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Parametric Excitations in Plasma in a Magnetic Field

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The coupled Vlasov equations for electrons and ions in the presence of a strong long-wavelength electromagnetic field are solved. Calculations of the growth rates for the excitation of the upper-hybrid, lower-hybrid, and the Bernstein modes are presented and discussed.

I. INTRODUCTION

Parametric excitation of plasma waves has been of considerable interest.[1] Here the parametric excitation describes the nonlinear coupling of a high-frequency electric field to low- and high-frequency density oscillation modes of the plasma. This nonlinear coupling can transfer energy from the high-frequency electric field to low-frequency modes and, therefore, drives low-frequency instabilities. Several authors have considered the excitation of electron plasma oscillations and ion acoustic waves via radiation or longitudinal fields.[1] In this paper we consider the parametric excitation of density waves for a plasma embedded in a dc magnetic field. The plasma may either be a collection of mobile electrons in a semiconductor, treated in the one-band effective mass approximation, or a high-temperature gas plasma, e.g., as obtained in a gas discharge. In semiconductors the electrons couple to the phonons [the ion (lattice) mode of excitation] through their self-consistent field. Similarly, in gas plasmas the electronic and ionic motions are coupled by their locally induced fields. The magnetic field adds a variety of new modes to the system. The spectrum of the collective oscillations of the electrons includes the upper-hybrid mode and the infinite set of Bernstein modes near the cyclotron harmonics. Since the magnetic field is also easily tunable in the laboratory, a study of the density excitations for a range of magnetic fields will give new and interesting information about the spectrum of fluctuations in the plasma. Although fluctuation can be excited at any angle relative to the magnetic field $\vec{B}$, we will only consider the particular geometry where the wave number $\vec{k}$ of the fluctuations is perpendicular to $\vec{B}$. This geometry is interesting because it allows us to study the collective effects over an extremely wide range of k. In fact, in this geometry, the boundary in k space between the collective and single-particle regions is not well defined. It is possible to follow the collective excitations to large k where they merge continuously into the single-particle resonance spectrum at ω_c, $2\omega_c$, $3\omega_c$, $\cdots$, etc.

In the general geometry, where $\vec{k}$ has a component parallel to the magnetic field $\vec{B}$, the collective modes are strongly damped for $k_{\parallel} \geq k_D$, where k_D is the Debye wave number. This damping, the Landau damping, is due to the coupling of the collective mode to the single-particle continuum. Unlike the case $\vec{k}$ perpendicular to $\vec{B}$ the single-particle excitations are not discrete, and the fluctuation spectrum changes qualitatively as k increases through k_D.

In the long-wavelength limit, the nonlinear coupling of a radiation field to the spectrum of the density fluctuations can be treated analytically. For $k/k_D \leq 1$ the parametric excitation of the lower- and upper-hybrid modes is dominant, since this has the highest growth rate. For the case of $k/k_D \leq 0.5$ parametric excitation of the mode at the cyclotron harmonics is possible for relatively low threshold fields. In fact we will show that the growth rate for excitation of the upper hybrid (lower hybrid) is independent of k/k_D for $k/k_D < 1$. On the contrary, the growth rate for excitation of the mode at the cyclotron harmonic is proportional to (k/k_D) for $(k/k_D) < 1$.

For the case $k/k_D \gtrsim 1$ we do not have closed-form analytic solutions for the growth rate of the parametrically excited density fluctuations. Here one should look for numerical solutions.

Our calculations were motivated by recent experiments which demonstrated that the lower- and upper-hybrid oscillations and the Bernstein modes in a plasma could be simultaneously excited by a radiation field.[2,3] We have calculated the growth rates for these nonlinear instabilities and obtained the expressions for the growth rates needed for the excitation of other modes in the plasma. Parametric excitation in a magnetic field was discussed by Aliev *et al.*[4]; however, they did not calculate the growth rates for these particular cases.

II. CALCULATIONS OF THE INSTABILITIES

Consider an electron-ion plasma in a homogeneous dc magnetic field. The dynamics of the plasma is described by the Vlasov equations for electrons and ions. We assume that the plasma is subject to a homogeneous oscillatory electric field of frequency ω_0. We solve for the response of the plasma to the electric field using a perturbation expansion. Our expansion parameter is the amplitude of the density fluctuations. The zeroth-order equation for the electron distribution function in the presence of the external electric field is

$$\frac{\partial f_0}{\partial t}+\omega_c(\vec{v}\times\vec{b})\cdot\frac{\partial f_0}{\partial\vec{v}}=\frac{e}{m}\vec{E}_0\sin\omega_0 t\cdot\frac{\partial f_0}{\partial\vec{v}}, \quad (1)$$

