represented by specified values of ϵ_2, μ_2, σ_2, and ϵ_1, μ_1, σ_1 for the permittivities, permeabilities, and conductivities of the scatterer and its environment, respectively. The known incident electric and magnetic fields will be denoted by $\mathbf{E}^i$ and $\mathbf{H}^i$. In addition, we shall assume that the time dependence of these fields (and hence also that of the scattered and transmitted fields) is of the form $e^{-i\omega t}$, which factor will hereafter be suppressed.[5] In this case, the propagation properties of the two media can be conveniently described in terms of the propagation number, given by

$$k_{1,2}^2=\omega^2\epsilon_{1,2}\mu_{1,2}+\omega\sigma_{1,2}\mu_{1,2}\,, \tag{1}$$

where the subscripts 1 and 2 refer to the exterior environment and the scatterer, respectively.

The geometrical shape of the scattering obstacle is represented by the equation of its surface, which in spherical coordinates takes the general form

$$r=r_s=a[1+\epsilon f(\theta,\varphi)]\,, \tag{2}$$

where ϵ is a suitably chosen "smallness parameter," $f(\theta,\varphi)$ is an arbitrary function, subject only to the conditions of single-valuedness and the requirement

$$|\epsilon f(\theta,\varphi)|<1,\quad 0\leq\theta<\pi,\quad 0\leq\varphi<2\pi, \tag{3}$$

and a represents the radius of the "unperturbed sphere."[6]

The complete electromagnetic field in the exterior consists of the sum of the given incident fields $(\mathbf{E}^i,\mathbf{H}^i)$ and the fields scattered by the obstacle $(\mathbf{E}^s,\mathbf{H}^s)$, whereas the total fields generated within the obstacle are given by the transmitted fields which will be denoted by $(\mathbf{E}^2,\mathbf{H}^2)$. Both the scattered and transmitted fields must obey the vector Helmholtz equation, as well as Maxwell's equations. As is well known,[7] the general solution of the vector Helmholtz equation in spherical coordinates can be written as a linear combination of the two so-called "unit fields"

$$\mathbf{M}_{\pm mn}=z_n(\rho)\mathbf{m}_{\pm mn}\,, \tag{4}$$

$$\mathbf{N}_{\pm mn}=\rho^{-1}z_n(\rho)\mathbf{p}_{\pm mn}+\rho^{-1}\frac{d}{d\rho}[\rho z_n(\rho)]\mathbf{n}_{\pm mn}\,, \tag{5}$$

where

$$\mathbf{m}_{\pm mn}=\mp\frac{m}{\sin\theta}P_n{}^m(x)\binom{\sin}{\cos}m\varphi\,\mathbf{e}_\theta - \frac{dP_n{}^m(x)}{d\theta}\binom{\cos}{\sin}m\varphi\,\mathbf{e}_\varphi\,, \tag{6}$$

$$\mathbf{n}_{\pm mn}=\frac{dP_n{}^m}{d\theta}\binom{\cos}{\sin}m\varphi\,\mathbf{e}_\theta\mp\frac{m}{\sin\theta}P_n{}^m(x)\binom{\sin}{\cos}m\varphi\,\mathbf{e}_\varphi\,, \tag{7}$$

$$\mathbf{p}_{\pm mn}=n(n+1)P_n{}^m(x)\binom{\cos}{\sin}m\varphi\,\mathbf{e}_r\,. \tag{8}$$

Here $x=\cos\theta$; $\rho=kr$ [where the value of k appropriate to each medium is given by Eq. (1)]; $\mathbf{e}_r$, $\mathbf{e}_\theta$, and $\mathbf{e}_\varphi$ are unit vectors along the directions of increasing r, θ, and φ, respectively; $P_n{}^m$ is an associated Legendre function; and $z_n(\rho)$ represents an appropriate spherical Bessel function. Thus, the scattered electric field in medium 1 may be written

$$\mathbf{E}^s=\sum_{m,n}(a_{\pm mn}\mathbf{M}_{\pm mn}{}^s+b_{\pm mn}\mathbf{N}_{\pm mn}{}^s)\,, \tag{9}$$

where the superscript s denotes the choice

$$z_n(\rho)=h_n{}^{(1)}(\rho_1)\,,$$

[3] Also sometimes known in the literature as "soft" scatterers.

[4] Similarly, a discussion of the background of this problem, together with relevant references to earlier work, is presented in I and will not be reiterated here.

[5] The situation where the given time dependence has a more complicated form can be reduced to the present case by means of well-known techniques of Fourier decomposition.

[6] The optimal choices for a and the location of the origin of the unperturbed sphere, as well as the class of shapes which are describable by an equation of the form (2), were discussed in detail in Appendix A of I. The generalization of the remarks made there to the present case is obvious. In brief, the only obstacles excluded from the present formalism are those whose surface cannot be described in terms of a single-valued function of the angular coordinates θ and φ. In particular, all convex bodies may be described by means of Eqs. (2) and (3), with a suitable choice of a and ϵ.

[7] See, for example, J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Co., Inc., New York, 1941).

$\rho_1=k_1r$, for the spherical Bessel function, and where $a_{\pm mn}$ and $b_{\pm mn}$ represent the undetermined "scattering coefficients."

By making use of the Maxwell equation

$$\nabla\times\mathbf{E}=-\mu(\partial\mathbf{H}/\partial t),$$

keeping in mind the $e^{-i\omega t}$ time dependence of all fields, and taking note of the easily verified relations $\nabla\times\mathbf{M}_{\pm mn}=k\mathbf{N}_{\pm mn}$ and $\nabla\times\mathbf{N}_{\pm mn}=k\mathbf{M}_{\pm mn}$, the scattered magnetic field is found to be

$$\mathbf{H}^s=\sum_{m,n}\chi_1(a_{\pm mn}\mathbf{N}_{\pm mn}{}^s+b_{\pm mn}\mathbf{M}_{\pm mn}{}^s),\quad \chi_1=k_1/i\omega\mu_1. \tag{10}$$

The total electromagnetic fields in the region outside the scatterer (medium 1) are then given by

$$\mathbf{E}^1=\mathbf{E}^i+\mathbf{E}^s,\quad \mathbf{H}^1=\mathbf{H}^i+\mathbf{H}^s. \tag{11}$$

Similarly, the transmitted electric field may in general be written

$$\mathbf{E}^2=\sum_{m,n}(c_{\pm mn}\mathbf{M}_{\pm mn}{}^2+d_{\pm mn}\mathbf{N}_{\pm mn}{}^2), \tag{12}$$

where the superscript 2 denotes the choice of $z_n(\rho)=j_n(\rho_2)$, $\rho_2=k_2r$, for the spherical Bessel function, and $c_{\pm mn}$ and $d_{\pm mn}$ represent undetermined coefficients. The corresponding transmitted magnetic field is easily found to be

$$\mathbf{H}^2=\sum_{m,n}\chi_2(c_{\pm mn}\mathbf{N}_{\pm mn}{}^2+d_{\pm mn}\mathbf{M}_{\pm mn}{}^2),\quad \chi_2=k_2/i\omega\mu_2. \tag{13}$$

In each of Eqs. (9), (10), (12), and (13), the summation extends over all non-negative integral values of n, and $m\leq n$, as well as over both the even $(+)$ and odd $(-)$ components.

