Correlations at the Critical Point of the Ising Model

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An analysis is made of the fluctuations in a finite part of an infinite, d-dimensional ferromagnetic Ising lattice at and near the critical point. This leads to a formal proof that $2d/(\delta+1)$ and $2d\beta/(\gamma'+2\beta)$ are lower bounds on the exponent characterizing the asymptotic decrease with distance of the two-body correlation function at the critical point, where β, δ, and γ' are exponents representing the asymptotic forms of the coexistence curve, critical isotherm, and isothermal susceptibility.

I. INTRODUCTION

Among statistical systems which exhibit critical phenomena, the Ising model of ferromagnetism is one of the most studied, and yet, with the exception of the two-dimensional model, little exact information is available concerning its asymptotic properties at the transition. [1] In higher dimensions, numerical analyses of series expansions have provided much approximate information, but all that is known exactly is that the exponents characterizing the various asymptotic properties must be subject to certain inequalities, which have been proved by general thermodynamic arguments. Some general properties of the spin-correlation functions have been proved by Griffiths[2] for a wide class of ferromagnetic Ising models in any number of dimensions.

In this paper, we make use of Griffiths's theorems and an analysis of the fluctuations in magnetization in a finite part of an infinite lattice to derive some new exact information. We prove the following limitations on the exponent n [defined in Eq. (18)], characterizing the asymptotic decrease with distance of the two-body correlation function at the critical point for a ferromagnetic Ising lattice in d dimensions:

$$n\geqslant 2d/(\delta+1), \tag{1}$$

$$n\geqslant 2d\beta/(\gamma'+2\beta), \tag{2}$$

where we have used the usual notation for the thermodynamic exponents, defined below in Eqs. (14)–(16). These two relations are, in principle, independent, but they become the same if $\delta-1=\gamma'/\beta$, as would result from the "homogeneity assumption."[3] In two dimensions, where it is known[1] at least in the case of interactions restricted to near neighbors, that $n=\frac{1}{4}$ and $\beta=\frac{1}{8}$, (1) and (2) lead to $\delta\geqslant 15$ and $\gamma'\geqslant 7/4$, results already known from the thermodynamic inequalities and the fact that the specific heat diverges logarithmically.

In the context of general critical transitions, we have previously discussed[4] the first inequality, using physical rather than algebraic arguments. Some consequences of the theorem were discussed, particularly the fact that the Ornstein-Zernike theory of the correlation function is not in general consistent with "classical" thermodynamic behavior. Some of the results in Ref. 4 can be readily extended with the use of the second inequality above.

We confine ourselves in this paper to presenting a formal proof of the relations (1) and (2) for the ferromagnetic Ising model. The class of lattices under consideration is defined in Sec. II, where we list several known results concerning these lattices and introduce a convenient cell division. In Sec. III we establish four lemmas central to our proof, which itself is given in Sec. IV. The lemmas are statements dealing with the properties of a finite part of an infinite lattice, and are essentially the following: The fluctuation in magnetization (1) is given by a double sum of the correlation function and (2) is bounded above by the isothermal susceptibility. (3) The mean square magnetization does not decrease as the magnetic field increases at fixed temperature, nor as the temperature decreases at fixed magnetic field. (4) The double sum of the translationally invariant correlation function is essentially reducible to a single sum. These lemmas are valid for finite regions "infinitely far" from any surface of the lattice, some care being required in the proof to eliminate surface effects. The problem of extending our results to a wider class of cooperative systems is briefly discussed in Sec. V.

II. PRELIMINARIES

A. Definitions and Notation

We consider a finite Ising lattice Ω_M in d dimensions, with a spin variable $\sigma_i=\pm 1$ associated with each lattice site i, where $i=1, 2, \cdots, M$. A configuration is an assignment of a spin value to each site, there being 2^M configurations in all. The position of the ith site is specified by the vector $\mathbf{r}_i$, with $\mathbf{r}_{ij}=\mathbf{r}_i-\mathbf{r}_j$ and r_{ij}

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[1] See, for example, M. E. Fisher, Rept. Progr. Phys. 30, Part II, 615 (1967), where other references are given.