with a similar equation for the ions. Here $\vec{b}=\vec{B}/B$, where $\vec{B}$ is the dc magnetic field and $\vec{E}_0$ is the external ac electric field. The distribution function f_0 is spatially uniform and describes electrons with a homogeneous oscillatory velocity field driven by the external field. The excitation of density fluctuations for the electrons may be found in the next order of the approximation. The distribution function obeys a linearized Vlasov equation:

$$\frac{\partial f_1}{\partial t}+\vec{v}\cdot\frac{\partial f_1}{\partial\vec{x}}+\omega_c(\vec{v}\times\vec{b})\cdot\frac{\partial f_1}{\partial\vec{v}}+\frac{e}{m}\frac{\partial\varphi_1}{\partial\vec{v}}\cdot\frac{\partial f_0}{\partial\vec{v}}=\frac{e}{m}\vec{E}_0\sin\omega_0 t\cdot\frac{\partial f_1}{\partial\vec{v}}. \quad (2)$$

The spatially dependent ion distribution function F_1 obeys a similar equation. The self-consistent field obeys the Poisson equation

$$\frac{\partial}{\partial\vec{x}}\left(\frac{\partial\varphi_1}{\partial\vec{x}}\right)=4\pi e n_0\int d\vec{v}(f_1-F_1), \quad (3)$$

where n_0 is the average electron and ion density. In the linearization procedure we have used, no assumption was made about the magnitude of the external field $\vec{E}_0$. The nonlinear term dropped from Eq. (2) is of the order $f_1\varphi_1$. Only the initial growth rate of the instability is given correctly by the solution of Eq. (2). It determines the spectrum of the instability, but does not describe the long-time behavior of the fluctuations. The solutions for f_0, $f_1(F_0, F_1)$ are given, for the electrons (ions), in Ref. (4). Although the transformed coordinates in Ref. 4 are given in implicit form, i.e., as a time integral these new coordinates can be determined explicitly when E_0 is homogeneous, we find them to be

$$\begin{aligned}\vec{u}&=\vec{v}-\vec{\xi}_e, & \vec{U}&=\vec{v}-\vec{\xi}_i,\\ \vec{r}&=\vec{x}-\vec{\xi}_e, & \vec{R}&=\vec{x}-\vec{\xi}_i,\\ t&=t, & t&=t,\end{aligned} \quad (4)$$

where

$$\vec{\xi}_e=\vec{\epsilon}_e\cos\omega_0 t+(1-\omega_c^2/\omega_0^2)^{-1}\times\{(\omega_c/\omega_0)(\vec{\epsilon}_e\times\vec{b})\sin\omega_0 t+(\omega_c^2/\omega_0^2)[\vec{b}\times(\vec{\epsilon}_e\times\vec{b})]\cos\omega_0 t\}, \quad (5)$$

$$\omega_0\vec{\xi}_e=-\vec{\epsilon}_e\sin\omega_0 t+(1-\omega_c^2/\omega_0^2)^{-1}\times\{(\omega_c/\omega_0)(\vec{\epsilon}_e\times\vec{b})\cos\omega_0 t-(\omega_c^2/\omega_0^2)[\vec{b}\times(\vec{\epsilon}_e\times\vec{b})]\sin\omega_0 t\}, \quad (6)$$

with $\vec{\epsilon}_e=e\vec{E}_0/m\omega_0$. (7)

Similarly one writes for the ions $\vec{\epsilon}_i=e\vec{E}_0/M\omega_0$ and defines $\vec{\xi}_i, \vec{\xi}_i$ as in Eqs. (5) and (6) with $\vec{\epsilon}_e\to\vec{\epsilon}_i$ and $\omega_c\to-\Omega_c$. The solution of the distribution functions is given by[4]

$$f_0=f_m(\vec{u}),\quad F_0=F_m(\vec{U}), \quad (8)$$

and

$$f_1=f_{\vec{k}}(\vec{u},t)e^{i\vec{k}\cdot\vec{r}},\quad F_1=F_{\vec{k}}(\vec{U},t)e^{i\vec{k}\cdot\vec{R}}. \quad (9)$$