The problem now consists of obtaining the coefficients $a_{\pm mn}$ and $b_{\pm mn}$, in terms of which all scattering quantities of interest, such as various cross sections, can easily be calculated. These are determined by applying the boundary conditions which must be satisfied at each point of the surface of the scatterer. For our case, the boundary conditions take the form

$$\mathbf{N}\times\mathbf{E}^1=\mathbf{N}\times\mathbf{E}^2,\quad r=r_s \tag{14}$$

$$\mathbf{N}\times\mathbf{H}^1=\mathbf{N}\times\mathbf{H}^2,\quad r=r_s \tag{15}$$

where $\mathbf{N}$ is any vector normal to the surface at each point. Such a vector is provided by

$$\mathbf{N}=(r_s/a)\nabla(r-r_s)\big|_{r=r_s}, \tag{16}$$

which upon expansion of the gradient in spherical coordinates can be written in the explicit form

$$\mathbf{N}=(1+\epsilon f)\mathbf{e}_r-\epsilon f_\theta\mathbf{e}_\theta-\epsilon(\sin\theta)^{-1}f_\varphi\mathbf{e}_\varphi, \tag{17}$$

where

$$f_\theta\equiv\frac{\partial f}{\partial\theta},\quad f_\varphi\equiv\frac{\partial f}{\partial\varphi}. \tag{18}$$

When this expression for $\mathbf{N}$ is used in the boundary condition (14), its three components become explicitly

$$(\sin\theta)^{-1}f_\varphi E_\theta{}^1-f_\theta E_\varphi{}^1=(\sin\theta)^{-1}f_\varphi E_\theta{}^2-f_\theta E_\varphi{}^2, \tag{19}$$

$$(1+\epsilon f)E_\varphi{}^1+\epsilon(\sin\theta)^{-1}f_\varphi E_r{}^1=(1+\epsilon f)E_\varphi{}^2+\epsilon(\sin\theta)^{-1}f_\varphi E_r{}^2, \tag{20}$$

$$(1+\epsilon f)E_\theta{}^1+\epsilon f_\theta E_r{}^1=(1+\epsilon f)E_\theta{}^2+\epsilon f_\theta E_r{}^2, \tag{21}$$

where it is understood that all field components are to be evaluated at the surface $r=r_s$. It is easily verified that of these three equations only two are independent. In our formulation we shall use Eqs. (20) and (21). If we multiply Eq. (20) by $\mathbf{e}_\varphi$, Eq. (21) by $\mathbf{e}_\theta$, and add the two resulting equations, the two boundary conditions (20) and (21) may be written as a single vector equation

$$(1+\epsilon f)\mathbf{E}_t{}^1+\epsilon\boldsymbol{\alpha}E_r{}^1=(1+\epsilon f)\mathbf{E}_t{}^2+\epsilon\boldsymbol{\alpha}E_r{}^2, \tag{22}$$

where the subscript t denotes the total *tangential* field[8] and the vector function $\boldsymbol{\alpha}$ is given by

$$\boldsymbol{\alpha}=f_\theta\mathbf{e}_\theta+(\sin\theta)^{-1}f_\varphi\mathbf{e}_\varphi. \tag{23}$$

In an exactly analogous manner the boundary condition (15) can be rewritten

$$(1+\epsilon f)\mathbf{H}_t{}^1+\epsilon\boldsymbol{\alpha}H_r{}^1=(1+\epsilon f)\mathbf{H}_t{}^2+\epsilon\boldsymbol{\alpha}H_r{}^2. \tag{24}$$

If we now make use of Eq. (11) and substitute for the scattered and transmitted fields from Eqs. (9), (10), (12), and (13), together with Eqs. (4)–(8), the two boundary conditions (22) and (24) take the explicit form

$$\begin{aligned}(1+\epsilon f)\left\{\mathbf{E}_t{}^i(\rho_{1s})+\sum_{m,n}\left[a_{\pm mn}h_n{}^{(1)}(\rho_{1s})\mathbf{m}_{\pm mn}+b_{\pm mn}\left(\frac{1}{\rho}\frac{d}{d\rho}[\rho h_n{}^{(1)}(\rho)]\right)_{\rho_{1s}}\mathbf{n}_{\pm mn}\right]\right\}\\
+\epsilon\boldsymbol{\alpha}\left[E_r{}^i(\rho_{1s})+\sum_{m,n}b_{\pm mn}\frac{h_n{}^{(1)}(\rho_{1s})}{\rho_{1s}}|\mathbf{p}_{\pm mn}|\right]=(1+\epsilon f)\sum_{m,n}\left[c_{\pm mn}j_n(\rho_{2s})\mathbf{m}_{\pm mn}\right.\\
\left.+d_{\pm mn}\left(\frac{1}{\rho}\frac{d}{d\rho}[\rho j_n(\rho)]\right)_{\rho_{2s}}\mathbf{n}_{\pm mn}\right]+\epsilon\boldsymbol{\alpha}\sum_{m,n}d_{\pm mn}\frac{j_n(\rho_{2s})}{\rho_{2s}}|\mathbf{p}_{\pm mn}|,\end{aligned} \tag{25}$$

[8] "Tangential" is used here in the sense of tangential to the original unperturbed sphere (or normal to $\mathbf{e}_r$), not to the surface of the actual scatterer.

