TABLE IX. Logarithmic derivatives of α and V_0 with respect to density at $\rho = 0.0218$ Å^{-3}.

Source	$\rho \frac{\partial \alpha_1}{\partial \rho}$	$\rho \frac{\partial \alpha_2}{\partial \rho}$	$\rho \frac{\partial \alpha}{\partial \rho}$	$\frac{\rho}{V_0} \frac{\partial V_0}{\partial \rho}$
Case M, theoretical c	−1.82	0.015	−1.81	−5.6
Case M, experimental c	−1.26	0.011	−1.25	−3.4
Case MW, experimental c	−1.45	0.225	−1.23	−3.7
Hard sphere model[2]	−1.13	...	−1.13	−2.8
Experimental (estimated in Ref. 2)	...	...	<−0.88	<−1.6

*Supported in part by NSF Grant GP-6210. The computational costs were met by NSF Grant G-22296.
†National Science Foundation Science-Faculty Fellow at Washington University (1966-67).

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PHYSICAL REVIEW VOLUME 178, NUMBER 1 5 FEBRUARY 1969

Entropy of the Superfluid Component of Helium*

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The specific entropy deficit $\Delta s = s - s_S$ of the superfluid component of liquid helium II has been determined over the temperature range 1.27°K to 1.93°K. Here s and s_S are the specific entropy of the bulk liquid and of the superfluid component, respectively. Superfluid was transferred into a thermally isolated reservoir from a constant temperature reservoir by film flow. Values of Δs were determined from flow rates obtained by driving the film by thermal means. Flow rates were measured with a capacitor depth gauge. The present results indicate that $\Delta s = s$ within ±3%.

INTRODUCTION

According to the two-fluid model of He II, mass transfer through a superleak is due to flow of the superfluid component alone. If this superfluid is then converted into bulk fluid by the absorption of heat (creation of normal fluid), the entropy increase, per gram of bulk fluid, created in the conversion process will be[1] $\Delta s = s - s_S$, where s is the specific entropy of the bulk fluid and s_S the specific entropy of the superfluid. The quantity Δs will be reffered to as the specific entropy deficit.

Landau[2] has argued that $s_S = 0$ and therefore $\Delta s = s$. The majority of experimental determinations[3,4] of Δs yield the result that Δs and the bulk specific entropy s, obtained by integration of specific heat data,[5] are the same within the combined experimental errors. These experiments are all fountain effect experiments. However, the values of Δs obtained by Chandrasekhar and Mendelssohn,[6] using a different technique, were larger than s by 15 - 20%. They used the He II film as a superleak between two thermally isolated reservoirs of liquid helium, and observed the

volume flow rate of superfluid transferred through the film by a measured power. No definite statement was made of the uncertainty in the values of Δs obtained in that experiment, but they included correction terms in their calculations amounting to 3 to 5% of Δs.

The authors have remeasured the specific entropy deficit in the direct manner of Chandrasekhar and Mendelssohn as one facet of a general investigation of mass transfer through the He II film.

THE APPARATUS

The apparatus (Fig. 1) consists of two concentric reservoirs R_i and R_0 separated by a stainless steel vacuum jacket V. Mass transfer between the reservoirs via the He II film takes place over surfaces S_2 and S_3 of the vacuum jacket. The outer wall of R_0 is a thin-walled brass can (BC) providing good thermal contact with the bath liquid in which the apparatus is submerged. Experiments can be performed in which R_i is thermally isolated from R_0 by lowering the stainless-steel plug P into its lapped conical seat at the top of R_i. Experiments with the plug seated will be referred to as adiabatic. Seating the plug prevents distillation between the reservoirs but apparently does not hinder the film flow (saturated adiabatic filling rates and isothermal gravitational emptying rates are found to be the same). With the plug raised isothermal experiments can be performed, heat transfer occurring by distillation. Mass transfer under adiabatic conditions can be initiated or sustained by supplying electrical energy to a Constantan heating coil mounted in the enlarged region of R_i. A carbon resistance thermometer is mounted in R_i for monitoring the temperature within R_i.

