$\Delta$ residue should be negative and small. Besides the well-established choosing-no-fixed-pole mechanism for the $N$ trajectory at $\alpha_N=-\frac{1}{2}$, $N$ must choose a fixed pole for $\alpha_N>-4.5$. The $\Delta$ trajectory chooses a fixed pole at $\alpha_\Delta=-1.5$, but should choose no fixed pole for all $\alpha_\Delta \leq -9.5$, and the $\Delta$ trajectory at least should go down to $\alpha_\Delta=-9.5$. (The statement about the $\Delta$ is of less certainty than that about the $N$.)

Unfortunately, so far there is no complete theory which gives good interpolation between high energy and low energy for any realistic reactions. Therefore, at this stage, we naively take the present Regge representation and extrapolate it to low energies in a simple way. In doing this we would like to make the following point clear. In the particular reaction that we studied—$\pi^- p \to \pi^0 n$—it happens that the combined $t$- and $u$-channel Regge poles can produce the resonance-type structure. However, there are reactions in which such phenomena do not happen. Thus, at this stage of phenomenological study, we take the following attitude: Whenever the present form of Regge representation predicts something in the low-energy region (like the dip-bump structure given by the $s^{\alpha(u)}$ term for fixed $t$), it should be qualitatively correct. So even if we do not expect $s^{\alpha(t)}+s^{\alpha(u)}$ to represent the whole amplitude, it is still interesting to notice that the $s^{\alpha(u)}$ term at fixed $t$ gives qualitative agreement. When the crossed-channel Regge poles cannot produce any structures in low $s$, we should be happy enough if they give the average.

Our study here is an attempt to see how far it is possible to go with the assumptions made. The lack of full agreement with the data seems very interesting because it is a measure (although a semiquantitative measure) of what is missed in the model.


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# Zero-Mass Bosons in $S$-Matrix Theory*

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We describe the soft coupling of zero-mass bosons to other particles, by considering the limit of a theory with a massive boson. With the standard $S$-matrix assumptions of analyticity and crossing for four-body helicity amplitudes, we demonstrate generally that in the limit of zero mass, a vector boson ($1^-$) couples to a conserved charge and a $2^+$ boson couples to the inertial mass. Bosons of other spin-parity combinations (with the exception of zero spin) have no zero-mass soft coupling. With this technique, we not only give a pedagogically interesting solution to gauge invariance and the kinematics of zero-mass particles, but suggest new applications to small-mass integral-spin systems. We speculate on the application of this technique to such problems as $\rho$ universality, the Adler-Weisberger relation, and the universality of leptonic couplings in a vector or axial-vector state.

## I. INTRODUCTION

RECENTLY, several authors[1] have studied the question of gauge invariance and zero-mass particles in $S$-matrix theory, and the related subject of small-mass mesons has also attracted some attention.[2] There exist two essentially distinct methods for the examination of the $S$-matrix theory of massless particles. One approach uses zero-mass particles from the beginning and entails the construction of certain amplitudes with the aid of the polarization four-vector of the zero-mass particle. The assumptions of Lorentz invariance, analyticity, and crossing are then introduced for these amplitudes. In this approach, the principle of gauge invariance—invariance under the addition of the light-like momentum vector to the polarization four-vector—is explicitly utilized. However, since it has been shown by Weinberg and Zwanziger that gauge invariance is a consequence of Lorentz invariance for zero-mass particles,[3] no new principle has in fact been introduced. This method has further been used by Weinberg to prove certain properties of the couplings of zero-mass particles, such as conservation of charge and the equivalence principle.

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[1] J. S. Ball, Phys. Rev. **124**, 2014 (1961); J. S. Ball and M. Jacob, CERN Report No. TH.825, 1967 (unpublished); E. Gotsman and U. Maor, Phys. Rev. **171**, 1495 (1968); J. P. Ader, M. Capdeville, and H. Navelet, Saclay Report, 1968 (unpublished); S. R. Cosslett, Phys. Rev. **176**, 1782 (1968); J. Daboul, *ibid.* (to be published).

[2] S. Mandelstam, Phys. Rev. **168**, 1884 (1968).

[3] D. Zwanziger, Phys. Rev. **133**, B1036 (1964); S. Weinberg, *ibid.* **135**, B1049 (1964).

In the second approach, which is the approach of this paper, one begins with the helicity amplitudes for massive particles and derives the zero-mass results by studying the limit of the amplitudes as one of the external masses is taken to zero. As we will show in this paper, this limiting process leads to all the results derived from gauge invariance. Furthermore, this approach may have the advantage of being more readily extended to the study of small-mass particles, or pairs of particles in a state of integer spin and low mass, such as the $J=1$ $\pi\pi$ state or lepton pairs $e\nu$ and $\mu\nu$. As far as the zero-mass limit of helicity amplitudes is concerned, three points have to be considered. We may demonstrate these points for the photon in the following way.

For a zero-mass vector particle, the states with helicity $\pm 1$ are completely decoupled from the zero-helicity state. That is, there is no Lorentz transformation (parity included) which mixes the two types of states. The photon occurs in the states with helicity $\pm 1$. However, there are as yet no proofs which would rule out the existence of a zero-helicity zero-mass particle. If such a particle existed in nature, it would behave like a spin-zero particle and would not have to be connected with the photon. Therefore, when the theory of photons is derived as the limit of a theory of massive vector particles nothing needs to be proved about the final behavior of the zero-helicity amplitude (denoted by $M_0$). Although $M_0$ plays an important role before the final limit is taken, at the end we are interested in the behavior of the amplitudes $M_{\pm 1}$.

The second question to be considered is the final form of the pole terms corresponding to the external particles (hereafter simply referred to as pole terms) for the helicity $\pm 1$ amplitudes. Although this question is well answered in the framework of perturbation theory, it has caused some confusion in the $S$-matrix approach. If one works with zero-mass particles to begin with, then the pole terms from the three channels of a four-body amplitude with one zero-mass particle all occur at the same point. Without the use of perturbation theory, it is not clear how these three poles should be represented. On the other hand, if one starts with the massive case, the $t$-channel nonsense amplitudes do not contain a $t$-channel pole. However, there is a kinematic singularity which in the limit of zero mass becomes a pole denominator. The problem of gauge invariance involves the normalization of the residue of this pole. Moreover, one has to show that in the limit of zero mass, this kinematic singularity has the characteristics of a dynamical pole. This means that the pole occurs only when the quantum numbers of the $t$ channel agree with those of the particle under consideration. For example, if one uses the Regge language, one has to show that the pole denominator multiplies only the Regge pole corresponding to the particle in question and does not occur in the other Regge contributions.[4] (As a kinematic singularity, before the mass is taken to zero, this factor does multiply the entire amplitude.) The problem of kinematic singularities has already been studied by other authors.[1,5]

The next property of zero-mass particles to be considered is the conditions on their couplings to other particles. These are the conditions we will be mostly concerned with in this paper. Using the properties of helicity amplitudes in the massive case and the extra assumption of the smoothness in the external mass (discussed in detail in Sec. II), we will demonstrate how, to zeroth order in its mass, a soft vector particle couples to a conserved quantity. We will also show that a soft, zero-mass $2^+$ particle has to couple to the inertial mass, and that massless particles of spin higher than 2, or a massless axial-vector particle, have no soft coupling. We have not been able to prove any results for the coupling of zero-mass spinless particles, except the existence of only one soft coupling (independent of helicities) to particles with spin.