$$(1+\epsilon f)\left\{\mathbf{H}_t^i(\rho_{1s})+\sum_{m,n}\chi_1\left[a_{\pm mn}\left(\frac{1}{\rho}\frac{d}{d\rho}[\rho h_n^{(1)}(\rho)]\right)_{\rho_{1s}}\mathbf{n}_{\pm mn}+b_{\pm mn}h_n^{(1)}(\rho_{1s})\mathbf{m}_{\pm mn}\right]\right\}$$

$$+\epsilon\alpha\left[H_r^i(\rho_{1s})+\sum_{m,n}\chi_1 a_{\pm mn}\frac{h_n^{(1)}(\rho_{1s})}{\rho_{1s}}|\mathbf{p}_{\pm mn}|\right]=(1+\epsilon f)\sum_{m,n}\chi_2\left[c_{\pm mn}\left(\frac{1}{\rho}\frac{d}{d\rho}[\rho j_n(\rho)]\right)_{\rho_{2s}}\mathbf{n}_{\pm mn}\right.$$

$$\left.+d_{\pm mn}j_n(\rho_{2s})\mathbf{m}_{\pm mn}\right]+\epsilon\alpha\sum_{m,n}\chi_2 c_{\pm mn}\frac{j_n(\rho_{2s})}{\rho_{2s}}|\mathbf{p}_{\pm mn}|, \quad (26)$$

where $\rho_{1s}=k_1r_s$ and $\rho_{2s}=k_2r_s$, the bars represent absolute magnitude, and the incident fields are now considered as functions of $\rho_1=k_1r$, θ, and φ. The problem now consists of determining the unknown coefficients $a_{\pm mn}$, $b_{\pm mn}$, $c_{\pm mn}$, and $d_{\pm mn}$.

III. GENERAL SOLUTION

The unknown coefficients $a_{\pm mn}$, $b_{\pm mn}$, $c_{\pm mn}$, and $d_{\pm mn}$ will now be determined from Eqs. (25) and (26) by applying the special boundary perturbation technique used in I and II. Accordingly, each coefficient is written in the form of a power series in ϵ, as follows:

$$(a,b,c,d)_{\pm mn}=\sum_{p=0}^{\infty}\frac{\epsilon^p(a,b,c,d)_{\pm mn}{}^p}{p!}, \quad (27)$$

where the coefficients $(a,b,c,d)_{\pm mn}{}^0$ represent the zero-order coefficients for the unperturbed sphere,[9] while $(a,b,c,d)_{\pm mn}{}^p$ represent the pth-order contributions of the boundary perturbation parameter ϵ.

Before proceeding with the analysis, it turns out to be convenient to multiply Eqs. (25) and (26) through by $\rho_{10}\equiv k_1a$. Noting that $\rho_{10}(1+\epsilon f)=\rho_{1s}=\kappa\rho_{2s}$, where we have defined $\kappa\equiv k_1/k_2$, Eqs. (25) and (26) may then be written[10]

$$\rho_{1s}\mathbf{E}_t^i(\rho_{1s})+\sum_{m,n}\left[a_{\pm mn}\rho_{1s}h_n^{(1)}(\rho_{1s})\mathbf{m}_{\pm mn}+b_{\pm mn}\left(\frac{d}{d\rho}[\rho h_n^{(1)}(\rho)]\right)_{\rho_{1s}}\mathbf{n}_{\pm mn}\right]+\epsilon\alpha\rho_{10}\left[E_r^i(\rho_{1s})+\sum_{m,n}b_{\pm mn}\frac{h_n^{(1)}(\rho_{1s})}{\rho_{1s}}|\mathbf{p}_{\pm mn}|\right]$$

$$=\kappa\sum_{m,n}\left[c_{\pm mn}\rho_{2s}j_n(\rho_{2s})\mathbf{m}_{\pm mn}+d_{\pm mn}\left(\frac{d}{d\rho}[\rho j_n(\rho)]\right)_{\rho_{2s}}\mathbf{n}_{\pm mn}\right]+\epsilon\alpha\rho_{10}\sum_{m,n}d_{\pm mn}\frac{j_n(\rho_{2s})}{\rho_{2s}}|\mathbf{p}_{\pm mn}|, \quad (28)$$

$$\rho_{1s}\mathbf{H}_t^i(\rho_{1s})+\sum_{m,n}\chi_1\left[a_{\pm mn}\left(\frac{d}{d\rho}[\rho h_n^{(1)}(\rho)]\right)_{\rho_{1s}}\mathbf{n}_{\pm mn}\right.$$

$$\left.+b_{\pm mn}\rho_{1s}h_n^{(1)}(\rho_{1s})\mathbf{m}_{\pm mn}\right]+\epsilon\alpha\rho_{10}\left[H_r^i(\rho_{1s})+\sum_{m,n}\chi_1 a_{\pm mn}\frac{h_n^{(1)}(\rho_{1s})}{\rho_{1s}}|\mathbf{p}_{\pm mn}|\right]$$

$$=\kappa\sum_{m,n}\chi_2\left[c_{\pm mn}\left(\frac{d}{d\rho}[\rho j_n(\rho)]\right)_{\rho_{2s}}\mathbf{n}_{\pm mn}+d_{\pm mn}\rho_{2s}j_n(\rho_{2s})\mathbf{m}_{\pm mn}\right]+\epsilon\alpha\rho_{10}\sum_{m,n}\chi_2 c_{\pm mn}\frac{j_n(\rho_{2s})}{\rho_{2s}}|\mathbf{p}_{\pm mn}|. \quad (29)$$

We now expand each term of the above equations explicitly as a power series in ϵ, which is involved in the coefficients $(a,b,c,d)_{\pm mn}$ as well as in the arguments ρ_{1s} and ρ_{2s}. This is accomplished by expanding all functions of ρ_{1s} and ρ_{2s} in the form of Taylor series about $r=a$. The required expansion coefficients are defined by the relations

$$\rho_{1s}h_n^{(1)}(\rho_{1s})=\sum_{p=0}^{\infty}\frac{\epsilon^p\beta_n{}^p}{p!}\rho_{10}{}^pf^p, \qquad \beta_n{}^p=\frac{d^p}{d\rho^p}[\rho h_n^{(1)}(\rho)]\bigg|_{\rho=\rho_{10}} \quad (30)$$

$$\rho_{2s}j_n(\rho_{2s})=\sum_{p=0}^{\infty}\frac{\epsilon^p\alpha_n{}^p}{p!}\rho_{20}{}^pf^p, \qquad \alpha_n{}^p=\frac{d^p}{d\rho^p}[\rho j_n(\rho)]\bigg|_{\rho=\rho_{20}} \quad (31)$$

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[9] These are known, inasmuch as the analytical solution of the scattering problem for the case of a sphere is well known; see, for example, Ref. 7.