Here $f_m(F_m)$ is the Maxwell-Boltzman distribution function for the electrons (ions). The equations of motion which f_k and F_k satisfy are given by

$$\frac{\partial f_{\vec{k}}}{\partial t}+i\vec{k}\cdot\vec{u}f_{\vec{k}}+\omega_c(\vec{u}\times\vec{b})\cdot\frac{\partial f_{\vec{k}}}{\partial\vec{u}}-\frac{\omega_p^2}{k^2}i\vec{k}\cdot\frac{\partial f_0}{\partial\vec{u}}[n_{\vec{k}}-N_{\vec{k}}e^{i\vec{k}\cdot(\vec{\xi}_e-\vec{\xi}_i)}]=0, \quad (10)$$

and

$$\frac{\partial F_{\vec{k}}}{\partial t}+i\vec{k}\cdot\vec{U}F_{\vec{k}}-\Omega_c(\vec{U}\times\vec{b})\cdot\frac{\partial F_{\vec{k}}}{\partial\vec{U}}-\frac{\Omega_p^2}{k^2}i\vec{k}\cdot\frac{\partial F_0}{\partial\vec{U}}[N_{\vec{k}}-n_k e^{-i\vec{k}\cdot(\vec{\xi}_e-\vec{\xi}_i)}]=0, \quad (11)$$

where n_k and N_k are defined by

$$n_k=\int d\vec{u}f_{\vec{k}}(\vec{u},t);\quad N_{\vec{k}}(t)=\int d\vec{U}F_{\vec{k}}(\vec{U},t). \quad (12)$$

If we note that $\xi_e/\xi_i\sim M/m$ we can approximate $\vec{\xi}_e-\vec{\xi}_i\approx\vec{\xi}_e$. We consider only the case when the density fluctuations propagate perpendicularly to the magnetic field. Let the magnetic field $\vec{B}$ be in the $\hat{z}$ direction, and without loss of generality choose $\vec{E}_0$ to be in the $\hat{x}$ direction, i.e., $\vec{E}_0(t)=\hat{x}E_0\sin\omega_0 t$ and $\vec{k}=(k_x,k_y,0)$. For this particular geometry we obtain

$$\vec{k}\cdot\vec{\xi}_e=-\lambda\sin(\omega_0 t+\varphi)\to-\lambda\sin\omega_0 t, \quad (13)$$

where

$$\lambda=\frac{eE_0}{m\omega_0^2}\left(1-\frac{\omega_c^2}{\omega_0^2}\right)^{-1}\left(k_x^2+k_y^2\frac{\omega_c^2}{\omega_0^2}\right)^{\frac{1}{2}}. \quad (14)$$

Here λ represents the ratio of the excursion of the electron relative to the ion under the effect of the ac external field to a characteristic wavelength of the oscillation. In Eq. (13) we have omitted the phase factor φ since the parametric excitation is independent of initial phases. The solution of the coupled Eqs. (10)–(12) may be obtained, using standard techniques.[5] One finds that the final result consists of an infinite set of coupled equations for the electron and ion density fluctuations[4] n_k and N_k:

$$\epsilon_e(\omega + l\omega_0)n(\omega + l\omega_0) + [\epsilon_e(\omega + l\omega_0) - 1] \times \sum_{S=-\infty}^{+\infty} J_{S-l}(\lambda)N(\omega + s\omega_0) = 0 , \quad (15)$$

$$\epsilon_i(\omega + l\omega_0)N(\omega + l\omega_0) + [\epsilon_i(\omega + l\omega_0) - 1] \times \sum_{S=-\infty} J_{S-l}(-\lambda)n(\omega + s\omega_0) = 0 . \quad (16)$$

Here ϵ_e and ϵ_i are respectively the well-known dielectric function in random-phase approximation for electrons and ions in our geometry for frequency ω and wave number $\vec{k}$. The wave number $\vec{k}$ is omitted for simplicity. Equations (15) and (16) are the exact analog of the well-known expressions for parametric excitation in the absence of the magnetic field. The effects of the magnetic field are buried in the behavior of the dielectric functions ϵ_i and ϵ_e.

III. CALCULATION OF THE INSTABILITIES

In order to extract useful information from the result given by Eqs. (15) and (16), let us limit ourselves to the two-mode instability. We now consider the case when only one low-frequency mode ω, and one high-frequency mode at ω-ω_0 are excited. Here ω_0 is the angular frequency of the radiation field. This approximation requires ω to be larger than the inverse lifetime of the mode at ω-ω_0. In this limit we obtain four coupled equations which read

$$\begin{aligned}
&\epsilon_e(\omega)n(\omega) + [\epsilon_e(\omega) - 1][J_0(\lambda)N(\omega) - J_1(\lambda)N(\omega - \omega_0)] = 0 ,\\
&\epsilon_e(\omega - \omega_0)n(\omega - \omega_0) + [\epsilon_e(\omega - \omega_0) - 1][J_0(\lambda)N(\omega - \omega_0) + J_1(\lambda)N(\omega)] = 0 ,\\
&\epsilon_i(\omega)N(\omega) + [\epsilon_i(\omega) - 1][J_0(\lambda)n(\omega) + J_1(\lambda)n(\omega - \omega_0)] = 0 ,\\
&\epsilon_i(\omega - \omega_0)N(\omega - \omega_0) + [\epsilon_i(\omega - \omega_0) - 1][J_0(\lambda)n(\omega - \omega_0) - J_1(\lambda)n(\omega)] = 0.
\end{aligned} \quad (17)$$