In the case of the coupling of a vector particle, we prove conservation of charge for a strong three-particle vertex. Conservation of charge for any amplitude can then be proved by induction if we assume that for an $n$-body reaction there exists an $(n-1)$-body reaction which is obtained from it by replacing two of the particles with some other communicating particle.

The method we use here, namely, the limiting procedure from the massive to the massless case, besides offering a clear proof of charge conservation, opens the possibility of other applications. For example, once we know that conservation of charge is not a peculiarity of the soft coupling of particles with exactly zero mass, but a result that is almost true for small-mass particles, we may ask if the approximate universality of the $\rho$ meson is somehow connected with the smallness of its mass. The scale of mass, of course, has to be somehow established. As indicated in Sec. V, we are also considering the possible extension of these techniques to diparticle states to discuss such problems as the Adler-Weisberger relation for the antisymmetric part of amplitudes involving two pions or problems involving lepton pairs or currents (results of conserved vector current and partially conserved axial-vector current). In a forthcoming paper, we will present our application of this method to the nonsense coupling of two photons to the Pomeranchuck trajectory in Compton scattering.

In Sec. II, we consider the problem of charge conservation for a strong vertex with spinless particles. In Sec.

[4] R. C. Brower and J. W. Dash, Phys. Rev. **175**, 2014 (1968). The Appendix gives the Reggeized treatment of this problem. Note that in Regge theory, the background with a given $s^\alpha$ behavior is absorbed into the $t$ dependence of $\beta$ and $\alpha$. Moreover, each Regge contribution is gauge-invariant by itself.

[5] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Ann. Phys. (N. Y.) **46**, 239 (1968). Throughout this paper we use the phase conventions of this article for the helicity amplitudes and the crossing matrices. Parity conservation for these amplitudes is written

$$M_{\lambda_3\lambda_4;\lambda_1\lambda_2}(s,t)=\prod_{i=1}^{4}\eta_i(-)^{s_i+\lambda_i}M_{-\lambda_3-\lambda_4;-\lambda_1-\lambda_2}(s,t).$$

III, we discuss the coupling of zero-mass particles of other spin-parity combinations. In Sec. IV, we generalize the proof of the conservation of charge to a vertex containing particles with arbitrary spin. Of course, in the process of the proof we will also give the correct expression for the pole terms in the final amplitudes. Also included in this section is a discussion of the kinematic singularities and constraints for photoproduction amplitudes in terms of the known results for the $\rho$ meson. These kinematical results have already been discussed by other authors.

The proof of Sec. IV involves a knowledge of the crossing properties of the helicities of a vertex function. Note that in $S$-matrix theory this vertex function is a number (the coupling constant) which depends on the external helicities. We discuss this crossing property which, besides its importance for our proof, is also useful in comparing coupling constants in factorized Regge residues which are related by the interchange of an internal pole with an external particle.

## II. ZERO-MASS VECTOR PARTICLE

In this section we will prove universality for the coupling of a soft vector particle to zeroth order in the mass of that particle. Conservation of charge, with charge defined as the coupling of a soft zero-mass photon, is then a rigorous consequence of our analysis. Since we always begin with a massive particle and take the limit of zero mass at the last stage of our proof, the main set of assumptions used here are the analyticity, crossing, and Lorentz invariance conditions usually assumed for massive particles. The only extra condition is an assumption of smoothness as the mass of the internal particle is taken to zero. The details of this assumption will be discussed in the process of the proof.

We consider a four-particle amplitude and define our channels as

$$\begin{aligned} s&: \quad Vs \to tu\,,\\ t&: \quad V\bar{t} \to \bar{s}u\,,\\ u&: \quad V\bar{u} \to t\bar{s}\,. \end{aligned}$$

We have denoted the particles in the initial states and the channels by the same symbol. We denote the masses of these particles by $m_V$, $m_s$, $m_t$, and $m_u$. In this section we consider spinless $s$, $t$, and $u$ particles, so that the physical arguments of our proof are not confused with spin complications. In Sec. IV, we give the outline of the proof in the general case. We define the $t$-channel helicity amplitudes $M_1{}^t$ and $M_0{}^t$, by

$$\begin{aligned} M_1{}^t&=M_{00;01}{}^t=-M_{00;0-1}{}^t\,,\\ M_0{}^t&=M_{00;00}{}^t\,, \end{aligned}$$

and similarly we define the $s$- and $u$-channel amplitudes.[5]

Our proof consists of essentially two parts. For nonzero $m_V$, the nonsense amplitude $M_1{}^t$ has no poles at $t=m_t{}^2$. However, it does contain the $s$ and the $u$ poles at $s=m_s{}^2$ and $u=m_u{}^2$, through crossing from the $s$- and $u$-channel sense amplitudes. The first step of the proof simply consists of writing a representation for the kinematic-singularity-free amplitude, $\bar{M}_1{}^t$, in terms of the $s$- and $u$-channel poles. The residues of the poles are normalized, through the crossing matrix, by the charges of the particles $s$ and $\bar{u}$ defined in the $s$- and $u$-channel sense amplitudes. Actually, immediately after this step we can set $m_V$ equal to zero and give a simple proof of conservation of charge by comparing $\bar{M}_1{}^t$ with $\bar{M}_1{}^s$. We will present this simple argument and then proceed to the second step of our original proof. This step involves a careful study of the behavior of $\bar{M}_1{}^t$ as $m_V$ goes to zero, and besides adding rigor to the discussion may lead to insights into such problems as the universality of the $\rho$ meson. Our procedure consists of using Lorentz invariance in the form of threshold and pseudothreshold relations[6] (TP relations) to normalize $\bar{M}_1{}^t$ near $t=m_t{}^2$ in terms of the $t$ charge defined in $M_0{}^t$. By comparing this with the representation of step I, we will prove that the sum of the $s$, $\bar{t}$, and $\bar{u}$ charges (denoted by $e_s$, $\bar{e}_t$, and $\bar{e}_u$) is of order $m_V$. It is important to point out that the comparison of the charges is made in the nonsense kinematic-singularity-free amplitude $\bar{M}_1{}^t$ which does not contain a $t$-channel pole. Therefore, we never write an amplitude as a sum of three poles in all three channels, so that we do not commit any double counting.