[10] This corresponds to the "alternative formulation" of II. The development analogous to the first formulation of II may also be carried out in the present case, but will be omitted for reasons of space. The comments made in II concerning the relative merits of the two possible formulations apply equally well here.

$$\frac{h_n^{(1)}(\rho_{1s})}{\rho_{1s}}=\sum_{p=0}^{\infty}\frac{\epsilon^p\eta_n{}^p}{p!}\rho_{10}{}^p f^p\,,\qquad \eta_n{}^p=\frac{d^p}{d\rho^p}\frac{h_n^{(1)}(\rho)}{\rho}\bigg|_{\rho=\rho_{10}} \tag{32}$$

$$\frac{j_n(\rho_{2s})}{\rho_{2s}}=\sum_{p=0}^{\infty}\frac{\epsilon^p\sigma_n{}^p}{p!}\rho_{20}{}^p f^p\,,\qquad \sigma_n{}^p=\frac{d^p}{d\rho^p}\frac{j_n(\rho)}{\rho}\bigg|_{\rho=\rho_{20}} \tag{33}$$

$$\rho_{1s}\mathbf{E}_t{}^i(\rho_{1s})=\sum_{p=0}^{\infty}\frac{\epsilon^p(\rho\mathbf{E}_t{}^i)^{(p)}}{p!}\rho_{10}{}^p f^p\,,\qquad (\rho\mathbf{E}_t{}^i)^{(p)}=\frac{\partial^p}{\partial\rho^p}[\rho\mathbf{E}_t{}^i(\rho,\theta,\varphi)]\bigg|_{\rho=\rho_{10}} \tag{34}$$

$$E_r{}^i(\rho_{1s})=\sum_{p=0}^{\infty}\frac{\epsilon^p(E_r{}^i)^{(p)}}{p!}\rho_{10}{}^p f^p\,,\qquad (E_r{}^i)^{(p)}=\frac{\partial^p}{\partial\rho^p}E_r{}^i(\rho,\theta,\varphi)\bigg|_{\rho=\rho_{10}} \tag{35}$$

$$\rho_{1s}\mathbf{H}_t{}^i(\rho_{1s})=\sum_{p=0}^{\infty}\frac{\epsilon^p(\rho\mathbf{H}_t{}^i)^{(p)}}{p!}\rho_{10}{}^p f^p\,,\qquad (\rho\mathbf{H}_t{}^i)^{(p)}=\frac{\partial^p}{\partial\rho^p}\mathbf{H}_t{}^i(\rho,\theta,\varphi)\bigg|_{\rho=\rho_{10}} \tag{36}$$

$$H_r{}^i(\rho_{1s})=\sum_{p=0}^{\infty}\frac{\epsilon^p(H_r{}^i)^{(p)}}{p!}\rho_{10}{}^p f^p\,,\qquad (H_r{}^i)^{(p)}=\frac{\partial^p}{\partial\rho^p}H_r{}^i(\rho,\theta,\varphi)\bigg|_{\rho=\rho_{10}}\,. \tag{37}$$

In addition, whenever we encounter a product of two infinite series, such as in the expanded version of the term $a_{\pm mn}\rho_{1s}h_n^{(1)}(\rho_{1s})$, the result is recast as a single power series in ϵ by making use of the easily verified theorem

$$\Big(\sum_{p=0}^{\infty}\epsilon^p a_p\Big)\Big(\sum_{p=0}^{\infty}\epsilon^p b^p\Big)=\sum_{p=0}^{\infty}\epsilon^p\sum_{q=0}^{p}a_q b_{p-q}\,. \tag{38}$$

If we now substitute the Taylor expansions (30)–(37) and make use of the theorem (38) wherever appropriate, the result of expanding Eqs. (28) and (29) as power series in ϵ becomes

$$\begin{aligned}
&\sum_{p=0}^{\infty}\frac{\epsilon^p(\rho\mathbf{E}_t{}^i)^{(p)}}{p!}\rho_{10}{}^p f^p+\sum_{p=0}^{\infty}\epsilon^p\sum_{m,n}\sum_{q=0}^{p}\left(\frac{a_{\pm mn}{}^q\beta_n{}^{p-q}}{(p-q)!}\rho_{10}{}^{p-q}f^{p-q}\mathbf{m}_{\pm mn}+\frac{b_{\pm mn}{}^q\beta^{p-q+1}}{(p-q)!}\rho_{10}{}^{p-q}f^{p-q}\mathbf{n}_{\pm mn}\right)\\
&\qquad+\epsilon\alpha\rho_{10}\sum_{p=0}^{\infty}\epsilon^p\left(\frac{(E_r{}^i)^{(p)}}{p!}\rho_{10}{}^p f^p+\sum_{m,n}\sum_{q=0}^{p}b_{\pm mn}{}^q\frac{\eta_n{}^{p-q}}{(p-q)!}\rho_{10}{}^{p-q}f^{p-q}|\mathbf{p}_{\pm mn}|\right)\\
&=\kappa\sum_{p=0}^{\infty}\epsilon^p\sum_{m,n}\sum_{q=0}^{p}\left(c_{\pm mn}{}^q\frac{\alpha_n{}^{p-q}}{(p-q)!}\rho_{20}{}^{p-q}f^{p-q}\mathbf{m}_{\pm mn}+d_{\pm mn}{}^q\frac{\alpha^{p-q+1}}{(p-q)!}\rho_{20}{}^{p-q}f^{p-q}\mathbf{n}_{\pm mn}\right)\\
&\qquad+\epsilon\alpha\rho_{10}\sum_{p=0}^{\infty}\epsilon^p\sum_{m,n}\sum_{q=0}^{p}d_{\pm mn}{}^q\frac{\sigma_n{}^{p-q}}{(p-q)!}\rho_{20}{}^{p-q}f^{p-q}|\mathbf{p}_{\pm mn}|\,,
\end{aligned} \tag{39}$$

$$\begin{aligned}
&\sum_{p=0}^{\infty}\frac{\epsilon^p(\rho\mathbf{H}_t{}^i)^{(p)}}{p!}\rho_{10}{}^p f^p+\sum_{p=0}^{\infty}\epsilon^p\sum_{m,n}\sum_{q=0}\chi_1\left(a_{\pm mn}{}^q\frac{\beta_n{}^{p-q+1}}{(p-q)!}\rho_{10}{}^{p-q}f^{p-q}\mathbf{n}_{\pm mn}+b_{\pm mn}{}^q\frac{\beta_n{}^{p-q}}{(p-q)!}\rho_{10}{}^{p-q}f^{p-q}\mathbf{m}_{\pm mn}\right)\\
&\qquad+\epsilon\alpha\rho_{10}\sum_{p=0}^{\infty}\epsilon^p\left(\frac{(H_r{}^i)^{(p)}}{p!}\rho_{10}{}^p f^p+\sum_{m,n}\sum_{q=0}^{p}\chi_1 a_{\pm mn}{}^q\frac{\eta_n{}^{p-q}}{(p-q)!}\rho_{10}{}^{p-q}f^{p-q}|\mathbf{p}_{\pm mn}|\right)\\
&=\kappa\sum_{p=0}^{\infty}\epsilon^p\sum_{m,n}\sum_{q=0}^{p}\chi_2\left(c_{\pm mn}{}^q\frac{\alpha_n{}^{p-q+1}}{(p-q)!}\rho_{20}{}^{p-q}f^{p-q}\mathbf{n}_{\pm mn}+d_{\pm mn}{}^q\frac{\alpha_n{}^{p-q}}{(p-q)!}\rho_{20}{}^{p-q}f^{p-q}\mathbf{m}_{\pm mn}\right)\\
&\qquad+\epsilon\alpha\rho_{10}\sum_{p=0}^{\infty}\epsilon^p\sum_{m,n}\sum_{q=0}^{p}\chi_2 c_{\pm mn}{}^q\frac{\sigma_n{}^{p-q}}{(p-q)!}\rho_{20}{}^{p-q}f^{p-q}|\mathbf{p}_{\pm mn}|\,.
\end{aligned} \tag{40}$$