The solution of Eq. (17) is straightforward: After some algebra we obtain the nonlinear dispersion relation for the case $\lambda \ll 1$ (as in realistic laboratory situations) to be

$$\epsilon(\omega)\epsilon(\omega - \omega_0) - \tfrac{1}{4}\lambda^2[\epsilon_e(\omega) - \epsilon_e(\omega - \omega_0)][\epsilon_i(\omega) - \epsilon_i(\omega - \omega_0)] = 0 \quad (18a)$$

with $\epsilon(\omega) = \epsilon_e(\omega) + \epsilon_i(\omega) - 1$. (19a)

For the electron-phonon system we obtain a similar result[6]:

$$\epsilon(\omega)\epsilon(\omega - \omega_0) + \tfrac{1}{4}\lambda^2(|\nu_k|^2/\varphi_k)[\epsilon_e(\omega) - \epsilon_e(\omega - \omega_0)][D(\omega - \omega_0) - D(\omega)] = 0, \quad (18b)$$

where $\epsilon(\omega) = \epsilon_e(\omega) - (|\nu_k|^2/\varphi_k)[1 - \epsilon_e(\omega)]D(\omega)$. (19b)

Here ν_k and φ_k are respectively the electron-phonon and electron-electron interactions and $D(\omega) = (\omega + i\eta)^2 - \Omega_k^2)$ is the phonon propagator. The solution of the dispersion relations, Eq. (18a), for $\lambda = 0$ is given by $\epsilon(\omega) = \epsilon(\omega - \omega_0) = 0$. Thus it follows that ω and $\omega - \omega_0$ respectively must be identified with a low- and a high-resonance frequency given by the zeros of the total dielectric function. In the case of small λ we may assume that the solution of Eq. (18a) is close enough to the zeros of $\epsilon(\omega)$. We next define the low- (high-) frequency roots of $\epsilon(\omega)$ by $\omega_L(\omega_H)$ and expand;

$$\epsilon(\omega) = \epsilon(\omega_L) + (\omega^2 - \omega_L^2)\left.\frac{\partial\epsilon(\omega)}{\partial\omega^2}\right|_{\omega^2=\omega_L^2} + \cdots ,$$

where $\epsilon(\omega_L) = 0$. Equation (18a) is given now to dominant order by

$$(\omega^2 - \omega_L^2)[(\omega - \omega_0)^2 - \omega_H^2] = \tfrac{1}{4}\lambda^2\chi , \quad (20)$$

$$\text{where}\quad \chi = [\epsilon_e(\omega_L) - \epsilon_e(\omega_H)][\epsilon_i(\omega_L) - \epsilon_i(\omega_H)] \bigg/ \frac{\partial\epsilon(\omega_L)}{\partial\omega_L^2}\frac{\partial\epsilon(\omega_H)}{\partial\omega_H^2} . \tag{21}$$

The dielectric function $\epsilon_e(\omega)$ $[\epsilon_i(\omega)]$ has been calculated in the random-phase approximation and is given for our geometry $(\vec{k} \perp \vec{B})$ by[7]

$$\epsilon_e(\omega) = 1 - \frac{\omega_p^2}{\omega_c^2}\frac{1}{z}\left([e^{-z}I_0(z) - 1] + 2\frac{\omega}{\omega_c}\sum_{n=1}^{\infty}\frac{e^{-z}I_n(z)}{(\omega/\omega_c)^2 - n^2}\right), \tag{22}$$

where ω_p and ω_c are respectively the electron-plasma and electron-cyclotron frequencies, $z = k^2 v_{th}^2/\omega_c^2$, and I_n is the Bessel function of the second kind. The ion dielectric function $\epsilon_i(\omega)$ is given by a similar equation with $\omega_p \to \Omega_p$, $\omega_c \to -\Omega_c$ and $z \to x = k^2 V_{th}^2/\Omega_c^2$. Here Ω_p and Ω_c are respectively the ion-plasma and ion-cyclotron frequencies. The growth rate of the parametrically excited density waves can be easily obtained from Eq. (20) for growth rates $\gamma < 2\omega_L$. We obtain using Eq. (14) the result

$$|\gamma| = \tfrac{1}{4}[ek_x E_0/m(\omega_0^2 - \omega_c^2)]\,(1 + k_y^2\omega_c^2/k_x^2\omega_0^2)^{1/2}\,(\chi/\omega_L\omega_H)^{1/2} . \tag{23}$$

The relation between γ and the condition for the onset of instability in dissipative systems is derived in the Appendix.