[2] R. B. Griffiths, J. Math. Phys. 8, 478 (1967); 8, 484 (1967). The proof of these theorems has been extended recently to the case of arbitrary spin [R. B. Griffiths (unpublished report)].

[3] B. Widom, J. Chem. Phys. 43, 3898 (1965).

[4] J. D. Gunton and M. J. Buckingham, Phys. Rev. Letters 20, 143 (1968).

$=|\mathbf{r}_{ij}|$. The Hamiltonian for the system is

$$\mathcal{H}=-\sum_{i<j}^{M} J_{ij}\sigma_i\sigma_j-H\sum_{i}^{M}\sigma_i, \tag{3}$$

where H is the magnetic field, and the pair-interaction energies J_{ij} are non-negative and possess the translational symmetry of the lattice; they must also satisfy some further conditions referred to below. The ensemble average of an operator x is

$$\langle x\rangle=\mathrm{Tr}(xe^{-\mathcal{H}/kT})/\mathrm{Tr}(e^{-\mathcal{H}/kT}), \tag{4}$$

where k is the Boltzmann constant, T is the temperature, and the trace is over all configurations. We will use the word "bulk" to denote the limit of the ensemble average as M tends to infinity, and will be interested specifically in the bulk magnetization, $\bar{m}=\bar{m}(t,H)$, the bulk correlation functions,[5] $g(\mathbf{r}_{ij},t,H)$ and $C(\mathbf{r}_{ij},t,H)$, and the bulk isothermal susceptibility $\chi(t,H)$, where the reduced temperature $t=(T-T_c)/T_c$ is defined in units of the critical temperature T_c. These bulk (thermodynamic) functions are defined as follows:

$$\bar{m}=\bar{m}(t,H)=\lim_{M\to\infty}\langle m\rangle, \tag{5}$$

where

$$m=M^{-1}\sum_{i}^{M}\sigma_i; \tag{6}$$

$$g(\mathbf{r}_{ij},t,H)=\lim_{M\to\infty}\langle\sigma_i\sigma_j\rangle, \tag{7}$$

$$C(\mathbf{r}_{ij},t,H)=\lim_{M\to\infty}\langle\tilde{\sigma}_i\tilde{\sigma}_j\rangle, \tag{8}$$

where (8) can be written as[6]

$$C(\mathbf{r}_{ij},t,H)=g(\mathbf{r}_{ij},t,H)-[\bar{m}(t,H)]^2, \tag{9}$$

and where we have introduced the notation

$$\tilde{x}=x-\langle x\rangle. \tag{10}$$

Finally,

$$\chi(t,H)=\lim_{M\to\infty}\chi_M(t,H), \tag{11}$$

where $\chi_M(t,H)$ is the isothermal susceptibility for the finite system, i.e.,

$$\chi_M(t,H)=kT(\partial\langle m\rangle/\partial H)_T. \tag{12}$$

Using (3) and the definitions, this becomes

$$\chi_M(t,H)=M^{-1}\sum_{i}^{M}\sum_{j}^{M}\langle\tilde{\sigma}_i\tilde{\sigma}_j\rangle. \tag{13}$$

[5] In the definition of the bulk correlation functions, Eqs. (7) and (8), the sites i and j are restricted to being infinitely far from the surface of Ω_M as M tends to infinity. This requirement is discussed later in this section.

[6] This result is only valid if $\bar{m}$ exists, in which case (see Ref. 2) $\bar{m}=\lim_{M\to\infty}\langle\sigma_i\rangle$ for any site i infinitely far, in the limit, from the surface of the lattice. The existence of $\bar{m}$ requires suitable restrictions on the interactions J_{ij} which have been discussed by R. B. Griffiths, J. Math. Phys. **5**, 1215 (1964). In particular any finite range interaction satisfies these conditions.

Care is to be taken in obtaining the value of these bulk functions in the case of zero magnetic field; in what follows we always mean that the limit $H\to\pm0$ is to be taken *after* the limit $M\to\infty$ of the ensemble average.