We now require that the boundary conditions (39) and (40) be satisfied term by term, i.e., that the coefficient of any given power of ϵ vanish identically.[11] Applying this requirement to the coefficient of ϵ^l, l arbitrary yields

$$\frac{(\rho\mathbf{E}_t^i)^{(l)}}{l!}\rho_{10}^l f^l+\sum_{m,n}\sum_{q=0}^{l}\left(a_{\pm mn}^q\frac{\beta_n^{l-q}}{(l-q)!}\rho_{10}^{l-q}f^{l-q}\mathbf{m}_{\pm mn}+b_{\pm mn}^q\frac{\beta_n^{l-q+1}}{(l-q)!}\rho_{10}^{l-q}f^{l-q}\mathbf{n}_{\pm mn}\right)$$
$$+\boldsymbol{\alpha}\left(\frac{(E_r^i)^{(l-1)}}{(l-1)!}\rho_{10}^l f^{l-1}+\sum_{m,n}\sum_{q=0}^{l-1}b_{\pm mn}^q\frac{\eta_n^{l-q-1}}{(l-q-1)!}\rho_{10}^{l-q}f^{l-q-1}|\mathbf{p}_{\pm mn}|\right)$$
$$=\kappa\sum_{m,n}\sum_{q=0}^{l}\left(c_{\pm mn}^q\frac{\alpha_n^{l-q}}{(l-q)!}\rho_{20}^{l-q}f^{l-q}\mathbf{m}_{\pm mn}+d_{\pm mn}^q\frac{\alpha^{l-q+1}}{(l-q)!}\rho_{20}^{l-q}f^{l-q}\mathbf{n}_{\pm mn}\right)$$
$$+\boldsymbol{\alpha}\sum_{m,n}\sum_{q=0}^{l-1}d_{\pm mn}^q\frac{\sigma_n^{l-q-1}}{(l-q-1)!}\rho_{20}^{l-q-1}f^{l-q-1}|\mathbf{p}_{\pm mn}|\,, \tag{41}$$

$$\frac{(\rho\mathbf{H}_t^i)^{(l)}}{l!}\rho_{10}^l f^l+\sum_{m,n}\sum_{q=0}^{l}\chi_1\left(a_{\pm mn}^q\frac{\beta_n^{l-q+1}}{(l-q)!}\rho_{10}^{l}f^{l-q}\mathbf{n}_{\pm mn}+b_{\pm mn}^q\frac{\beta_n^{l-q}}{(l-q)!}\rho_{10}^{l-q}f^{l-q}\mathbf{m}_{\pm mn}\right)$$
$$+\boldsymbol{\alpha}\left(\frac{(H_r^i)^{(l-1)}}{(l-1)!}\rho_{10}^l f^{l-1}+\sum_{m,n}\sum_{q=0}^{l-1}\chi_1 a_{\pm mn}^q\frac{\eta_n^{l-q-1}}{(l-q-1)!}\rho_{10}^{l-q}f^{l-q-1}|\mathbf{p}_{\pm mn}|\right)$$
$$=\kappa\sum_{m,n}\sum_{q=0}^{l}\chi_2\left(c_{\pm mn}^q\frac{\alpha_n^{l-q+1}}{(l-q)!}\rho_{20}^{l-q}f^{l-q}\mathbf{n}_{\pm mn}+d_{\pm mn}^q\frac{\alpha_n^{l-q}}{(l-q)!}\rho_{20}^{l-q}f^{l-q}\mathbf{m}_{\pm mn}\right)$$
$$+\boldsymbol{\alpha}\rho_{10}\sum_{m,n}\sum_{q=0}^{l-1}\chi_2 c_{\pm mn}^q\frac{\sigma_n^{l-q-1}}{(l-q-1)!}\rho_{20}^{l-q-1}f^{l-q-1}|\mathbf{p}_{\pm mn}|\,. \tag{42}$$

We next explicitly isolate the highest-order perturbation coefficients (corresponding to $q=l$ in the summation over q) occurring in each of Eqs. (41) and (42). After some rearrangements, this yields

$$\sum_{m,n}\left[(a_{\pm mn}^l\beta_n^0-\kappa c_{\pm mn}^l\alpha_n^0)\mathbf{m}_{\pm mn}+(b_{\pm mn}^l\beta_n^1-\kappa d_{\pm mn}^l\alpha_n^1)\mathbf{n}_{\pm mn}\right]=\mathbf{S}_l\,, \tag{43}$$

$$\sum_{m,n}\left[(\chi_1 b_{\pm mn}^l\beta_n^0-\kappa\chi_2\alpha_n^0 d_{\pm mn}^l)\mathbf{m}_{\pm mn}+(\chi_1 a_{\pm mn}^l\beta_n^1-\kappa\chi_2 c_{\pm mn}^l\alpha_n^1)\mathbf{n}_{\pm mn}\right]=\mathbf{T}_l\,, \tag{44}$$

where

$$\mathbf{S}_l=\sum_{q=0}^{l-1}\sum_{m,n}\frac{f^{l-q}}{(l-q)!}\Bigg[(\kappa c_{\pm mn}^q\alpha_n^{l-q}\rho_{20}^{l-q}-a_{\pm mn}^q\beta_n^{l-q}\rho_{10}^{l-q})\mathbf{m}_{\pm mn}+(\kappa d_{\pm mn}^q\alpha_n^{l-q-1}\rho_{20}^{l-q}-b_{\pm mn}^q\beta_n^{l-q+1}\rho_{10}^{l-q})\mathbf{n}_{\pm mn}$$
$$+\boldsymbol{\alpha}\rho_{10}\frac{l-q}{f}|\mathbf{p}_{\pm mn}|(d_{\pm mn}^q\sigma_n^{l-q-1}\rho_{20}^{l-q-1}-b_{\pm mn}^q\eta_n^{l-q-1}\rho_{10}^{l-q-1})\Bigg]-\frac{(\rho\mathbf{E}_t^i)^{(l)}}{l!}\rho_{10}^l f^l-\boldsymbol{\alpha}\frac{(E_r^i)^{(l-1)}}{(l-1)!}\rho_{10}^l f^{l-1}, \tag{45}$$