Helium is admitted to R_0 through the spring-loaded needle valve U. Leakage through U causes negligible level changes in R_0. Valve L allows 4.2°K liquid into R_i to speed the initial cool down and filling procedures, and permits, when desired, the rapid equalization of liquid levels in the two reservoirs. The inner reservoir is almost entirely filled by the hollow stainless-steel center post (CP) which is positioned and supported by electrically insulating supports. The radial separation of the precisely machined and polished cylindrical surfaces of CP and S_2 is 0.016 in. These surfaces function as a capacitor depth gauge, the capacitor thus formed being 12 in. long and 1.0 in. in diameter. The small annular area in this region of R_i allows the transfer of small volumes to be detected. The position of the liquid in R_0 is determined by viewing with a cathetometer the liquid level in a glass stand-pipe which is in constant communication with R_0.

The experimental capacitor and a fixed inductance determine the frequency ($\sim$ 90 kHz) of a transistor oscillator circuit. The output of the oscillator is fed to a Hewlett Packard Model 5243L Electronic Counter. The counter counts an internal crystal controlled 10 MHz signal for the duration of 10^4 cycles from the transistor oscillator. Period determinations are made at a repetition rate of approximately one per second and are recorded by a digital recorder. Changes in the period are recorded by a strip-chart recording potentiometer by monitoring the output of a digital to analog converter. Level changes in R_i are inferred by calibration of the transistor oscillator. Under good conditions the noise level of the detection system corresponds to a liquid level change of about $\pm$ 1/80 mm, whereas the

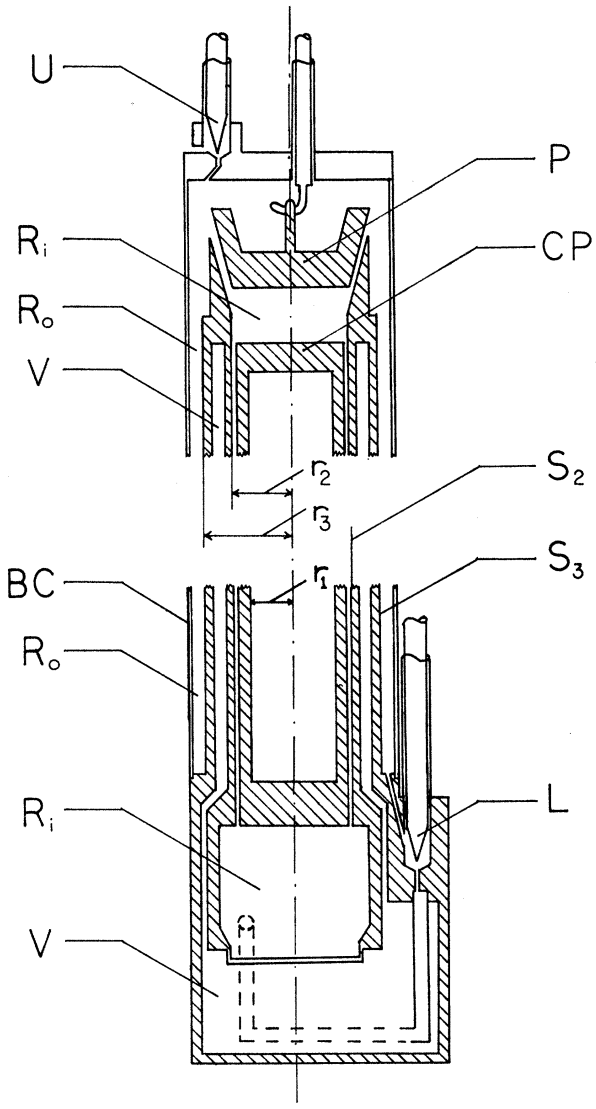

FIG. 1. The film flow apparatus. Mass transfer occurs between reservoirs R_0 and R_i through the adsorbed film on surfaces S_2 and S_3. The reservoirs are separated by the evacuated jacket V. The reservoir R_i is thermally isolated when plug P is seated.

phenomena usually investigated represent level changes within R_i of a few tenths of a millimeter to a few millimeters.

The carbon resistance thermometer is one arm of an ac Wheatstone bridge circuit[7] which is balanced at a reference temperature with P raised. Bridge off-balance signals, phase-detected and displayed on a strip-chart recorder, are converted to temperature variations through calibration using the fountain effect. Under ideal conditions temperature changes of ± 1 μdeg can be observed. The bath temperature is regulated within $\pm 1 - 10$ μdeg by an ac Wheatstone bridge temperature regulator.