### Step I

We define the kinematic factors $\mathcal{T}$ and $\mathcal{T}'$ as

$$\begin{aligned} \mathcal{T}^2&=[t-(m_t+m_V)^2][t-(m_t-m_V)^2]\,,\\ \mathcal{T}'^2&=[t-(m_s+m_u)^2][t-(m_s-m_u)^2]\,, \end{aligned} \tag{2.1}$$

and a similar definition of $\mathcal{S}$ and $\mathcal{S}'$. In writing crossing matrices we will use the conventions of Ref. 5:

$$M_{00;0\lambda}{}^t=\sum_{\lambda'}(-1)^{\lambda'}d_{\lambda'\lambda}{}^1(-\chi_V)M_{00;0\lambda'}{}^s. \tag{2.2}$$

The angle $\chi_V$ is given by

$$\cos\chi_V=P_V(t,s)/\mathcal{S}\mathcal{T},\quad \sin\chi_V=2m_V(\Phi)^{1/2}/\mathcal{S}\mathcal{T}, \tag{2.3}$$

where

$$\begin{aligned} P_V(t,s)=(s+m_V{}^2-m_s{}^2)(t+m_V{}^2-m_t{}^2)\\ -2m_V{}^2(m_V{}^2+m_u{}^2-m_t{}^2-m_s{}^2) \end{aligned}$$

and $\Phi$ is the Kibble function, with $(\Phi)^{1/2}$ defined to be positive in the $s$ physical region. The kinematic-singularity-free amplitudes are given by

$$M_1{}^t=[(\Phi)^{1/2}/\mathcal{T}]\bar{M}_1{}^t,\quad M_0{}^t=(1/\mathcal{T})\bar{M}_0{}^t, \tag{2.4}$$

and by a similar relation for the $s$-channel amplitudes. Substituting these results in Eq. (2.2), we have

$$\bar{M}_1{}^t=-(1/\mathcal{S}^2)[\sqrt{2}m_V\bar{M}_0{}^s+P_V(s,t)\bar{M}_1{}^s]. \tag{2.5}$$

[6] J. D. Jackson and G. E. Hite, Phys. Rev. **169**, 1248 (1968); J. E. Mandula, *ibid*. **174**, 1948 (1968).

The charge of the particle $s$ is defined by

$$\lim_{s\to m_s^2}(s-m_s^2)M_0{}^s=-i(4m_s{}^2-m_V{}^2)^{1/2}ge_s\,, \quad (2.6)$$

where $g$ is the coupling of the $s$, $t$, and $u$ particles at the other vertex ($g$ has the units of mass). The constant factor, $-i(4m_s{}^2-m_V{}^2)^{1/2}$, is inserted to make our definition of charge the same as the conventional one. With the aid of Eq. (2.5), we obtain the residue of $\bar{M}_1{}^t$:

$$\lim_{s\to m_s^2}(s-m_s^2)\bar{M}_1{}^t=+\sqrt{2}ge_s\,. \quad (2.7)$$

Similarly, from the crossing relation from the $u$ channel we find[7]

$$\lim_{u\to m_u^2}(u-m_u{}^2)\bar{M}_1{}^t=-\sqrt{2}g\bar{e}_u\,. \quad (2.8)$$

For $s$ near $m_s{}^2$ and values of $t$ such that $u$ is also near $m_u{}^2$, we can write

$$\bar{M}_1{}^t=\frac{\sqrt{2}ge_s}{s-m_s{}^2}-\frac{\sqrt{2}g\bar{e}_u}{u-m_u{}^2}+B_1{}^t\,. \quad (2.9)$$

For the simple version of the proof, we set $m_V$ equal to zero at this point and use the same crossing matrix again. In this limit, for general $s$ and $t$, the crossing matrix becomes diagonal and we have

$$(t-m_t{}^2)\bar{M}_1{}^s=-(s-m_s{}^2)\bar{M}_1{}^t\,. \quad (2.10)$$

We can write a representation similar to that of Eq. (2.9) for the $s$-channel amplitude $\bar{M}_1{}^s$:

$$\bar{M}_1{}^s=\frac{\sqrt{2}g\bar{e}_t}{t-m_t{}^2}-\frac{\sqrt{2}g\bar{e}_u}{u-m_u{}^2}+B_1{}^s\,. \quad (2.11)$$

By putting Eqs. (2.9)–(2.11) together, we find that if $g$ is not zero, and as long as the backgrounds are not as singular as a pole, we must have

$$e_s+\bar{e}_t+\bar{e}_u=0 \quad (2.12)$$

and

$$B_1{}^t(s,t)\to 0\,, \quad \text{as} \quad t\to m_t{}^2\,. \quad (2.13)$$

This proves the conservation of charge for a strong vertex,[8] but in order to examine the behavior of our amplitudes more carefully we will now proceed to the second step of our original proof (this latter technique is the one we generalize to the discussion of the conservation of charge for vertices with arbitrary spin).

### Step II

For the purposes of this discussion it is best to derive the TP relations from $s,t$ crossing. From the inverse of Eq. (2.2) we can write

$$\bar{M}_0{}^s=(+1/\mathcal{T}^2)[P_V(s,t)\bar{M}_0{}^t-2\sqrt{2}m_V\Phi\bar{M}_1{}^t]\,, \quad (2.14)$$

$$\bar{M}_1{}^s=(-1/\mathcal{T}^2)[\sqrt{2}m_V\bar{M}_0{}^t+P_V(s,t)\bar{M}_1{}^t]\,. \quad (2.15)$$

At threshold and pseudothreshold $[t=(m_t\pm m_V)^2=t^\pm]$, $\mathcal{T}^2$ becomes infinite and we must have

$$\bar{M}_1{}^t(s,t^\pm)=[-\sqrt{2}m_V/P_V(s,t^\pm)]\bar{M}_0{}^t(s,t^\pm)\,. \quad (2.16)$$