$$\mathbf{T}_l=\sum_{q=0}^{l-1}\sum_{m,n}\frac{f^{l-q}}{(l-q)!}\Bigg[(\kappa\chi_2 c_{\pm mn}^q\alpha_n^{l-q+1}\rho_{20}^{l-q}-\chi_1 a_{\pm mn}^q\beta_n^{l-q+1}\rho_{10}^{l-q})\mathbf{n}_{\pm mn}$$
$$+(\kappa\chi_2 d_{\pm mn}^q\alpha_n^{l-q}\rho_{20}^{l-q}-\chi_1 b_{\pm mn}^q\beta_n^{l-q}\rho_{10}^{l-q})\mathbf{m}_{\pm mn}+\boldsymbol{\alpha}\rho_{10}\frac{l-q}{f}|\mathbf{p}_{\pm mn}|(\chi_2 c_{\pm mn}^q\sigma_n^{l-q-1}\rho_{20}^{l-q-1}$$
$$-\chi_1 a_{\pm mn}^q\eta_n^{l-q-1}\rho_{10}^{l-q-1})\Bigg]-\frac{(\rho\mathbf{H}_t^i)^{(l)}}{l!}\rho_{10}^l f^l-\boldsymbol{\alpha}\frac{(H_r^i)^{(l-1)}}{(l-1)!}\rho_{10}^l f^{l-1}. \tag{46}$$

It is important to note that the expressions for $\mathbf{S}_l$ and $\mathbf{T}_l$ involve only known coefficients and perturbation coefficients of order lower than l.

The highest-order perturbation coefficients

$$(a,b,c,d)_{\pm mn}^l$$

may now be obtained by making use of the known

[11] For the mathematical justification of this procedure, see the comments made in I.

orthogonality properties[7]

$$\int_0^{2\pi}\int_0^{\pi}\mathbf{m}_{\pm mn}\cdot\mathbf{n}_{\pm m'n'}\sin\theta d\theta d\varphi=0\,,\qquad \text{all } m,\, n,\, m',\, n'\,, \tag{47}$$

$$\int_0^{2\pi}\int_0^{\pi}\mathbf{m}_{\pm mn}\cdot\mathbf{m}_{\pm m'n'}\sin\theta d\theta d\varphi = \int_0^{2\pi}\int_0^{\pi}\mathbf{n}_{\pm mn}\cdot\mathbf{n}_{\pm m'n'}\sin\theta d\theta d\varphi=\xi_{mn}\delta_{mm'}\delta_{nn'}\,, \tag{48}$$

where

$$\xi_{mn}=\frac{2n(n+1)(n+m)!}{(2n+1)(n-m)!}(1+\delta_{m0})\pi\,, \tag{49}$$

and where it is understood that the right-hand side of Eq. (48) should be replaced by zero for every integral of a product of an even (+) function with an odd (−) function, as well as for the product of $\mathbf{m}_{-0n}\cdot\mathbf{m}_{-0n}$, which corresponds to $m=m'=0$. Thus, if in the usual manner we successively dot Eqs. (43) and (44) into $\mathbf{m}_{\pm m'n'}$ and $\mathbf{n}_{\pm m'n'}$, and integrate over solid angle, we obtain

$$\beta_n{}^0 a_{\pm mn}{}^l-\kappa\alpha_n{}^0 c_{\pm mn}{}^l = (\xi_{mn})^{-1}\int_0^{2\pi}\int_0^{\pi}\mathbf{S}_l\cdot\mathbf{m}_{\pm mn}\sin\theta d\theta d\varphi\,, \tag{50}$$

$$\beta_n{}^1 b_{\pm mn}{}^l-\kappa\alpha_n{}^1 d_{\pm mn}{}^l = (\xi_{mn})^{-1}\int_0^{2\pi}\int_0^{\pi}\mathbf{S}_l\cdot\mathbf{n}_{\pm mn}\sin\theta d\theta d\varphi\,, \tag{51}$$

$$\chi_1\beta_n{}^0 b_{\pm mn}{}^l-\kappa\chi_2\alpha_n{}^0 d_{\pm mn}{}^l = (\xi_{mn})^{-1}\int_0^{2\pi}\int_0^{\pi}\mathbf{T}_l\cdot\mathbf{m}_{\pm mn}\sin\theta d\theta d\varphi\,, \tag{52}$$

$$\chi_1\beta_n{}^1 a_{\pm mn}{}^l-\kappa\chi_2\alpha_n{}^1 c_{\pm mn}{}^l = (\xi_{mn})^{-1}\int_0^{2\pi}\int_0^{\pi}\mathbf{T}_l\cdot\mathbf{n}_{\pm mn}\sin\theta d\theta d\varphi\,. \tag{53}$$

Equations (50)–(53) represent a set of four simultaneous linear algebraic equations for the lth-order perturbation coefficients $(a,b,c,d)_{\pm mn}{}^l$, which is trivially solved. In particular, the scattering coefficients are given explicitly by

$$a_{\pm mn}{}^l=[\xi_{mn}(\chi_1\alpha_n{}^0\beta_n{}^1-\chi_2\alpha_n{}^1\beta_n{}^0)]^{-1}\times\int_0^{2\pi}\int_0^{\pi}(\alpha_n{}^0\mathbf{T}_l\cdot\mathbf{n}_{\pm mn}-\chi_2\alpha_n{}^1\mathbf{S}_l\cdot\mathbf{m}_{\pm mn})\times\sin\theta d\theta d\varphi\,, \tag{54}$$

$$b_{\pm mn}{}^l=[\xi_{mn}(\chi_1\alpha_n{}^1\beta_n{}^0-\chi_2\alpha_n{}^0\beta_n{}^1)]^{-1}\times\int_0^{2\pi}\int_0^{\pi}(\alpha_n{}^1\mathbf{T}_l\cdot\mathbf{m}_{\pm mn}-\chi_2\alpha_n{}^0\mathbf{S}_l\cdot\mathbf{n}_{\pm mn})\times\sin\theta d\theta d\varphi\,. \tag{55}$$

Equations (54) and (55) represent explicit analytical expressions for the scattering coefficients $(a,b)_{\pm mn}{}^l$ in terms of coefficients of lower order, and may thus be used to calculate successively the scattering coefficients up to any desired order. As is well known, all scattering quantities of interest can be readily calculated in terms of these coefficients. Thus, Eqs. (54) and (55), together with the easily obtained corresponding expressions for the coefficients $(c,d)_{\pm mn}{}^l$, and in conjunction with Eqs. (45), (46), and (27), represent the complete analytical solution for the general scattering problem under consideration.