We are concerned here with the thermodynamic behavior of a system possessing a critical point, and primarily in the immediate neighborhood of that point, $t=0$, $H=0$. We therefore introduce a set of exponents to characterize the asymptotic form of the various functions. Thus we write for the asymptotic form[7] of the spontaneous magnetization

$$\bar{m}(t,0)\sim(-t)^{\beta},\quad t<0,\, H\to+0, \tag{14}$$

the critical isotherm

$$\bar{m}(0,H)\sim H^{1/\delta},\quad t=0,\, H>0, \tag{15}$$

the isothermal susceptibility[8] at zero field, $H\to+0$,

$$\begin{aligned}\chi(t,0)&\sim t^{-\gamma}, && t>0\\ &\sim(-t)^{-\gamma'}, && t<0\end{aligned} \tag{16}$$

and, as follows from (15),

$$\chi(0,H)\sim H^{-(\delta-1)/\delta},\quad t=0,\, H>0. \tag{17}$$

Finally, we ignore any anisotropy in the bulk correlation function for large spatial separations, and characterize its critical-point behavior by

$$C(r_{ij},0,0)\sim r_{ij}^{-n}. \tag{18}$$

Although we have taken a simple power-law description for these asymptotic properties, this restriction is not essential in any of the arguments presented in this paper. Thus, should such a description prove inadequate the results presented here could be restated to include other functional forms.

B. Griffiths's Theorems

A number of useful properties of the correlation functions have been proved by Griffiths[2] for Ω_M. In particular,

$$\langle\sigma_i\sigma_j\rangle\geqslant0 \tag{19}$$

and

$$\langle\tilde{\sigma}_i\tilde{\sigma}_j\rangle\geqslant0. \tag{20}$$

Furthermore, the bulk limits of $\langle\sigma_i\sigma_j\rangle$ and $\langle\tilde{\sigma}_i\tilde{\sigma}_j\rangle$ [given by (7) and (8)] exist, provided that the sites i and j are eventually (as $M\to\infty$) infinitely far from the surface of Ω_M. Griffiths also proved that the value of the bulk correlation function $g(\mathbf{r}_{ij},t,H)$ at given $\mathbf{r}_{ij}$

[7] We use the symbol $\sim$ to mean "asymptotically equal to" in the usual sense, i.e., $f(x)\sim g(x)$ means that the limit as $x\to x_0$ of the ratio $f(x)/g(x)$ is a finite number. In Eqs. (14)–(17), x_0 is zero, whereas in (18) it is infinity.

[8] The form (17) results immediately if χ is taken as $(\partial\bar{m}/\partial H)_T$ where $\bar{m}$ has involved the limit $M\to\infty$ [Eq. (5)], rather than as defined in Eq. (11). These definitions are equivalent if the operations $\lim(M\to\infty)$ and $\partial/\partial H$ commute. That this is so in the one phase region for the Ising model with finite range of interaction has been proved by C. N. Yang and T. D. Lee, Phys. Rev. **87**, 404 (1952); **87**, 410 (1952).

is monotonically nondecreasing in H for fixed t and monotonically nonincreasing in t for fixed H; i.e.,

$$g(\mathbf{r}_{ij},t,H)\leqslant g(\mathbf{r}_{ij},\,t,\,H+h)\leqslant g(\mathbf{r}_{ij},\,t-\tau,\,H+h)\,,\quad \tau,\,h\geqslant 0. \quad (21)$$

The requirement that the sites be infinitely far from the surface is an important one and reflects itself in the proof of our lemmas in that we focus attention only on interior cells (cells eventually infinitely far from the surface). It is in this sense that we mentioned eliminating "surface effects" in the introduction.

C. Cell Division

We divide our system Ω_M of M sites into a set of $\nu=M/N$ identically shaped cells, each cell labelled by an index α (with $\alpha=1,\,2,\,\cdots,\,\nu$) and containing N sites. We choose the shape of Ω_M and the cells to be "hypercubes", of side length $M^{1/d}$ and $N^{1/d}$, respectively. This choice is appropriate for a lattice structure which is itself "hypercubic", with unit lattice spacing. This restriction to a particular type of lattice structure and cell division is for convenience only and does not affect the generality of our results.