The identity

$$[P_V(s,t)]^2=\mathcal{S}^2\mathcal{T}^2-4m_V{}^2\Phi \quad (2.17)$$

ensures that both crossing relations lead to Eq. (2.16). The charge of $\bar{t}$ is defined in the same way as the $s$ and $\bar{u}$ charges:

$$\lim_{t\to m_t^2}(t-m_t{}^2)M_0{}^t=-i(4m_t{}^2-m_V{}^2)^{1/2}g\bar{e}_t\,. \quad (2.18)$$

Using the definition of the kinematic-singularity-free amplitudes, Eq. (2.4), we have,

$$\bar{M}_0{}^t=\frac{m_V(4m_t{}^2-m_V{}^2)g\bar{e}_t}{t-m_t{}^2}+B_0(s,t)\,. \quad (2.19)$$

In order to normalize the value of $\bar{M}_1{}^t(s,m_t{}^2)$ in terms of $\bar{e}_t$ and then compare the result with the representation of step I, we substitute Eq. (2.19) in the TP relations of Eq. (2.16). We will be concerned with the region of $t$ between $t^-$ and $t^+$. This interval is of order $m_V$. Our smoothness conditions consist of assuming that as $m_V$ goes to zero, $\bar{e}_t$ is bounded and that the quantities $B_0(s,t^+)-B_0(s,t^-)$ and $\bar{M}_1(s,t^+)-\bar{M}_1(s,t^-)$ are of order $\epsilon(m_V)$, where $\epsilon(m_V)$ denotes any quantity that goes to zero as $m_V$ is taken to zero.[9] Note that although these assumptions may seem plausible, a rigorous proof of them in $S$-matrix theory would entail a close examination of the unitarity condition. We have not addressed ourselves to such a problem in this paper.

Substituting Eq. (2.19) in Eq. (2.16), and using the fact that $P(s,t^\pm)=\pm 2m_Vm_t(s-m_s{}^2)+O(m_V{}^2)$, we obtain

$$(s-m_s{}^2)\bar{M}_1{}^t(s,t^\pm)=-\sqrt{2}\bar{e}_tg\mp(1/\sqrt{2}m_t)B_0(s,t^\pm)+O(m_V)\,. \quad (2.20)$$

By adding and subtracting the relations at $t^+$ and $t^-$ and

[7] Here we have assumed there are no particles in the $Vs$, $V\bar{t}$, and $V\bar{u}$ channels degenerate in mass with particles $s$, $\bar{u}$, and $\bar{t}$. (For the photon, $c=-1$ rules out all couplings to self-charge-conjugate particles like the $\pi^0$.) With this assumption and the assumption of zero spin for $s$, $\bar{t}$, and $\bar{u}$, we have the same coupling $g$ at all three strong vertices $\langle tu|s\rangle$, $\langle t\bar{s}|\bar{u}\rangle$, and $\langle \bar{s}u|\bar{t}\rangle$.

[8] The exact condition on the charges is $e_s+\bar{e}_t+\bar{e}_u+(s-m_s{}^2)B_1{}^t+(t-m_t{}^2)B_1{}^s=0$ for all $s$ and $t$. For the proof of charge conservation any singularity in $B_1{}^t$ can be tolerated except a simple pole at $s=m_s{}^2$ and $t=m_t{}^2$. However, the condition on the zero helicity amplitude is $m_VM_0{}^t\to 0$ as $m_V\to 0$ to obtain Eq. (2.10) as the limit of Eq. (2.5).

[9] We have introduced the assumption of smoothenss on quantities with $s$ and $u$ poles; however, it can be shown that the pole terms above satisfy our condition for fixed $s\neq m_s{}^2$. Hence, the condition really applies to the infinite set of cuts in the continuum. One can see explicitly the conditions on the cuts, by separating the $s$ and $u$ poles from $B_0$ and then substituting this decomposition of $\bar{M}_0(s,t^\pm)$ and the decomposition of $\bar{M}_1(s,t^\pm)$ [Eq. (2.19)] into the TP relations [Eq. (2.16)]. In the interest of clarity, we use an approach that severely limits the algebra. By any approach, the lesson is the same; with proper conditions on the cuts, Lorentz invariance and crossing require charge conservation in the limit of zero mass.

with the aid of the assumptions mentioned above, we find that for $t^- < t < t^+$

$$B_0(s,t) = \epsilon(m_V) \tag{2.21}$$

and

$$(s-m_s^2)\bar{M}_1{}^t(s,t) = -\sqrt{2}\bar{e}_t g + \epsilon(m_V)\,. \tag{2.22}$$

Note that $\bar{M}_1{}^t(s,t)$ and $B_0(s,t)$ have poles at $s=m_s^2$ and $u=m_u^2$, hence it is convenient to consider a function $f(s,t)$ defined by

$$f(s,t) = (s-m_s^2)(u-m_u^2)\bar{M}_1(s,t)\,. \tag{2.23}$$

In terms of this function Eq. (2.22) at $t=m_t^2$, with the help of the identity $(u-m_u^2) = -(s-m_s^2)-(t-m_t^2)-m_V^2$, now becomes

$$f(s,m_t^2) = \sqrt{2}\bar{e}_t g(s-m_s^2) + \epsilon(m_V)\,. \tag{2.24}$$

This expression is a partial expansion in $(s-m_s^2)$ and $m_V$ that can be continued to any value of $s$, including $s=m_s^2$ and $u=m_u^2$. A fuller expansion would explicitly introduce the $s$ dependence in $\epsilon(m_V)$, $\epsilon(m_V)=\epsilon_0(m_V)+\epsilon_1(m_V,s)(s-m_s^2)$, and the $m_V$ dependence in $\bar{e}_t$.

From the representation of step I, Eq. (2.9), we obtain another expansion

$$f(s,m_t^2) = -\sqrt{2}g\bar{e}_u(s-m_s^2) - \sqrt{2}ge_s(s-m_s^2) - (s-m_s^2)^2 B_1{}^t(s,m_t^2) + O(m_V)\,. \tag{2.25}$$

Comparing Eqs. (2.24) and (2.25) we conclude that if $g$ is nonzero, the quantity $(e_s+\bar{e}_t+\bar{e}_u)$ is of order $\epsilon(m_V)$ so that the small mass vector particle couples approximately to a conserved quantity.