IV. REDUCTION OF THE GENERAL SOLUTION FOR SPECIAL CASES

A. First-Order Solution

For purposes of comparison with earlier results we consider the special case of the first-order solution explicitly. This may readily be obtained by substituting $l=1$ into the general solution obtained above. Making this substitution in Eqs. (45) and (46) for $\mathbf{S}_l$ and $\mathbf{T}_l$ reduces all sums over the index q to the single term $q=0$ and we accordingly obtain

$$\mathbf{S}_1=\sum_{m,n} f[(\kappa c_{\pm mn}{}^0\alpha_n{}^1\rho_{20}-a_{\pm mn}{}^0\beta_n{}^1\rho_{10})\mathbf{m}_{\pm mn}+(\kappa d_{\pm mn}{}^0\alpha_n{}^2\rho_{20}-b_{\pm mn}{}^0\beta_n{}^2\rho_{10})\mathbf{n}_{\pm mn} + \boldsymbol{\alpha} f^{-1}\rho_{10}|\mathbf{p}_{\pm mn}|(d_{\pm mn}{}^0\sigma_n{}^0-b_{\pm mn}{}^0\eta_n{}^0)]-(\rho\mathbf{E}_t{}^i)^{(1)}\rho_{10}f-\boldsymbol{\alpha}E_r{}^i(\rho_{10})\rho_{10}\,, \tag{56}$$

$$\mathbf{T}_1=\sum_{m,n} f[(\kappa\chi_2 c_{\pm mn}{}^0\alpha_n{}^2\rho_{20}-\chi_1 a_{\pm mn}{}^0\beta_n{}^2\rho_{10})\mathbf{n}_{\pm mn}+(\kappa\chi_2 d_{\pm mn}{}^0\alpha_n{}^1\rho_{20}-\chi_1 b_{\pm mn}{}^0\beta_n{}^1\rho_{10})\mathbf{m}_{\pm mn} + \boldsymbol{\alpha}\rho_{10}f^{-1}|\mathbf{p}_{\pm mn}|(\chi_2 c_{\pm mn}{}^0\sigma_n{}^0-\chi_1 a_{\pm mn}{}^0\eta_n{}^0)]-(\rho\mathbf{H}_r{}^i)^{(1)}\rho_{10}f-\boldsymbol{\alpha}\rho_{10}H_r{}^i(\rho_{10})\,, \tag{57}$$

where the zero-order coefficients $(a,b,c,d)_{\pm mn}{}^0$ correspond to the known solutions for the sphere.[9] These expressions for $\mathbf{S}_1$ and $\mathbf{T}_1$ may be conveniently reformulated in terms of the zero-order fields themselves by referring to Eqs. (9), (10), (12), and (13). We may thus write

$$\mathbf{S}_1=\rho_{10}{}^2 f[\kappa^{-1}\mathbf{E}_t{}^{02}(\rho_2)-\mathbf{E}_t{}^{0s}(\rho_1)-\mathbf{E}_t{}^i(\rho_1)]'_{r=a} + \boldsymbol{\alpha}\rho_{10}[E_r{}^{02}(\rho_2)-E_r{}^{0s}(\rho_1)-E_r{}^i(\rho_1)]_{r=a}\,, \tag{58}$$

$$\mathbf{T}_1=\rho_{10}^2 f[\kappa^{-1}\mathbf{H}_t^{02}(\rho_2)-\mathbf{H}_t^{0s}(\rho_1)-\mathbf{H}_t^{i}(\rho)]'_{r=a} +\alpha\rho_{10}[H_r^{02}(\rho_2)-H_r^{0s}(\rho_1)-H_r^{i}(\rho_1)]_{r=a}. \quad (59)$$

Here the superscript 0 on the fields represents the zero-order fields for the case of a sphere, the prime represents differentiation with respect to the appropriate argument (ρ_1 or ρ_2) of the various functions, and use has been made of the boundary conditions known to be satisfied by the zero-order fields, to wit,

$$\mathbf{E}_t^{02}-\mathbf{E}_t^{0s}-\mathbf{E}_t^{i}|_{r=a}=0,\quad \mathbf{H}_t^{02}-\mathbf{H}_t^{0s}-\mathbf{H}_t^{i}|_{r=a}=0. \quad (60)$$

The first-order expressions (58) and (59), together with Eqs. (50)–(53), agree in their essentials with the results given by Yeh.[12]

B. Scattering of a Plane Wave

The development above was carried out for the general case of an arbitrary specified incident radiation field. The case of most frequent practical interest is that where the incident radiation field corresponds to that of a plane-polarized plane wave. Accordingly, we shall here give the explicit reduction of the general solution for this case.

Without loss of generality, we assume that the incident plane wave is traveling in the positive z direction with its electric field polarized along the x axis. In addition, we shall assume a field of unit magnitude. As is well known,[7] the incident electric and magnetic fields may then be written

$$\mathbf{E}^i=e^{ik_1z}\mathbf{e}_x=\sum_{n=1}^{\infty}\nu_n(\mathbf{M}_{-1n}{}^i-i\mathbf{N}_{+1n}{}^i), \quad (61)$$

$$\mathbf{H}^i=i\chi_1e^{ik_1z}\mathbf{e}_y=\sum_{n=1}^{\infty}\chi_1\nu_n(\mathbf{N}_{-1n}{}^i-i\mathbf{M}_{+1n}{}^i), \quad (62)$$

where

$$\nu_n=i^n(2n+1)/n(n+1) \quad (63)$$

and the superscript i denotes the choice $z_n(\rho)=j_n(\rho_1)$ for the spherical Bessel functions in Eqs. (4)–(8). The coefficients corresponding to the incident fields which enter into the general expressions (45) and (46) for $\mathbf{S}_l$ and $\mathbf{T}_l$ may then be explicitly written

$$(\rho E_t{}^i)^{(l)}=\sum_{n=1}^{\infty}\nu_n(\bar{\alpha}_n{}^l\mathbf{m}_{-1n}-i\bar{\alpha}_n{}^{l+1}\mathbf{n}_{+1n}), \quad (64)$$

$$(E_r{}^i)^{(l-1)}=-i\sum_{n=1}^{\infty}\nu_n\bar{\sigma}_n{}^{l-1}|\mathbf{p}_{+1n}|, \quad (65)$$

$$(\rho\mathbf{H}_t{}^i)^{(l)}=\sum_{n=1}^{\infty}\chi_1\nu_n(\bar{\alpha}_n{}^{l+1}\mathbf{n}_{-1n}-i\bar{\alpha}_n{}^l\mathbf{m}_{+1n}), \quad (66)$$

$$(H_r{}^i)^{(l-1)}=\sum_{n=1}^{\infty}\chi_1\nu_n\bar{\sigma}_n{}^{l-1}|\mathbf{p}_{-1n}|, \quad (67)$$

where we have defined

$$\bar{\alpha}_n{}^p=\frac{d^p}{d\rho^p}\rho j_n(\rho)\bigg|_{\rho=\rho_{10}},\quad \bar{\sigma}_n{}^p=\frac{d^p}{d\rho^p}\frac{j_n(\rho)}{\rho}\bigg|_{\rho=\rho_{10}}. \quad (68)$$