We identify a surface "shell" whose thickness l_M is such that

$$\lim_{M\to\infty} l_M=\infty \quad (22)$$

but

$$\lim_{M\to\infty} (l_M/M^{1/d})=0. \quad (23)$$

For example, we could choose $l_M=M^{1/2d}$. We define an *interior* cell as a cell none of whose sites lies within the surface shell. The first condition ensures that any pair of sites within any particular interior cell satisfies Griffiths's criterion for the existence of the limiting bulk correlation function as $M\to\infty$. The second condition ensures that the cells in the shell constitute only a negligible fraction of the total number of cells for large M. We will require both of these properties in the arguments below.

We define a magnetization operator m_α for cell α by

$$m_\alpha=N^{-1}\sum_{i\in\alpha}^{N}\sigma_i\,, \quad (24)$$

where the summation is over all N sites contained in the cell α. The magnetization operator for Ω_M [defined by (6)] is thus

$$m=\nu^{-1}\sum_{\alpha=1}^{\nu} m_\alpha. \quad (25)$$

D. Fluctuations in a Cell

We see from (24) that the fluctuation of magnetization in any given cell in Ω_M is given by

$$N\langle\tilde{m}_\alpha{}^2\rangle=N^{-1}\sum_{i\in\alpha}^{N}\sum_{j\in\alpha}^{N}\langle\tilde{\sigma}_i\tilde{\sigma}_j\rangle. \quad (26)$$

From (6) and (13) the fluctuation for the whole system is given by

$$M\langle\tilde{m}^2\rangle=M^{-1}\sum_{i}^{M}\sum_{j}^{M}\langle\tilde{\sigma}_i\tilde{\sigma}_j\rangle=\chi_M(t,H)\,, \quad (27)$$

where $\chi_M(t,H)$ is the isothermal susceptibility for Ω_M.

The fluctuation of magnetization in an *interior* cell α' (containing N sites) embedded in an *infinite* lattice is the bulk limit of (26), which we denote by $X_N(t,H)$:

$$X_N(t,H)=\lim_{M\to\infty} N\langle\tilde{m}_{\alpha'}{}^2\rangle\,, \quad (28)$$

with N fixed and finite. We have introduced the prime in (28) to denote that α' labels an interior cell. It is immediately seen in the next section that $X_N(t,H)$ is independent of which interior cell α' is taken.

We are considering here two quantities which, although somewhat similar, refer to quite distinct physical situations. The function $\chi_N(t,H)$ [Eq. (27) with $M=N$] represents the fluctuation of the magnetization of a finite system of N spins, Ω_N. On the other hand, $X_N(t,H)$ represents the fluctuation of magnetization for a similar set of N spins when they form part of an infinite lattice. These functions differ in two respects. Surface effects are included only in $\chi_N(t,H)$. (These may be expected to become unimportant for large enough N.) More significant is the fact that the effects of correlations from distant regions of an infinite lattice are included only in $X_N(t,H)$ and these are *not* expected to be unimportant near enough to the critical point. For large enough N and far enough from the critical point, the functions might be expected to become equal.

This is essentially a statement of the "fluctuation theorem" which in the present notation would be the assertion

$$\psi(t,H)=\chi(t,H)\,,$$

where

$$\psi(t,H)=\lim\nolimits_{N\to\infty} X_N(t,H)\,.$$

While the theorem is often assumed there appears to be no rigorous proof in the literature.[9] We do not make this assumption for the purposes of the present paper, although its use would simplify some of the arguments and would permit their extension to some of the other inequalities discussed in Ref. 4.