Note that by comparing the coefficients of $(s-m_s^2)^2$ in the two representations we have shown that $B_1{}^t$ $(t=m_t^2)$ goes to zero as $m_V$ goes to zero. Therefore in this limit, $\bar{B}_1{}^t(s,t) \equiv B_1{}^t(s,t)/(t-m_t^2)$ has no pole at $t=m_t^2$. The full helicity amplitude $M_1$ can then be written as

$$M_1{}^t = \frac{\sqrt{\Phi}}{t-m_t^2}\left(\frac{\sqrt{2}ge_s}{s-m_s^2} - \frac{\sqrt{2}g\bar{e}_u}{u-m_u^2}\right) + (\sqrt{\Phi})\bar{B}_1{}^t(s,t)\,. \tag{2.26}$$

Using charge conservation, we find the residue of the $t$ pole is $i\sqrt{2}m_t g e_t$. Hence the existence of the pole now depends on whether the quantum numbers of the $t$ channel allow the existence of the $t$ pole in the sense amplitude before $m_V$ is taken to zero. In this sense the Born term of $M_1{}^t$ now has the characteristics of a dynamical pole.

## III. MASSLESS PARTICLE OF GENERAL SPIN AND PARITY

In this section we will prove that the coupling of a soft massless $2^+$ particle, denoted by $G$, to a spin-zero particle is proportional to the rest mass of that particle. We will also discuss the case of other spin-parity combinations. We define our channels and the kinematic factors in the same way as in Sec. I. We still confine ourselves to spinless $s$, $t$, and $u$ particles. The proof we will discuss here will be analogous to the one at the end of step I Sec. II. The amplitudes under consideration are $M_\lambda$, where $\lambda$ refers to the helicity of the $2^+$ particle and parity gives $M_{-\lambda}=(-1)^\lambda M_\lambda$. The kinematic-singularity-free amplitudes are given by

$$M_\lambda{}^t = [(\Phi)^{1/2}]^{|\lambda|}/\mathcal{T}^2\bar{M}_\lambda{}^t \tag{3.1}$$

and by similar definitions for the $s$ and $u$ channels. The couplings of the $2^+$ particle are again defined in the sense amplitudes of the respective channels by

$$\lim_{t\to m_t^2}(t-m_t^2)M_0{}^t = m_t f_t g\,. \tag{3.2}$$

As in Sec. II, the bar denotes the coupling to the antiparticle. After defining $f_s$ and $\bar{f}_u$ in a similar way, we can use the crossing matrix to obtain the residue of the $s$ and $u$ poles of $\bar{M}_2{}^t$. We find

$$\begin{aligned}\lim_{s\to m_s^2}(s-m_s^2)\bar{M}_2{}^t &= -\tfrac{1}{4}(6)^{1/2}f_s g/m_s\,,\\ \lim_{u\to m_u^2}(u-m_u^2)\bar{M}_2{}^t &= -\tfrac{1}{4}(6)^{1/2}\bar{f}_u g/m_u\,.\end{aligned} \tag{3.3}$$

We can thus write the following representation for $\bar{M}_2{}^t$ when $s$ and $u$ are near $m_s^2$ and $m_u^2$, respectively ($t$ is near $m_t^2$):

$$\bar{M}_2{}^t = \tfrac{1}{4}(6)^{1/2}\frac{f_s g}{m_s}\frac{1}{s-m_s^2} - \tfrac{1}{4}(6)^{1/2}\frac{\bar{f}_u g}{m_u}\frac{1}{u-m_u^2} + B^t\,. \tag{3.4}$$

Repeating the same procedure for $\bar{M}_2{}^s$, we find

$$\bar{M}_2{}^s = -\tfrac{1}{4}(6)^{1/2}\frac{f_t g}{m_t}\frac{1}{t-m_t^2} - \tfrac{1}{4}(6)^{1/2}\frac{f_u g}{m_u}\frac{1}{u-m_{u2}} + B^s\,. \tag{3.5}$$

If we now put $m_G=0$, the crossing relation reduces to

$$(s-m_s^2)^2\bar{M}_2{}^t = (t-m_t^2)^2\bar{M}_2{}^s\,. \tag{3.6}$$

Substituting Eqs. (3.4) and (3.5) in Eq. (3.6), we find for the pole terms

$$\begin{aligned}&[(f_s/m_s)-(\bar{f}_u/m_u)](s-m_s^2)^2 + (f_s/m_s)(s-m_s^2)(t-m_t^2) = +(f_t/m_t)(s-m_s^2)(t-m_t^2)\\ &\qquad + [(\bar{f}_t/m_t)-(\bar{f}_u/m_u)](t-m_t^2)^2\,.\end{aligned} \tag{3.7}$$

This equation is satisfied if

$$f_s/m_s = \bar{f}_t/m_t = \bar{f}_u/m_u\,. \tag{3.8}$$

Note that we have found the same sign for the coupling of the $2^+$ to spinless particles and antiparticles. If the zero-mass particle has spin higher than 2, and natural $J$ parity, the kinematic singularities are given by ($s$, $t$, $u$ spinless)

$$m_\lambda{}^t = \frac{[(\Phi)^{1/2}]^{|\lambda|}}{\mathcal{T}^J}\bar{M}_\lambda{}^t\,. \tag{3.9}$$

In the equation analogous to Eq. (3.7), there will be terms proportional to $f_s(s-m_s^2)^{J-1}(t-m_t^2)$ and

$\bar{f}_t(t-m_t^2)^{J-1}(s-m_s^2)$ which can not be matched. Therefore, the soft couplings of zero-mass particles with spin higher than 2 must vanish.

If the zero-mass particle has unnatural parity, then its coupling to spinless particles is zero because of conservation of angular momentum and parity. We studied these couplings to particles with spin with the techniques introduced in Sec. IV. One can prove, for example, that all the soft couplings of an axial-vector zero-mass particle must vanish. We can extend this result to all unnatural-parity particles except the $0^-$.

## IV. CHARGE CONSERVATION FOR ARBITRARY SPIN

Here we generalize the discussion of charge conservation to particles of arbitrary spin and parity at the strong vertex. Only an outline of the proof is given, since it closely parallels the discussion of Sec. II for spinless particles. However, enough of the formalism is given to indicate the essential differences. In addition to the proof, this formalism provides a complete solution to the problem of kinematic singularities and gauge invariance for the four-particle photoproduction amplitudes in terms of the known results for massive particles.[5,6] With this formalism it is easy to translate the standard kinematic singularities, conspiracy relations, and TP relations for helicity amplitudes with a massive $\rho$ into results for the helicity amplitudes of photoproduction.[1] Also we note a useful crossing relation between coupling constants for helicity amplitudes. This relates the sense couplings at a vertex to the sense couplings resulting from the interchange of the internal pole with one external particle.