THE EXPERIMENT

Figure 2 presents typical chart recordings of the temperature T and liquid level z within R_i obtained in an experiment combining both adiabatic and isothermal conditions. The experimental procedure is as follows: With the plug raised and valve L open, the thermometer bridge is balanced at the reference bath temperature T_0. The plug is lowered and valve L closed, and baselines are established. At time t_1 a constant power, $\dot{Q}$, small enough not to saturate the flow, is supplied to R_i. The power is turned off at t_2. In Region I the temperature and level in R_i rise steadily; inertial effects cause an oscillation to be superimposed on the steady rise. The oscillation is seen to be damped. When the power is turned off at t_2, the momentum of the film causes the level to overshoot and a concomitant temperature decrease. In Region II of the figure, oscillations having the same frequency and damping as those in Region I are superimposed on an exponential decay of the level within R_i. The exponential decay is due to a small heat leak present between the two reservoirs. Region II is allowed to be of sufficient duration that the oscillation damps out; this permits the decay time of the liquid level τ_1, to be determined.

At time t_3 the plug is raised. The temperature within R_i rapidly drops to that of R_0 by distillation. The inner level empties by isothermal gravitational film flow. The discontinuity in z is due to the change in capacitance when the plug is raised. The momentum of the film when the liquid levels are equalized by film flow again causes the inner level to overshoot and a damped isothermal oscillation occurs. Region III is allowed to be of sufficient duration that the oscillation nearly damps out. Valve L is then opened so the levels can be brought

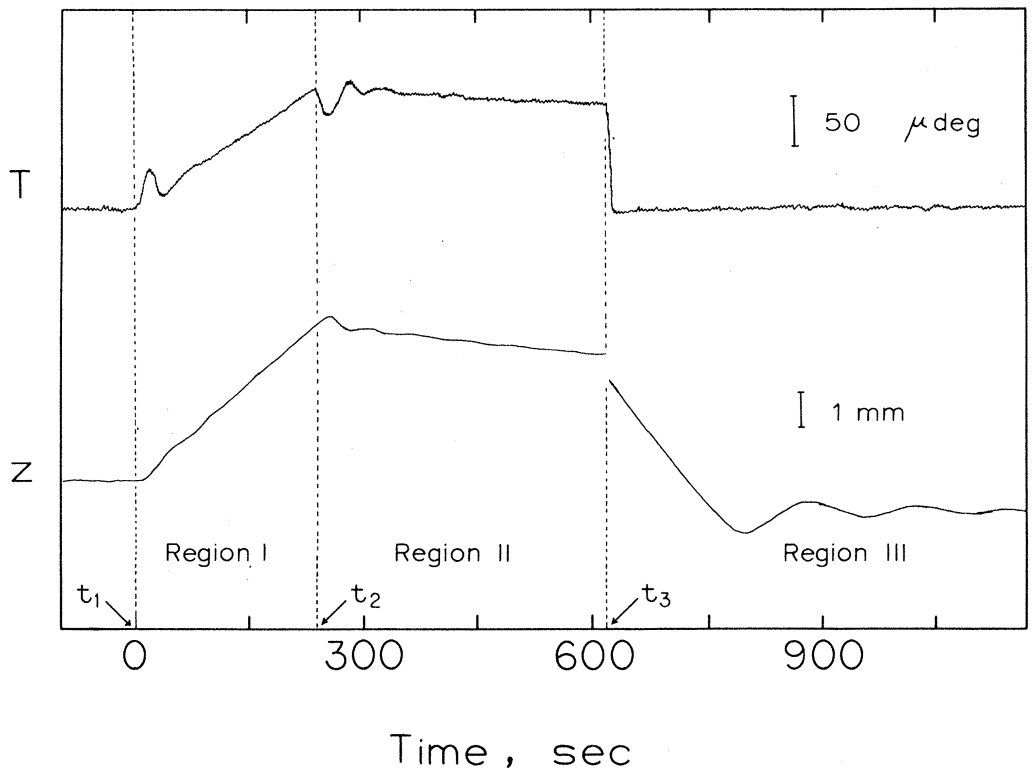

FIG. 2. Typical results of a run in which flow data is obtained under adiabatic (plug seated) and isothermal (plug raised) conditions. T refers to the temperature within R_i, and z refers to the liquid level within R_i. With the plug seated, constant power to R_i is turned on at t_1 and turned off at t_2. The plug is raised at time t_3. Values of T and z increase from the bottom to the top in the figure.

into equilibrium before the next experiment. The data from which the specific entropy deficit is computed is obtained from the Regions I and II of the figure.