III. STATEMENT AND PROOF OF LEMMAS CONCERNING FLUCTUATIONS

Any two sites in an interior cell α' satisfy Griffiths's criterion, so we may make use of (8) when taking the limit as $M\to\infty$ of the finite sum in (26). With (28),

[9] See, e.g., remarks by M. E. Fisher, in *Lectures in Theoretical Physics*, VIIC, edited by W. E. Brittin (University of Colorado Press, Boulder, 1965) p. 72. The validity of the fluctuation theorem for the Ising model may be implied in some recent work by J. Lebowitz and O. Penrose (J. Lebowitz, private communication).

this gives

Lemma 1:

$$X_N(t,H)=N^{-1}\sum_{i\in\alpha'}^{N}\sum_{j\in\alpha'}^{N}C(\mathbf{r}_{ij},t,H). \qquad (29)$$

This lemma is thus a simple corollary of Griffiths's proof of the existence of the bulk correlation function.

We observe that an equation corresponding to (29) *cannot* be deduced for cells which remain near the surface as $M\to\infty$, since the limiting correlation function for sites within such cells is not, in general, $C(\mathbf{r}_{ij},t,H)$. Due to the translational invariance of the bulk correlation function and the identical structure of the cells, the fluctuation in an interior cell is in fact independent of the cell considered. That is, $X_N(t,H)$ is independent of α', and is, in particular, the limit (as $M\to\infty$) of the minimum value of the fluctuations over the set of interior cells. By a similar argument (see Ref. 6), it follows that in the limit as $M\to\infty$ the mean magnetization for any interior cell is just $\bar{m}(t,H)$. We make use of this fact and Eq. (9) to rewrite (29) as an expression for the mean square magnetization, namely,

$$\lim_{M\to\infty}N\langle m_{\alpha'}{}^2\rangle=X_N(t,H)+N(\bar{m}(t,H))^2$$
$$=N^{-1}\sum_{i}^{N}\sum_{j}^{N}g(\mathbf{r}_{ij},t,H), \qquad (30)$$

a result which will prove useful later.

We now prove that the fluctuation in an interior cell is bounded above, i.e.,

Lemma 2:

$$X_N(t,H)\leqslant\chi(t,H) \qquad (31)$$

for all t, H, and N.

The expression (27) giving the fluctuation in Ω_M can be related to the fluctuations in the individual cells [given by (26)] by splitting the sum in (27) into two parts: The first part contains those terms for which i and j are in the same cell, while the second part contains those terms for which i and j are in different cells. Thus

$$\chi_M(t,H)=M\langle\tilde{m}^2\rangle=\nu^{-1}\sum_{\alpha}^{\nu}[N^{-1}\sum_{i\in\alpha}^{N}\sum_{j\in\alpha}^{N}\langle\tilde{\sigma}_i\tilde{\sigma}_j\rangle]$$
$$+M^{-1}\sum_{\alpha\neq\beta}^{\nu}\sum^{\nu}[\sum_{i\in\alpha}^{N}\sum_{j\in\beta}^{N}\langle\tilde{\sigma}_i\tilde{\sigma}_j\rangle]. \qquad (32)$$

We obtain an inequality from (32) by dropping the second term on the right-hand side and restricting the sum over α in the first term to include only interior cells. Since from (20) $\langle\tilde{\sigma}_i\tilde{\sigma}_j\rangle\geqslant0$, this operation can only reduce the value of the right-hand side of (32), so that

$$\chi_M(t,H)\geqslant\nu^{-1}\sum_{\alpha'}^{\nu'}[N^{-1}\sum_{i\in\alpha'}^{N}\sum_{j\in\alpha'}^{N}\langle\tilde{\sigma}_i\tilde{\sigma}_j\rangle], \qquad (33)$$

where ν' is the number of interior cells. Furthermore

$$\chi_M(t,H)\geqslant\frac{\nu'}{\nu}N^{-1}\sum_{i\in\alpha^*}^{N}\sum_{j\in\alpha^*}^{N}\langle\tilde{\sigma}_i\tilde{\sigma}_j\rangle, \qquad (34)$$

where α^* denotes any particular interior cell for which the double sum in brackets in (33) achieves its minimum value over the set of interior cells. Taking the limit[10] of (34) as $M\to\infty$, keeping N fixed and finite, and noting that the sum for a cell α^* becomes equal to that for any interior cell α', we obtain

$$\chi(t,H)\geqslant N^{-1}\sum_{i\in\alpha'}^{N}\sum_{j\in\alpha'}^{N}C(\mathbf{r}_{ij},t,H). \qquad (35)$$

With (29), this proves the lemma (31).