To the particles $s$, $t$ and $u$ of Sec. II, we now assign the spins $J_s$, $J_t$, and $J_u$ and the parities $\eta_s$, $\eta_t$, and $\eta_u$. The $t$-channel helicity amplitudes are related to the kinematical-singularity-free amplitudes $\bar{F}_{\lambda_u\lambda_s;\lambda_t\lambda_V}{}^{\pm}$ by

$$M_{\lambda_u\lambda_s;\lambda_t\lambda_V}{}^t(s,t) = (1-z_t)^{|\lambda-\mu|/2}(1+z_t)^{|\lambda+\mu|/2}[K^+(t)\bar{F}_{\lambda_u\lambda_s;\lambda_t\lambda_V}{}^+(s,t) + K^-(t)\bar{F}_{\lambda_u\lambda_s;\lambda_t\lambda_V}{}^-(s,t)], \quad (4.1)$$

where, for nonzero $m_V$, the kinematic singularities are

$$K^{\pm\eta}(t) = \frac{1}{(t^{1/2})^l}\frac{[t-(m_u+m_s)^2]^{p\pm}[t-(m_u-m_s)^2]^{q\pm}}{(\mathcal{T})^{J_t+1-m-\frac{1}{2}(1\mp1)}(\mathcal{T}')^{J_u+J_s-m}}. \quad (4.2)$$

In the case of all unequal masses $l=|\lambda|+|\mu|$,

$$p^\pm=\tfrac{1}{2}[1\mp\eta\eta_s\eta_u(-)^{J_s+J_u-v}],$$
$$q^\pm=\tfrac{1}{2}[1\mp\eta\eta_s\eta_u(-)^{\epsilon_{su}(J_u-J_s)-v}],$$

with

$$\epsilon_{su}=(m_u-m_s)/|m_u-m_s|, \quad\text{and}\quad \eta=\eta_t(-)^{J_t-v}. \quad (4.3)$$

The rule for modifying the kinematics in the limit $m_V\to 0$ is very simple. The half-angle factors $(1\pm z_t)$, the factors of $t^{1/2}$ and $[t-(m_u\pm m_s)^2]$ are not affected at all by the limit. Also, the conspiracy relations at $t=0$, and the TP relations at the massive particle vertex at $t=(m_u\pm m_s)^2$, carry over without any changes, since they only relate amplitudes with the same value of $\lambda_t$ and $\lambda_V$. As we will see in the process of the proof, all the factors of $1/\mathcal{T}$ go over into kinematical factors of $1/(t-m_t^2)$, *except* one factor of $1/\mathcal{T}$, in $K^\eta(t)$, which becomes a dynamical pole if such a pole is permitted by the conservation laws. The resultant TP relations at $t=m_t^2$ will be indicated at the end of this section.

As in Sec. II, we must first define the charges in all the sense amplitudes. We do this through the partial-wave amplitudes of appropriate parity $\eta=\eta_t(-)^{J_t-v}$ ($v=\frac{1}{2}$ for fermions) by the limit:

$$\lim_{t\to m_t^2}(t-m_t^2)a_{\lambda_u\lambda_s;\lambda_t\lambda_V}{}^{J_t\eta} = -2im_t g_{\lambda_u\lambda_s}{}^t e_{\lambda_t\lambda_V}{}^t(-i)^{k_t}. \quad (4.4)$$

The phase $(-i)^{k_t}$ is chosen so that the pole's contribution is in accord with the real analyticity of $\bar{F}^\pm$. From the partial-wave expression

$$K^{\pm\eta}(t)\bar{F}_{\lambda_u\lambda_s;\lambda_t\lambda_V}{}^{\pm\eta}(s,t) = 2\sum_J [a_{\lambda_u\lambda_s;\lambda_t\lambda_V}{}^{J\eta}(t)e_{\lambda\mu}{}^{J\pm}(z_t) + a_{\lambda_u\lambda_s;\lambda_t\lambda_V}{}^{J-\eta}(t)e_{\lambda\mu}{}^{J\pm}(z_t)], \quad (4.5)$$

we see that the phase of $(-i)^{k_t+1}$ must be the phase of $(\mathcal{T}\mathcal{T}')^{J_t-m}K^\eta(t)$ at $t=m_t^2$. With the above definition of charge, we can write the sense helicity amplitudes as a pole plus a smooth background in the neighborhood of $t=m_t^2$:

$$M_{\lambda_u\lambda_s;\lambda_t\lambda_V}{}^t(s,t) = \frac{4m_V m_t^2}{\mathcal{T}}(-i)^{k_t}\frac{g_{\lambda_u\lambda_s}{}^t e_{\lambda_t\lambda_V}{}^t}{t-m_t^2}d_{\lambda\mu}{}^{J_t}(z_t) + \frac{(1-z_t)^{|\lambda-\mu|/2}(1+z_t)^{|\lambda+\mu|/2}}{(\mathcal{T})^{J_t+1-m}}\bar{B}_{\lambda_u\lambda_s;\lambda_t\lambda_V}(s,t). \quad (4.6)$$

Similar expressions hold in the $s$ and $u$ channel. Note that as $t\to t^\pm$, $z_t\to[2m_t^2(s-m_s^2)+O(m_V^2)]/\mathcal{T}\mathcal{T}'$, so that both the pole term and the background have explicitly exhibited the proper kinematical singularity of $1/(\mathcal{T})^{J_t+1}$ at $t=t^\pm$. The more distant singularities in $t$ have not been made explicit, since they do not affect our discussion. It will be useful to express the function $d_{\lambda\mu}(z)$ by the equation,

$$d_{\lambda\mu}{}^J(z) = \sum_\tau u_{\lambda\tau}{}^*(J)e^{i\tau\theta}u_{\mu\tau}(J), \quad (4.7)$$

so that expansions can be made when $t\to t_\pm$. The $u$ matrices are the unitary, symmetric transformations from helicity to transversity given by

$$u_{\lambda\tau}(J) = e^{-i\lambda\pi/2}d_{\lambda\tau}{}^J(\pi/2)e^{i\tau\pi/2}. \quad (4.8)$$