The growth rate γ can be calculated analytically for the long-wavelength case. Here, for z, $x < 1$ we may write

$$\begin{aligned} \epsilon_e &= 1 - \omega_p^2[(1-z)/\omega^2 - \omega_c^2) + z/(\omega^2 - 4\omega_c^2)] , \\ \epsilon_i &= 1 - \Omega_p^2[(1-x)/(\omega^2 - \Omega_c^2) + x/(\omega^2 - 4\Omega_c^2)] , \end{aligned} \tag{24}$$

and thus obtain for χ using $x, z \ll 1$ and $\omega_c, \omega_p, \omega_H \gg \Omega_c, \Omega_p, \omega_L$

$$\chi = \frac{\left[\frac{1}{\omega_c^2} + \left(\frac{1-z}{\omega_H^2 - \omega_c^2} + \frac{z}{\omega_H^2 - 4\omega_c^2}\right)\right]\left(\frac{1-x}{\omega_L^2 - \Omega_c^2} + \frac{x}{\omega_c^2 - 4\Omega_c^2}\right)}{\left(\frac{1-z}{(\omega_H^2 - \omega_c^2)^2} + \frac{z}{(\omega_H^2 - 4\omega_c^2)^2}\right)\left(\frac{1-x}{(\omega_L^2 - \Omega_c^2)^2} + \frac{x}{(\omega_L^2 - 4\Omega_c^2)^2}\right)} . \tag{25}$$

Now ω_H and ω_L are given by the zeros of the dielectric function $\epsilon(\omega) = 0$. We obtain, using Eqs. (19a), (24), and (25), four roots given by

$$\omega_H^2 = \omega_p^2 + \omega_c^2[1 - 3\omega_p^2 z/(3\omega_c^2 - \omega_p^2)] , \tag{26}$$

$$\omega_H^2 = 4\omega_c^2[1 + \tfrac{3}{4}\,\omega_p^2 z/(3\omega_c^2 - \omega_p^2)] , \tag{27}$$

$$\omega_L^2 = \bar{\Omega}_p^2 + \Omega_c^2[1 - 3\bar{\Omega}_p^2 x/(3\Omega_c^2 - \bar{\Omega}_p^2)] , \tag{28}$$

$$\omega_L^2 = 4\Omega_c^2[1 + \tfrac{3}{4}\bar{\Omega}_p^2 x/(3\Omega_c^2 - \bar{\Omega}_p^2)] , \tag{29}$$

$$\text{where}\quad \bar{\Omega}_p^2 = \Omega_p^2/[1 + (\omega_p/\omega_c)^2(1 - \tfrac{5}{4}z)] . \tag{30}$$

We consider now the growth rate for the upper-hybrid and lower-hybrid modes given in Eqs. (26) and (28). Using Eq. (25) we find to a good approximation that $\chi = \omega_p^2\Omega_p^2$. This is the result obtained for plasmas free of magnetic field for the electron-plasma and ion-acoustic modes. Since χ represents the strength of the nonlinear coupling between the radiation field and the density fluctuations, we conclude that this coupling is not altered by the presence of the magnetic field. In the calculation of the growth rate, Eq. (23), we substitute $\omega_0 = \omega_L + \omega_H \approx \omega_H \approx (\omega_p^2 + \omega_c^2)^{1/2}$. The coupling of the electric field E_0 to the electronic motion is proportional to $(\omega_0^2 - \omega_c^2)^{-1}$ and is larger than for the case of no magnetic field. The growth rate for parametric excitation of the upper and lower hybrid reads

$$\gamma = \tfrac{1}{4}(ek_x E_0/m\omega_p^{\,2})(1+k_y^{\,2}\omega_c^{\,2}/k_x^{\,2}\omega_0^{\,2})^{1/2}\,\omega_p\Omega_p/[(\omega_p^{\,2}+\omega_c^{\,2})\Omega_c^{\,2}+\Omega_p^{\,2}\omega_c^{\,2}]^{1/2}\quad . \tag{31a}$$

The growth rate depends mildly on the magnetic field, i.e., $\gamma \sim \omega_c^{-1/2}$.