Our use of Lemma 2 will involve taking advantage of the fact that it applies in particular for values of t or H which are N-dependent and would vanish as $N\to\infty$. The lemma provides us with an upper bound on the fluctuations even when they are becoming anomalously large.

We now observe that the monotonic properties of the correlation function $g(\mathbf{r}_{ij},t,H)$, as expressed by (21), ensure that the mean square magnetization of a finite part of an infinite system does not decrease as H increases at fixed t, nor as t decreases at fixed H. That is, upon inserting (21) into (30) we find[11]

Lemma 3:

$$X_N(t,H)+N[\bar{m}(t,H)]^2\leqslant X_N(t-\tau,\,H+h)$$
$$+N[\bar{m}(t-\tau,\,H+h)]^2 \qquad (36)$$

for all N and for any $\tau,h\geqslant0$.

Our last lemma is concerned with the reduction of the double sum of the translationally invariant bulk cor-

[10] Even when $\chi(t,H)$ does not exist as a finite number (for example, at the critical point), (35) is still true, since the right-hand side is necessarily finite for finite N. Thus (31) applies for all values of t, H and N. It is perhaps worth pointing out in connection with (31) that the quantity $X_N(t,H)$ can itself be thought of as a kind of isothermal susceptibility. Thus, consider applying in addition to the homogeneous magnetic field H a spatially varying field which is zero everywhere except in some interior cell α, wherein the field is constant and equal to h. Then the "inhomogeneous isothermal susceptibility" for the cell is $kT(\partial\langle m_\alpha\rangle/\partial h)_{H,T}=N\langle m_\alpha{}^2\rangle$, so that from Eq. (28), $X_N(t,H)$ is the limit of this susceptibility as h tends to zero and M tends to infinity. Hence the ordinary isothermal susceptibility is an upper bound for the inhomogeneous isothermal susceptibility.

[11] In Ref. 4, [Eq. (5)], a property A_ϵ of a distribution at some temperature T and symmetric at $\bar{m}=0$ was defined as follows: If the mean square deviation of m (in a cell) from its mean value $\bar{m}$ is σ^2 when $\bar{m}=0$, its value at $\bar{m}=\epsilon\sigma$ is given by $A_\epsilon\sigma^2$. On the grounds that the distribution of m is insensitive to changes of $\bar{m}$ small compared with the "noise level" σ, it was claimed that A_ϵ remains near unity for $\epsilon\ll1$. It was stated that for a two-δ-function distribution corresponding to two-phase equilibrium $A_\epsilon=1-\epsilon^2$, and for a Gaussian distribution $A_\epsilon=1$. From Lemma 3 above, it follows that indeed for the Ising Model $A_\epsilon\geqslant1-\epsilon^2$ for all temperatures.

relation function which occurs in (29), and is written in two parts.[12]

Lemma 4a:

$$(1-\xi)^d \sum_j^{(\xi^d N)} C(\mathbf{r}_{0j},t,H) \leqslant N^{-1}\sum_i^N \sum_j^N C(\mathbf{r}_{ij},t,H) \leqslant \sum_j^{(2^d N)} C(\mathbf{r}_{0j},t,H), \quad (37)$$

where the subscript 0 denotes a particular site at the center of the hypercube containing the sites included in the sum, and where ξ is any number (which could depend on N) such that $0\leqslant\xi\leqslant 1$.

Lemma 4b:

$$\lim_{N\to\infty} N^{-1}\sum_i^N \sum_j^N C(\mathbf{r}_{ij},t,H) = \sum_j^{\infty} C(\mathbf{r}_{0j},t,H) \quad (38)$$

so long as the sums converge.