As in step II of the spinless case, we find the value of the nonsense amplitude $M_{\lambda_u\lambda_s;J_t,-1}{}^t$ in terms of the charges for the sense amplitude by the use of TP relations. With the help of Eq. (4.7), we rewrite the crossing relation of Ref. 5 in the form

$$M_{\lambda_u\lambda_s;\lambda_t\lambda_V}{}^t=\sum_{\tau_t\tau_V} u_{\lambda_t\tau_t}u_{\lambda_V\tau_V}e^{-i\chi_t\tau_t}e^{-i\chi_V\tau_V}F_{\tau_t\tau_V}, \quad (4.9)$$

where all the factors that are not singular at $t=t_\pm$ are in $F_{\tau_t\tau_V}(s,t;\lambda_u\lambda_s)$

$$F_{\tau_t\tau_V}=-(-)^{J_s}e^{-i\pi\lambda_u}\sum_{\lambda_u'\lambda_s'\lambda_t'\lambda_V'}(-)^{\lambda_V'}u_{\lambda_t'\tau_t}{}^*u_{\lambda'_V\tau_V}{}^*$$
$$\times d_{\lambda_u'\lambda_u}{}^{J_u}(\chi_u)d_{\lambda_s'\lambda_s}{}^{J_s}(-\chi_s)M_{\lambda_u'\lambda_s';\lambda_t'\lambda_V'}{}^s. \quad (4.10)$$

By expressing the singular terms $e^{+i\chi_t\tau_t}$ and $e^{\mp i\chi_V\tau_V}$ in the neighborhood of $t\simeq t^\pm$, and $s=m_s{}^2$ as

$$e^{i\chi_t}\simeq\frac{-4m_t{}^2(s-m_s{}^2)}{\mathcal{T}\mathcal{S}'};\quad e^{\mp i\chi_V}\simeq\left(\frac{4m_V}{\mathcal{T}}\right), \quad (4.11)$$

we can easily derive the TP relations,[10]

$$\mathcal{T}^{J_t+1}M_{\lambda_u\lambda_s;\lambda_t\lambda_V}{}^t\big|_{t=t^\pm}$$
$$=u_{\lambda_t-J_t}u_{\lambda_V\pm1}\left(-\frac{m_t{}^2(s-m_s{}^2)}{\mathcal{S}'}\right)^{J_t}(4m_V)F_{-J_t\pm1}. \quad (4.12)$$

By taking ratios of the above relations for various values of $\lambda_t$ and $\lambda_V$, and substituting for the sense amplitudes their expression in terms of the pole plus background [Eq. (4.6)], several important results are obtained. We show (a) that the background for the helicity-zero amplitudes goes to zero as we take $m_V$ to zero and also (b) that there is only one soft photon coupling $\bar{e}_t$ in the limit $m_V\to0$.

(a) $B_{\lambda_u\lambda_s;\lambda_t0}(s,t\simeq m_t)\propto\epsilon(m_V)$: This is proved by taking the ratio of $\mathcal{T}^{J_t+1}M_{\lambda_u\lambda_s;\lambda_t0}{}^t$ to the nonsense amplitude $\mathcal{T}^{J_t+1}M_{\lambda_u\lambda_s;J_t-1}{}^t$ at $t=t^\pm$. Using the relation $u_{0\pm1}/u_{1\pm1}=\pm i\sqrt{2}$, we eliminate the pole term, as in the spinless example, to obtain

$$B_{\lambda_u\lambda_s,\lambda_t0}(t^+)+B_{\lambda_u\lambda_s,\lambda_t0}(t^-)\propto\mathcal{T}^{J_t+1}M_{\lambda_u\lambda_s;J_t,-1}\big|_{t^+}$$
$$-\mathcal{T}^{J_t+1}M_{\lambda_u\lambda_s;J_t,-1}\big|_{t^-}. \quad (4.13)$$

The assumption of smoothness for the nonsense amplitude gives our result.

(b) $e_{\lambda_t\lambda_V}{}^t=\delta_{0\lambda_V}e_t+\epsilon(m_V)$: By using the ratio of $\mathcal{T}^{J_t+1}M_{\lambda_u\lambda_s;\lambda_t0}{}^t$ to $\mathcal{T}^{J_t+1}M_{\lambda_u\lambda_s;\lambda_t'0}$ at $t=t^\pm$ and result (a), one shows that all zero-helicity photon couplings are the same. For $\lambda_V=\pm1$, one takes the ratio of $\mathcal{T}^{J_t+1}\times M_{\lambda_u\lambda_s;\lambda_t\pm1}$ to $\mathcal{T}^{J_t+1}M_{\lambda_u\lambda_s;\lambda_t0}$ for threshold and pseudothreshold. Between these equations $B_{\lambda_u\lambda_s;\lambda_t\pm1}(s,t)$ can be eliminated to zeroth order in $m_V$ to obtain $e_{\lambda_t\pm1}\propto\epsilon(m_V)$.

Finally, we can normalize the nonsense amplitude

$$\bar{M}_{\lambda_u\lambda_s,J_t,-1}{}^t=M_{\lambda_u\lambda_s;J_t,-1}{}^t/(1-z_t)^{|\lambda-\mu|/2}(1+z_t)^{|\lambda+\mu|/2}$$

at $t=t^\pm$ by taking its ratio to a sense amplitude $M_{\lambda_u\lambda_s;\lambda_t0}$ and using the above results. To zeroth order in $m_V$, we obtain

$$(s-m_s{}^2)\bar{M}_{\lambda_u\lambda_s;J_t,-1}{}^t=-[\mathcal{T}'(m_t{}^2)/2m_t]$$
$$\times\sqrt{2}d_{\mu J_t}{}^{J_t}(\pi/2)g_{\lambda_u\lambda_s}{}^te_t(-i)^{k_t}, \quad (4.14)$$

which for the spinless case reduces exactly to the result of Sec. II, Eq. (2.20).

To complete the proof we use a representation for the nonsense amplitude similar to the one of step I, Eq. (2.9), for the spinless case:

$$\bar{M}_{\lambda_u\lambda_s;J_t,-1}{}^t(s,t)=\frac{r_s(t)}{s-m_s{}^2}+\frac{r_u(t)}{u-m_u{}^2}$$
$$+\bar{B}_{\lambda_u\lambda_s;J_t,-1}{}^t(s,t). \quad (4.15)$$

The residues $r_s(t)$ and $r_u(t)$ are determined through crossing. Since $r_s(t)$ and $r_u(t)$ are analytic for $t\simeq m_t{}^2$, they can be expanded, and only the first term $r_s(m_t{}^2)$ and $r_t(m_t{}^2)$ enter into the proof. To demonstrate the procedure, we will find $r_s(m_t{}^2)$.