THEORY

Atkins[8] has treated thoroughly the theory of the oscillatory behavior of two reservoirs of liquid helium connected via the helium film under isothermal conditions. Let z be the height of the inner liquid level above the equilibrium liquid level of the two reservoirs. If no dissipation of energy occurs under these conditions the behavior of z would be described by: $\ddot{z} = -\omega_0^2 z$, where ω_0 is the natural oscillation frequency determined essentially by the film profile and the geometry of the apparatus. For the present geometry this natural frequency is

$$\omega_0^2 = 2\pi g \frac{\rho_s}{\rho} \frac{r_2 + r_3}{r_2 r_3} \frac{a_0 + a_i}{a_0 a_i} \left(\int_0^{h_0} \frac{dh}{d(h)} \right)^{-1}, \quad (1)$$

where a_i and a_0 are the cross sectional areas of the inner and outer reservoirs, r_2 and r_3 the inner and outer radii of the wall separating the two reservoirs (Fig. 1), ρ_s and ρ are the densities of the superfluid component and of the bulk fluid, respectively, and $d(h)$ the thickness of the film at height h above the liquid surface. The upper limit, h_0, is the height of the rim of the vacuum jack above the equilibrium liquid level.

It is found experimentally,[8–10] however, that the oscillations are exponentially damped. A physical understanding of this damping is not yet available; to account for it phenomenologically, a damping term, $2\dot{z}/\delta_I$, is introduced into the equation of motion, viz.

$$\ddot{z} + 2\dot{z}(\delta_I)^{-1} + \omega_0^2 z = 0. \quad (2)$$

Under adiabatic conditions the mechanical equation of motion is modified to include thermal effects. Following Robinson[11]

$$\ddot{z} + \omega_0^2 z - \omega_0^2 \gamma(\Delta s)(T - T_0)/g = 0, \quad (3)$$

where $$\gamma = 1 - (dp/dT)_s/(\rho \Delta s). \quad (4)$$

Here Δs is the specific entropy deficit and $(dp/dT)_s$ the derivative of the pressure along the saturated vapor curve; T is the temperature within the isolated inner reservoir and T_0 is the regulated temperature within the outer reservoir.

In addition to the mechanical equation of motion, a heat balance equation must be satisfied[11]

$$\dot{Q} = \rho(\Delta s) T_0 a_i \gamma \dot{z} + C\dot{T} + \kappa(T - T_0), \quad (5)$$

where C is the heat capacity of the inner reservoir (2 - 20 J/°K), κ the overall coefficient of thermal conductivity (0.02 - 0.3 J/sec °K), and $\dot{Q}$ is the rate at which heat is admitted to the inner reservoir by the electrical heater.

Experimentally it is observed that if $z = \dot{z} = \ddot{z} = 0$ at the instant the power is turned on, then z has the form of a damped oscillation superimposed on a steady rise, provided $\dot{Q}$ is not so large as to saturate $\dot{z}$. Further, when the power is turned off, z is observed to have the form of a damped oscillation superimposed on an exponential decay. Robinson indicates that the form of Eqs. (3) and (5) leads to damping of the oscillations because of the thermal conductivity. The observed damping of the adiabatic oscillations, however, is larger than the predicted by Robinson's result using observed values of the level decay time and the oscillation frequency. (Preliminary measurements by Manchester and Brown[12] on the damping of adiabatic oscillations through a wire-packed superleak also indicate damping in excess of that expected using Robinson's result.) To describe this observation the damping term, $2\dot{z}/\delta_A$, is added to Eq. (3) yielding

$$\ddot{z} + 2\dot{z}(\delta_A)^{-1} + \omega_0^2 z - \omega_0^2 \gamma(\Delta s)(T - T_0)/g = 0. \quad (6)$$