Since it follows from (8) and (20) that $C(\mathbf{r}_{ij},t,H)\geqslant 0$, adding a region of summation to the double sum can only increase its value, while subtracting a region can only decrease it. Thus, to obtain an upper bound on the double sum we choose, for each particular site i in the sum, to extend the region of summation over j beyond the original sites to include all those (2^dN) sites in the hypercube of side length $2N^{1/d}$ centered about i. Since $C(\mathbf{r}_{ij},t,H)$ is translationally invariant, this sum over j is now independent of the site i, so that the sum over i gives merely a factor N. Thus we obtain

$$N^{-1}\sum_i^N \sum_j^N C(\mathbf{r}_{ij},t,H) \leqslant \sum_j^{(2^d N)} C(\mathbf{r}_{0j},t,H), \quad (39)$$

where the site 0 is at the center of a hypercube of 2^dN sites. Similarly, to obtain a lower bound we first discard terms in the sum for which the sites i do not lie within a centrally located hypercube of reduced side length $(1-\xi)N^{1/d}$, where ξ is any number (possibly N-dependent) satisfying $0\leqslant\xi\leqslant 1$. Then, for any site i in this reduced hypercube, we choose a hypercube of length $\xi N^{1/d}$ centered about i (all of whose sites lie within the original summation), to which we restrict the sum over j. Again, because of the translational invariance of $C(\mathbf{r}_{ij},t,H)$, this sum over j is independent of i so that performing the sum over i this time gives a factor $(1-\xi)^dN$. Therefore

$$(1-\xi)^d \sum_j^{(\xi^d N)} C(\mathbf{r}_{0j},t,H) \leqslant N^{-1}\sum_i^N \sum_j^N C(\mathbf{r}_{ij},t,H), \quad (40)$$

where again 0 labels the site at the center of the hypercube of summation.

The results (39) and (40) establish (37), which we now examine in the limit as $N\to\infty$. Since (37) is true for any value of ξ satisfying $0\leqslant\xi\leqslant 1$, including an N-dependent ξ, we can choose ξ to be a function of N such that $\xi\to 0$ but $(\xi^dN)\to\infty$ as $N\to\infty$. For example, we could choose $\xi^d=N^{-a}$, with $0<a<1$. Taking the limit as $N\to\infty$ of (37) then immediately yields (38), since the upper and lower bounds both equal the single sum so long as it converges.

IV. PROOF OF CORRELATION-FUNCTION INEQUALITIES

It is now an easy matter to make use of the above lemmas to establish our theorem. We combine (29) and (37) to obtain upper and lower bounds on the fluctuations in an interior cell:

$$(1-\xi)^d \sum_j^{(\xi^d N)} C(\mathbf{r}_{0j},t,H) \leqslant X_N(t,H) \leqslant \sum_j^{(2^d N)} C(\mathbf{r}_{0j},t,H). \quad (41)$$

We consider the value of $X_N(t,H)$ at the critical point $t=0$, $H=0$, where the sums appearing in (41) would tend to infinity as $N\to\infty$. This divergence is due to the asymptotic spatial dependence of $C(\mathbf{r}_{ij},0,0)$, which we take to be given by (18); only for $n>d$ would the correlation function decrease sufficiently rapidly for the sums to converge. Since we are interested in obtaining the asymptotic N dependence of $X_N(0,0)$ from (41), we choose ξ to be a number between zero and unity which is *independent* of N. Then, as can be seen by replacing the sums in (41) by their corresponding integrals, we find that for large enough N (the precise value depending on the detailed form of the correlation function) there exist upper and lower bounds for $X_N(0,0)$, both proportional to $N^{(d-n)/d}$. Thus there is a finite positive number A, independent of N, such that

$$X_N(0,0)\geqslant AN^{(d-n)/d}. \quad (42)$$

We now combine Lemmas 2 and 3 [inequalities (31) and (36)] and note that $\bar{m}(0,0)=0$ to obtain an upper bound on $X_N(0,0)$:

$$X_N(0,0)\leqslant X_N(-\tau,h)+N[\bar{m}(-\tau,h)]^2 \leqslant \chi(-\tau,h)+N[\bar{m}(-\tau,h)]^2 \quad (43)$$

in the domain $\tau,h\geqslant 0$. At the critical point itself ($\tau=0$, $h=0$) the inequality (43), while valid gives no information about $X_N(0,0)$ since $\chi(-\tau,h)$ becomes infinite. Moving away from the critical point along a path in any direction in the domain $\tau,h\geqslant 0$, $\bar{m}^2$ increases from zero while χ decreases from its infinite value at the critical point, say, like $\bar{m}^{-q}$. For the extreme paths