Although the above modal expansions for these coefficients may sometimes be convenient, because of the analytical simplicity of the incident fields for the case of a plane wave, these coefficients may also be written in the closed form

$$(\rho\mathbf{E}_t{}^i)^{(l)}=(i\cos\theta)^{l-1}e^{i\rho_{10}\cos\theta}(l+i\rho_{10}\cos\theta)\times(\cos\theta\cos\varphi\,\mathbf{e}_\theta-\sin\varphi\,\mathbf{e}_\varphi), \quad (69)$$

$$(E_r{}^i)^{(l-1)}=(i\cos\theta)^{(l-1)}e^{i\rho_{10}\cos\theta}\sin\theta\cos\varphi, \quad (70)$$

$$(\rho\mathbf{H}_t{}^i)^{(l)}=i\chi_1(i\cos\theta)^{(l-1)}e^{i\rho_{10}\cos\theta}\times(\cos\theta\sin\varphi\,\mathbf{e}_\theta+\cos\varphi\,\mathbf{e}_\varphi), \quad (71)$$

$$(H_r{}^i)^{(l-1)}=i\chi_1(i\cos\theta)^{l-1}e^{i\rho_{10}\cos\theta}\sin\theta\sin\varphi. \quad (72)$$

V. CONCLUSION

The present paper represents the final paper in a three-part series devoted to the problem of the scattering of electromagnetic radiation by obstacles of irregular geometric shape. In I, we obtained an analytical solution to this problem for the case of cylindrical symmetry with the incident radiation field corresponding to that of a plane wave. This solution was generalized in II to apply to the case of an obstacle of a general arbitrary shape as well as an arbitrary incident radiation field. However, the results of both I and II were restricted to perfectly conducting scatterers. This restriction is removed in the present paper, which presents the solution for the case where both the scattering obstacle of arbitrary shape and its environment may have arbitrary, though homogeneous, electromagnetic properties as represented by specified values of the permittivity, permeability, and conductivity of the two media.

In each case, the solution was obtained by means of a special boundary perturbation technique, in which the scatterer of irregular shape is viewed as a perturbation of a fictitious "unperturbed sphere" whose radius and origin may be chosen such as to optimize the rapidity of convergence of the series solution. The cornerstone of this perturbation technique consists of replacing the boundary condition to be satisfied at the surface of the irregular obstacle by an infinite set of boundary conditions at the surface of the unperturbed sphere in a consistent manner. It should be emphasized that the perturbation technique was used only as a tool in obtaining the final solutions, which are exact inasmuch as no mathematical or physical approximations were introduced. Accordingly, they are equally valid in the near and far zones, as well as for all values of the incident radiation frequency. Moreover, unlike solutions obtained by other methods, such as variational

[12] C. Yeh, Phys. Rev. **135**, A1193 (1964). Our expressions for $\mathbf{S}_1$ and $\mathbf{T}_1$ are related to Yeh's vector functions $\mathbf{u}$ and $\mathbf{v}$ by means of $\mathbf{S}_1=\rho_{10}\mathbf{u}$ and $\mathbf{T}_1=\rho_{10}\mathbf{v}$, once the errors in Yeh's functions are corrected (for details, see Ref. 7 of II).

techniques, the solutions developed here are complete in that they provide exact analytical expressions not only for the various scattering cross sections but also for the fields themselves at all points of space. Such analytically complete solutions in three dimensions have heretofore been available only for the cases of a sphere and an infinite circular cylinder.

The nature of the final solutions is similar to that of the well-known Mie series for the case of the sphere, although it is, of course, more complicated in that each individual scattering coefficient is itself expressed in terms of a perturbation series. However, unlike in most perturbation techniques, we were able to obtain an exact analytical expression for the contribution of *every* order in the perturbation. This makes it possible to program the analytical results for purposes of numerical evaluation in a straightforward way such that numerical results can be obtained to any desired degree of accuracy in a completely routine and systematic manner. Although the final analytical expressions may at first sight appear forbiddingly complex, it should be kept in mind that for most "reasonably shaped" scatterers of practical interest, the infinite series will terminate after a finite number of terms by virtue of the intrinsic orthogonality properties of the circular and associated Legendre functions.

The general method presented in this series of papers for treating the problem of scattering by obstacles of arbitrary shape dealt with the scattering of electromagnetic wave fields, i.e., solutions of the vector Helmholtz equation, in the framework of a spherical coordinate system for which the sphere represents the natural choice for the "unperturbed shape." Evidently this method can readily be adapted to other coordinate systems in which the vector Helmholtz equation is separable. Thus, for example, a similar analytic treatment can be developed for the scattering of electromagnetic waves from infinite dielectric cylinders of arbitrary cross section, for which the choices of a cylindrical coordinate system and the infinite circular cylinder as the unperturbed shape are appropriate.[13] Moreover, it should also be possible to apply similar methods to the case of inhomogeneous scatterers, although in that case the radial solutions would no longer be represented by spherical Bessel functions.[14] Finally, the analytical techniques presented here may be modified for boundary value problems in the presence of arbitrarily shaped boundaries which are governed by partial differential equations other than the vector Helmholtz equation, e.g., the scattering of various types of scalar waves. It is thus to be expected that such techniques, appropriately generalized, may be found to be useful not only in the field of electromagnetic scattering, but also in areas of physics such as acoustics, astrophysics, nuclear scattering, and biophysics. The adaptation of the present boundary perturbation method to other problems, as well as the numerical evaluation of the analytical expressions for particular geometrical shapes, will be presented elsewhere.

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[13] Results for this case limited to the first order in the perturbation have been presented by C. Yeh, J. Math. Phys. **6**, 2008 (1965).

[14] P. J. Wyatt, Phys. Rev. **127**, 1837 (1962); C. Yeh, *ibid.* **131**, 2350 (1963).