Equations (5) and (6) are combined to yield an equation for z

$$d(\ddot{z} + \beta\dot{z} + \omega^2 z)/dt = \alpha - \omega_0^2 \beta z, \quad (7)$$

where $$\alpha \equiv \gamma a_0 (\Delta s)\dot{Q}\omega_0^2/(a_0 + a_i)Cg, \quad (8)$$

$$\omega^2 \equiv \omega_0^2[1 + \rho\, a_i T_0 \gamma^2 (\Delta s)^2 a_0/(a_0 + a_i)Cg], \quad (9)$$

and $$\beta \equiv \kappa/C. \quad (10)$$

Equation (7) allows solutions of the form found experimentally

$$z = \alpha(\beta\omega_0^2)^{-1} - Ae^{-t/\tau_1} + Be^{-t/\tau_2}\sin(\omega_* t + \phi), \quad (11)$$

where $$\beta + 2\delta_A^{-1} = 2\tau_2^{-1} + \tau_1^{-1}, \quad (12)$$

$$\omega^2 + 2\beta/\delta_A = (1 + \omega_*^2 \tau_2^2)\tau_2^{-2} + 2/\tau_1\tau_2, \quad (13)$$

and $$\omega_0^2 \beta = (1 + \omega_*^2 \tau_2^2)/\tau_2^2 \tau_1. \quad (14)$$

If an experiment is performed for which the initial conditions are $z = \dot{z} = \ddot{z} = 0$, one can write for z

$$z(t) = z_0(t) + Be^{-t/\tau_2}\sin(\omega_* t + \phi), \quad (15)$$

where

$$z_0(t) = [1 - \epsilon - \exp(-t/\tau_1)]\,\alpha/\beta\omega_0^2(1-\epsilon), \quad (16)$$

and

$$\epsilon \equiv [\tau_2(2 - \tau_2/\tau_1)/\tau_1]/(\omega_*^2\tau_2^2 + 1). \quad (17)$$

Equation (16) yields

$$\omega_0^2\beta\tau_1 = \alpha/(1-\epsilon)\dot{z}_0(0), \quad (18)$$

where $\dot{z}_0(0)$ is the derivative of $z_0(t)$ evaluated at $t = 0$.

We find Δs to be given by

$$\Delta s = \frac{\dot{Q}}{\rho a_i T_0 (1-\epsilon)\dot{z}_0(0)} \times \left(1 - \frac{1}{\beta\tau_1} - \frac{\tau_2}{\tau_1}\,\frac{(\beta\tau_1 - 1)(2 - \beta\tau_2)}{\omega_*^2\tau_2^2 + 1}\right) - \frac{1}{\rho}\left(\frac{dp}{dT}\right)_s. \quad (19)$$

$\beta\tau_1$ is given by

$$\beta\tau_1 = a_0\gamma\dot{Q}(\Delta s)/(a_0 + a_i)Cg\dot{z}_0(0)(1-\epsilon). \quad (20)$$

CALCULATIONS AND RESULTS

Equation (19) is used to compute Δs. This equation expresses Δs in terms of quantities determined from the experiment and in terms of Δs itself through $\beta\tau_1$. However, the terms involving $\beta\tau_1$ are small compared to unity, and represent correction terms to the basic expression. For this reason values of $\beta\tau_1$ for insertion into (19) are computed using for Δs values of the bulk specific entropy s. The values of s are those calculated by Ter Harmsel and Van Dijk[13] from the specific heat data of Kramers, Wasscher, and Gorter[5] and of Keesom and Keesom[5] using the 1958 He^4 temperature scale. The values of the flow quantities, $\dot{z}_0(0)$, τ_1, τ_2, ϵ, and ω_* are determined by fitting relations (11) and (15) to the data. $\dot{Q}$ is measured using a standard potentiometric method. The T_0 is determined by measuring the vapor pressure above the surrounding liquid bath, and is expressed on the 1958 He^4 scale. The values of $(dp/dT)_S$ are obtained from the National Bureau of Standards Monograph 10.[14] The density values are those of Kerr.[15] The values of the specific heat of liquid helium are those of Kramers, Wasscher, and Gorter.[5] The heat capacity C contains a contribution from the stainless steel[16] in the center post and inner wall of the vacuum jacket. This is necessary because the skin depth for temperature oscillations at the frequencies observed is larger than the thickness of the steel. The heat capacity of the steel is about 15% of the total heat capacity at 1.27°K and only about 2% of the total at 1.93°K. The annular areas a_0 and a_i are computed from room temperature measurements of the radii involved with necessary temperature corrections taken into account.