For the excitation of the upper hybrid and a phonon, we obtain the growth rate

$$|\gamma| \approx \tfrac{1}{4}(1-\epsilon_\infty/\epsilon_0)(ek_x E_0/m^*\omega_p^{\,2})\omega_e\omega_p/(\omega_1\omega_2)^{1/2}. \tag{31b}$$

Here ω_1 and ω_2 are given by

$$\omega_{1,2}^{\,2} = \tfrac{1}{2}(\omega_p^{\,2}+\omega_c^{\,2}+\omega_l^{\,2}) \pm \tfrac{1}{2}(\omega_p^{\,2}+\omega_c^{\,2}-\omega_l^{\,2})\{1+[4(\omega_p^{\,2}+\omega_c^{\,2})\omega_l^{\,2}/\omega_p^{\,2}+\omega_c^{\,2}-\omega_l^{\,2}](1-\epsilon_\infty/\epsilon_0)\}^{1/2},$$

where ϵ_∞ and ϵ_0 are respectively the infinite and zero frequencies dielectric constant.

We next compare our result Eq. (31a) with the growth rate obtained for parametric excitation of plasma and ion-acoustic waves in plasmas without a magnetic field. The result for this growth rate reads

$$\gamma = \tfrac{1}{4}(ekE_0/m\omega_p^{\,2})[\omega_p\Omega_p]^{1/2}(k_D/k)^{1/2}. \tag{32}$$

The growth rate for density excitation in plasmas with or without magnetic field turn out to be comparable since (k/k_D) cannot be very small. Otherwise the frequency of the acoustic wave which is proportional to (k/k_D) goes to zero, and the two-mode analysis fails. (The two-mode analysis is valid only if $\tau_H\omega_L > 1$.) Also for the case of $(k/k_D) \lesssim 1$ the rate of absorption of the waves increases largely due to Landau damping, and therefore a larger growth rate is needed to start the instability. However, for density waves in plasmas embedded in a magnetic field and propagating perpendicular to the magnetic field, the collision frequency is independent of (k/k_D), and one can study the excitation of the density waves regardless of their wave number. The magnetic field also provides a frequency tuning which can help in the experiment.

For the instability of the mode at the cyclotron harmonic, we calculate the growth rate for the two modes at

$$\omega_L \approx \Omega_c^{\,2} + \Omega_p^{\,2}\omega_c^{\,2}/\omega_c^{\,2}+\omega_p^{\,2},\quad \omega_H^{\,2} = 4\omega_c^{\,2}[(1+\tfrac{3}{4}\omega_p^{\,2}z/(3\omega_c^{\,2}-\omega_p^{\,2})],\quad \text{and } \omega_0 = \omega_L + \omega_H .$$

Using Eqs. (23) and (25) we can, to leading order in z, obtain the growth rate for these modes (for $3\omega_c^2 \neq \omega_p^2$);

$$\gamma = \frac{1}{4}\left(\frac{ekE_0}{m3\omega_c^{\,2}}\right)\left(1+\frac{k_y^{\,2}}{4k_x^{\,2}}\right)^{1/2}\left(\frac{3\omega_c^{\,2}}{3\omega_c^{\,2}-\omega_p^{\,2}}\right)\left(\frac{\omega_c^{\,2}}{\omega_p^{\,2}+\omega_c^{\,2}}\right)^{1/2}\frac{\omega_p\Omega_p(z)^{1/2}}{\{4\omega_c^{\,2}[\Omega_c^{\,2}+\Omega_p^{\,2}\omega_c^{\,2}/(\omega_p^{\,2}+\omega_c^{\,2})]\}^{1/4}}. \tag{33}$$

In Eq. (33) the growth rate is smaller by a factor of $z^{1/2} = kv_{\text{th}}/\omega_c$ than that given by Eqs. (31a) and (32). We therefore conclude that the parametric excitation of the cyclotron harmonic is possible provided kv_{th}/ω_c can be made of the order of unity.

We next consider the parametric excitation of the two high-frequency modes at the upper hybrid and the cyclotron harmonics [see Eqs. (26) and (27)]. For this case $\omega_0 = 2\omega_c + (\omega_c^2+\omega_p^2)^{1/2}$, and if $\omega_0 \neq n\omega_c$, n being an integer, our two-mode analysis is valid. The coupling between the two electronic modes is done via the ionic motion and we approximate in Eqs. (15) and (16) $\epsilon_i(\omega+s\omega_0) \approx 1$ and $\epsilon_i(\omega+s\omega_0) - 1 \approx \Omega_p^2/(\omega+s\omega_0)^2$. The dispersion relation for the two-mode instability is given using Eqs. (15) and (16) and reads

$$\epsilon_e(\omega)\epsilon_e(\omega-\omega_0) + \tfrac{1}{4}\lambda^2[\epsilon_e(\omega)-1][\epsilon_e(\omega-\omega_0)-1][\Omega_p^{\,2}/\omega^2 - \Omega_p^{\,2}/(\omega-\omega_0)^2]^2 = 0. \tag{34}$$