[12] This reduction of the double sum of a translationally invariant function to a single sum, as given by (38), is of course very reasonable and is often made without proof in the literature. The proof given here can be extended easily to include translationally invariant functions which are bounded, but whose sign is not restricted to being positive, so long as the sum is absolutely convergent, or indeed so long as the sum of such a function minus a constant is absolutely convergent.

corresponding to the critical isotherm and the coexistence curve χ decreases as $\bar{m}$ increases as $\bar{m}^{-(\delta-1)}$ and $\bar{m}^{-\gamma'/\beta}$, respectively, as can be seen from the definitions (11)–(14). If we evaluate (43) at a point[13] on the path where $\bar{m}$ is proportional to $N^{-1/(2+q)}$, the two terms on the right are both proportional to the same power, $q/(2+q)$ of N. Writing this power as $(d-\hat{n})/d$, i.e., defining $\hat{n}=2d/(2+q)$ the inequality (43) with (42) shows that $n \geqslant \hat{n}$. For the critical isotherm and the coexistence curve $\hat{n}$ takes the values $2d/(\delta+1)$ and $2d\beta/(\gamma'+2\beta)$, which are thus lower bounds on the index n proving our results (1) and (2).

V. DISCUSSION

We have now completed the proof that for a ferromagnetic Ising lattice the rate of decrease with distance of the correlation function at the critical point is subject to certain restrictions embodied in relations (1) and (2). Before concluding we discuss briefly the possibility of rigorously proving this theorem for more general cooperative systems.[14]

We first note that our considerations for the Ising model say nothing of the conditions that may be necessary for the existence of a critical point. They merely result in the statement that if such a singularity exists, it must conform to the restrictions imposed by the theorem. Griffiths's theorems and the lemmas happen to be true for any thermodynamic state, whether or not it is near the critical point. This lack of specific reference to the critical region is a source of difficulty in attempts to generalize to a wider class of systems, since then the conditions used in the proof are certainly not all true for all thermodynamic states. Indeed some are not even precisely true in the critical region. At the same time they are by no means *necessary* conditions for the validity of the theorem either, although we have seen that they are sufficient. In fact the conditions can be relaxed considerably. For example, the positivity property of the correlation function, which was a very useful condition in our proof for the Ising model, is clearly far from a necessary one. Negative excursions of $C(\mathbf{r})$ have little effect so long as they do not overwhelm the positive contributions to the various summations involved in the analysis.

What *are* crucial to our proof are two assertions concerning the fluctuations in a finite part (containing N sites or atoms) of an infinite system:

(i) The fluctuations have an upper bound which is a multiple F times the isothermal susceptibility:

$$X_N(t,H) \leqslant F\chi(t,H).$$

For the Ising lattice, $F=1$, but it is sufficient for the proof of the theorem that F does not increase without limit as $N \to \infty$, for states very near the critical point. At the critical point itself, of course, the condition is obviously satisfied.

(ii) Along appropriate thermodynamic paths (increasing $\bar{m}$ at constant or decreasing temperature) the mean square magnetization has a lower bound which is a multiple f of its value when the mean magnetization is zero. For the Ising lattice $f=1$ but it is sufficient that f does not vanish as $N \to \infty$.

Eminently reasonable as these assertions are, we have not succeeded in rigorously proving them for critical transitions in general.

[13] It should be noted that the thermodynamic state corresponding to this point is in the one-phase region. Its distance from the critical point would, of course, vanish in the limit $N \to \infty$, but we deal here with large yet finite values of N.

[14] Several inequalities, including (1), have been obtained recently by T. R. Choy and F. H. Rees (unpublished report) for the gas-liquid system, on the assumption that a certain series expansion for the cell-pair correlation function is bounded in each term and is convergent.