The computed values of Δs at nine temperatures are presented in Table I, as are the bulk specific-entropy values s due to Ter Harmsel and Van Dijk. Δs is the average of two to ten determinations at each temperature. The Δs values of the present experiment are compared to these bulk specific-entropy values because both sets of values are based on the 1958 temperature scale. The uncertainties in s are estimated to be about ± 2%. The uncertainties in Δs due to uncertainties in $\dot{z}_0(0)$, C, s, and a_i range from about 1% to about 4% and are presented in the table also. It is seen that the values of Δs and s agree within 1 to 3%, the Δs values being slightly smaller than the s values. The preliminary calculations of $\beta\tau_1$ assuming $\Delta s = s$ are justified.

TABLE I. The present values of the specific entropy deficit Δs compared to Ter Harmsel and Van Dijk's values of the bulk specific entropy s.

T (°K)	Δs (J/g °K)	s (J/g°K)	$(\Delta s - s)/s$ (%)
1.270	0.072± 0.003	0.074	−2.7
1.401	0.135± 0.002	0.133	1.5
1.455	0.166± 0.003	0.166	0.0
1.501	0.197± 0.003	0.199	−1.0
1.553	0.238± 0.004	0.242	−1.6
1.573	0.253± 0.004	0.260	−2.7
1.655	0.337± 0.004	0.347	−2.9
1.812	0.556± 0.005	0.565	−1.6
1.933	0.79 ± 0.01	0.79	0.0

It is concluded from the above results that in the reversible flow of liquid helium through the He II film the specific-entropy deficit of the superfluid equals the bulk specific entropy.

Investigation of the isothermal oscillations and their damping is continuing.

ACKNOWLEDGMENTS

The authors wish to thank Professor William Zimmermann, Jr. for the loan of the electronic temperature regulator, and for helpful discussions during the course of the work.

*This research was supported by the U. S. Air Force Office of Scientific Research and the Graduate School of the University of Minnesota.

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PHYSICAL REVIEW VOLUME 178, NUMBER 1 5 FEBRUARY 1969

Inelastic Boltzmann Equation for Slightly Ionized Gases

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(Received 16 September 1968)

By expanding in a Taylor series the inelastic scattering probability in the collision term of the Boltzmann transport equation, an approximate form of the Boltzmann equation for slightly ionized gases in the presence of a time-dependent electric field, which takes into account elastic and inelastic electron-molecule collisions, is obtained in a rigorous fashion. The inelastic collision term is obtained for arbitrary collision frequency, and it is shown how this expression can be simplified under the assumption of a low inelastic (in comparison with the elastic) collision frequency, or low molecular excitation energy in comparison with the average kinetic energy of the electrons, or both. Making both these assumptions and further assuming a low inelastic energy transfer, a formal solution of this equation is effected by an iterative procedure. This solution indicates explicitly that inelastic collisions are responsible for a small correction term to the distribution which is obtained if one considers only elastic collisions.

I. INTRODUCTION

In many investigations into ionospheric and gas-discharge phenomena, one is frequently confronted with the problem of determining the electron-velocity distribution for a slightly ionized gas in the presence of a relatively weak time-dependent electric field. The Boltzmann equation for the electron-distribution function, $f(\vec{v},t)$, in this situation is (assuming spatial homogeneity and the external electric field $\vec{E}$ to be in the x direction)

$$\partial f/\partial t + (eE/m)\partial f/\partial v_x = (\partial f/\partial t)_c , \tag{1}$$

where e and m are the charge and mass of the electron, respectively, and $(\partial f/\partial t)_c$ is the rate of change of the distribution function due to collisions. To solve this equation one often makes the following, quite reasonable assumptions:

(a) $f(\vec{v},t)$ is isotropic except for a time varying perturbation in the x direction caused by the electric field, i.e.,