By looking at the residues of the $s$ pole at $t=m_t{}^2$, the crossing matrix becomes especially simple. Furthermore, $z_t$ and $z_s$ go to zero at this point. Hence we have

$$r_s(m_t{}^2)=e^{i\pi(J_s+J_t-\lambda_u)}\sum_{\lambda_u'\lambda_s'\lambda_t'\lambda_V'}d_{\lambda_u'\lambda_u}{}^{J_u}(0)$$
$$\times d_{\lambda_s'\lambda_s}{}^{J_s}(-\pi/2)d_{\lambda_t'J_t}{}^{J_t}(\pi/2)$$
$$\times d_{\lambda_V'-1}{}^1[-\chi_V(m_s{}^2,m_t{}^2)]2m_s(-i)^{k_s+1}$$
$$\times g_{\lambda_u\lambda_s'}{}^se_{\lambda_t'\lambda_V'}{}^sd_{\lambda_s'-\lambda_V',\lambda_u'-\lambda_t'}{}^{J_s}(-\pi/2). \quad (4.16)$$

Now using TP relations in the $s$ channel, we obtain $e_{\lambda_s'\lambda_V'}{}^s=\delta_{\lambda_V'0}e_s+\epsilon(m_V)$. This relation and the special properties of the $d$ function at 0 and $\frac{1}{2}\pi$ allow all the sums to be done and lead to the following equation:

$$r_s(m_t{}^2)=[\mathcal{T}(m_t{}^2)/2m_t]\sqrt{2}d_{\mu J_t}{}^{J_t}(\pi/2)$$
$$\times(-1)^{J_u-\lambda_u}g_{\lambda_u,\lambda_u-\lambda_s}{}^se_s(-i)^{k_s}(-)^{J_s+J_t-J_u}. \quad (4.17)$$

With the observation that $(-i)^{k_t}/(-i)^{k_s}$ is real and that

$$g_{\lambda_u\lambda_s}{}^t=\pm(-)^{J_u-\lambda_u}g_{\lambda_u,\lambda_u-\lambda_s}, \quad (4.18)$$

we have the desired result up to an undetermined sign.

The last ingerdient to this outline [Eq. (4.18)] can easily be understood by considering the coupling constant $g_{\lambda_u',\lambda_s'}{}^t$ as the on-mass-shell vertex function $\langle\lambda_u'p_u',\lambda_s'p_s'|\lambda_t'p_t'\rangle$, $(p_t'=p_u'+p_s',\ \lambda_t'=\lambda_u'-\lambda_s')$. The coupling constant $g_{\lambda_u'\lambda_s'}{}^t$ is defined in the rest frame of particle $\bar{t}$, or equivalently the center-of-mass frame of $u$ and $\bar{s}$, if $\bar{t}$ is a resonance. With $\mathbf{p}_u'$ taken along the positive $z$ axis, $\lambda_t'$ is the $z$ component of spin for particle $\bar{t}$. To cross this function, we first boost all particles along the positive $z$ axis until we get to the rest frame of particle $\bar{s}$ (laboratory frame). The $z$ component of spin is not affected by the boost, so in the new frame (denoted by double prime) $\lambda_u''=\lambda_u'$, $\lambda_t''=\lambda_t'$, and the $z$ component of spin for $s$ is $-\lambda_s'$. Now we cross the particles

[10] This derivation is essentially the one given in Ref. 5.

$\bar{s}$ and $\bar{t}$ to $s$ and $t$. In order to conserve the $z$ component of spin both before and after crossing, the $z$ component must flip for either the $s$ and $t$ particles or the $u$ particle. The resultant coupling is $g_{\lambda_u \lambda_t}{}^{*}$ (up to a real phase) with $\lambda_u = \pm\lambda_u'$ and $\lambda_t = \pm\lambda_t' = \pm(\lambda_u - \lambda_s)$. Note that these two couplings have the same magnitude if parity is conserved.

In addition to the TP relations used in the above proof, there are the derivative relations obtained by expanding the right-hand side of Eq. (4.12) to higher orders in $(t-t^{\pm})$. For the photon, these additional relations survive at $t=m_t^2$ and are responsible for eliminating the kinematical factors of $1/(t-m_t^2)^{2J_t}$ from the unpolarized cross section so only the "dynamical pole" at $t=m_t^2$ survives. In terms of the amplitudes $(t-m_t^2)^{J_t} \times M_{\lambda_u\lambda_s,\lambda_t\lambda_V}{}^t(s,t)$ that contain only dynamical poles the new TP relations are given by

$$\sum_{\lambda'_t} u_{\tau\lambda'_t}{}^{*}(t-m_t^2)^{J_t} M_{\lambda_u\lambda_s;\lambda'_t\lambda_V} \propto (t-m_t^2)^{J_t+\tau}. \tag{4.19}$$

This completes the description of the kinematical constraints discussed earlier in this section for the photoproduction amplitudes.

## V. OTHER APPLICATIONS

Before we discuss the possibility of other applications of this method, we should again emphasize that the assumed behavior of amplitudes in the external mass, which is so essential to our proof, needs a careful examination in itself. As the mass of an external particle goes to zero, an infinite number of branch points approach the point $s=m_s^2$, $t=m_t^2$. As mentioned before, a knowledge of the behavior of the amplitude in this limit probably entails a detailed examination of the unitarity condition. A very interesting question that arises in this connection is the relation of the strength of the coupling to this behavior. We have to find out whether our smoothness assumptions hold only for special cases of small couplings such as electrodynamics, or whether they are also true when the small mass particle participates in strong interactions. If this latter alternative is true, we can proceed and examine some of the applications of this method to strongly interacting particles.

The first application concerns the problem of $\rho$ universality. Usually $\rho$ universality is proved with the aid of the concept of $\rho$ dominance in, for example, the electromagnetic form factor. It is possible that our method may be used to deduce some approximate universality without the use of $\rho$ dominance. Note that one can easily generalize the mechanics of our proof of charge conservation to the case of couplings involving isospin or other symmetries. This problem of $\rho$ universality and the related problem of its application to Compton scattering is now being investigated. We should not expect $\rho$ universality to be very accurate, because of the long extrapolation in the mass of the $\rho$.

Another interesting set of problems may be the application of this method to the scattering of pions off other particles or the coupling of currents in weak interactions. In the proof of this paper, we used a four-body amplitude in which the small mass particle was one of the external lines. The question arises whether we can substitute, for example, the external vector particle with a $p$ wave, $\pi\pi$ channel. If this can be done, we can then prove results such as the Adler-Weisberger relation for the antisymmetric part of $\pi\pi$ amplitudes. More promising perhaps is the application to weak interactions, since the coupling is small and local. However, we feel that in order to make rigorous statements about these problems, a careful study of the five-line connected part and the related questions of TP relations for subenergies is needed. Such a study may also lead to other results associated with current algebra, partially conserved axial-vector current (PCAC), and the smallness of certain masses.

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