In deriving Eq. (34) we assumed that $\lambda < 1$ and also that $\Omega_p < \omega_c, \omega_p$. The growth rate is given by [see Eqs. (18) to (23)]:

$$\gamma = \frac{1}{4}\left(\frac{ekE_0}{m(\omega_0^{\,2}-\omega_c^{\,2})}\right)\left(1+\frac{k_y^{\,2}\omega_c^{\,2}}{k_x^{\,2}\omega_0^{\,2}}\right)^{1/2}\Omega_p^{\,2}\left|\frac{1}{\omega_1^{\,2}}-\frac{1}{\omega_2^{\,2}}\right|\left(\omega_1\frac{\partial\epsilon(\omega_1)}{\partial\omega_1^{\,2}}\,\omega_2\frac{\partial\epsilon(\omega_2)}{\partial\omega_2^{\,2}}\right)^{-1/2} \tag{35}$$

In Eq. (35) we identify ω_1^2 with the upper hybrid, i.e., $\omega_1^2 = \omega_p^2+\omega_c^2$. The mode at

$$\omega_2{}^2 = 4\omega_c{}^2([1 + \tfrac{3}{4}\omega_p{}^2 z/(3\omega_c{}^2 - \omega_p{}^2)])$$

is the Bernstein mode. The growth rate for these modes is found to be

$$\gamma \approx \frac{2}{16\sqrt{2}} \left(\frac{ekE_0}{m(\omega_p{}^2 + 2\omega_c{}^2)}\right) \left(1 + \frac{k_y{}^2}{k_x{}^2} \frac{\omega_c{}^2}{\omega_p{}^2 + 5\omega_c{}^2}\right)^{1/2} \frac{m}{M} \left(\frac{\omega_p{}^2}{\omega_p{}^2 + \omega_c{}^2}\right)^{5/4} \frac{\omega_p}{\omega_c} (z\omega_c\omega_p)^{1/2} . \tag{36}$$

We can see that the growth rate for parametric excitation of the upper hybrid and the cyclotron harmonics is proportional to $z^{1/2} = kv_{\text{th}}/\omega_c$ as expected. We next compare the growth rate for excitation of two high-frequency modes, Eq. (36), with the growth rates for excitation of high- and low-frequency modes, Eqs. (31a) and (33). If we assume $z \lesssim 1$ we obtain, apart from terms of order unity, that ratio of the growth rates γ in Eq. (36), to γ in Eqs. (31a) and (33) is $(m/M)^{3/4}$. This small ratio comes from general consideration of parametric oscillators where the electron ion coupling is done by the self-consistent field. The electrons are driven by the ions and the external field with coupling proportional to $\omega_p{}^2$. Similarly the ions are driven by the electrons and the external field with coupling proportional to $\Omega_p{}^2$ (see Eq. 17). The effective coupling is $\chi \simeq \omega_p{}^2\Omega_p{}^2$ as we found using our Eq. (25). The growth rate in our case for the lower and upper hybrids is given by $|\gamma| \sim (\chi/\omega_L\omega_H)^{1/2}$, i.e.,

$$\gamma \sim \omega_p\Omega_p(\omega_L\omega_H)^{-1/2} \sim (\omega_p\Omega_p)^{1/2} \sim (m/M)^{1/4}\omega_p .$$

For the excitation of any of the high-frequency electron modes we depend on the ion fluctuations [see Eq. (34)]. We obtain $\chi \sim \Omega_p{}^4$ if all of the high frequencies considered are of the same order of magnitude. The growth rate γ' for this case is given by $\gamma' \sim (\Omega_p{}^4/\omega_p{}^2)^{1/2}$. Here ω_p is typically the order of the high frequencies in our problem. We therefore obtain that $\gamma' \sim (m/M)\omega_p$ and the ratio of γ' to γ is $(m/M)^{3/4}$ as we have found.

The ratio of $(m/M)^{3/4}$ is very small; even for helium, this is of order 10^{-3}. We find, however, from order-of-magnitude considerations, that when the full electromagnetic radiation field is considered, the ions can be ignored, but parametric excitation with growth rate $\gamma'' = (v/c)\,\gamma$ takes place. Here v is a typical phase velocity of the density fluctuations which in our estimate can be taken to be of the order of v_{th}. Therefore in gaseous plasmas, where v_{th}/c is of the order 10^{-2}, the coupling via the ions may be ignored. However, for electron-hole plasmas in semiconductors, where v_{th}/c is typically 10^{-4} but (m/M) could be of order $\frac{1}{30}$ (e.g., InSb), the coupling between the electronic upper-hybrid and the first Bernstein mode will take place via the holes.

CONCLUSIONS

In conclusion we have calculated the growth rates for excitation of density waves for plasmas in a magnetic field. We find that the growth rate for excitation of low- and high-frequency modes is comparable with the excitation of electron-plasma and ion acoustic waves in the absence of a magnetic field. The growth rate for excitation of two high-frequency modes is smaller and will be strongly dependent on the electron-ion mass ratio.

APPENDIX

We derive here the relation between our calculated growth rate γ and the condition for the onset of instability. Our starting point is Eq. (20). For the case when $\gamma < 2\omega_L$ we obtain

$$\gamma = \tfrac{1}{4}\lambda(\chi/\omega_L\omega_H)^{1/2} , \tag{A1}$$

where λ, χ, ω_L, and ω_H are defined in the text. We next generalize Eq. (20) to include phenomenologically the dissipation by introducing, respectively, γ_L and γ_H, the dissipation rates of the linear modes at ω_L and ω_H. Equation (20) now reads

$$[(\omega + i\gamma_L)^2 - \omega_L{}^2][(\omega - \omega_0 + i\gamma_H)^2 - \omega_H{}^2] = \lambda^2\chi/4 . \tag{A2}$$

Using the condition that γ_L, $\gamma_H < \omega_L$ we solve Eq. (A2) and obtain the growth rate $\tilde{\gamma}$, in the presence of dissipation, to be

$$\tilde{\gamma} = \tfrac{1}{2}\{-(\gamma_L + \gamma_H) + [(\gamma_L - \gamma_H)^2 + 4\gamma^2]^{1/2}\} . \tag{A3}$$

Here γ is given in Eq. (A1). From Eq. (A3) it is clear that $\tilde{\gamma}$ is positive (the system is unstable) if the growth rate γ, calculated in the text, is larger than the "effective" dissipation rate of the two modes given by

$$\gamma \geqslant \gamma_{\text{eff}} = (\gamma_L\gamma_H)^{1/2} . \tag{A4}$$

This is a well-known result of parametric oscillator theory and is the basis for calculations of the threshold field $(\gamma \sim E_0)$ needed to obtain the instability.

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Ground-State Energy of Solid He^3†

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The ground-state energy of solid He^3 is computed using a cluster expansion terminated in third order. A 6-9 potential is used that gives a better fit to virial data than the 6-12 Lennard-Jones potential does. A substantial improvement in energy for a particle localization corresponding to the crystal is obtained, although the crystal is not stable against further particle delocalization. Thus, for this potential, the interaction is too weak to give a stable crystal structure. Some simplifying features of the cluster expansion are also demonstrated.

I. INTRODUCTION

Several calculations of the ground-state energy of solid He^3 have been carried out using a variational method based on a trial wave function of the Jastrow type[1–4]

$$\Psi(\vec r_1, \dots, \vec r_n) = \prod_{i=1}^{N} \varphi(|\vec r_i - \vec R_i|) \prod_{j<k} f_{jk}(|\vec r_k - \vec r_j|) \ . \quad (1.1)$$

The φ are single-particle wave functions describing the localization of particle i with coordinate $\vec r_i$ about the lattice site $\vec R_i$. These wave functions are usually assumed to be Gaussian. The f_{jk} are two-particle correlation functions, going to zero for small r_{jk}, and going to a constant for large r_{jk}. In some calculations the f_{jk} are assumed to be independent[2,3] of j, k and then simply denoted by f; in other calculations they depend on j, k. The ground-state energy may then be expressed as a cluster expansion,[5] although this is not done in a unique way by the above mentioned authors.[1,2] Introducing

$$V(r) \equiv v(r) - (\hbar^2/2m)\nabla^2 \ln f(r) \quad (1.2)$$

enables the energy terms in the cluster expansion to be expressed in a convenient way, with an effective potential Vf^2. In the previous calculations $v(r)$ has been assumed to be the Lennard-Jones potential

$$v(r) = 4\epsilon\left[(\sigma/r)^{12} - (\sigma/r)^6\right]. \quad (1.3)$$

The cluster development terminated in low order will include the short-range correlations. The long-range correlations associated with the zero-point fluctuations of the phonon-degrees of freedom is not, however, included in the low-order terms in the expansion. A calculation of the long-range correlations has been performed by Koehler.[6]

The phonon corrections to the ground-state energy may also be calculated readily by perturbation methods. The ring diagrams in the