# Study of Scattering Amplitudes with Regge Poles Only

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The object of this paper is a study of the idea, recently emphasized, that crossed-channel Regge poles only can well express a given amplitude not only asymptotically but also at any energy. The present study is a phenomenological one and the reaction examined is $\pi N$ charge exchange. All leading Regge trajectories which have been found by previous research to be necessary for accurate agreement with the data (namely, $\Delta$, $N$, $\rho$, $\rho'$) are introduced. In order to cover the region for which we do not have detailed knowledge of Regge-pole parameters, the assumption is made that trajectories are straight lines. Special care is taken in discussing the residues which are the most relevant quantities. The $\alpha$ factors included in the residues are discussed in some detail. The various possible choices of $\alpha$ factors are effective both on the $s$-channel Argand diagrams and also in providing the resonance dip and bump structures in $d\sigma/dt|_{t=0}(\pi^- p \to \pi^0 n)$ and $\sigma_{\text{total}} = \sigma(\pi^- p) - \sigma(\pi^+ p)$. Throughout our calculations, exact formulas are always used. Finally, the result of a similar study of the elastic $\pi p$ reactions is discussed.

## I. INTRODUCTION

INCREASING interest has recently been devoted to the problem of writing scattering amplitudes containing only Regge-pole contributions at all energies. In other words, the background integral should be mainly reexpressed by some choice of a few or an infinite number of Regge poles.

Basically, in this spirit, two ideas have been elaborated: The older is that of interference between the direct-channel resonances and the crossed-channel Regge pole[1]; the newer, which has been emphasized in different ways by Schmid,[2] by Khuri,[3] and by Veneziano,[3] is that of duality (using the name introduced by Chew and Pignotti[4]). The duality approach seems very much simpler for bootstrap purposes. Unfortunately, so far there is no complete theory which gives interpolation between the high-energy and low-energy regions for any realistic reactions.[5] Therefore, at this stage, our study of the duality principle can only be a phenomenological one. We argue that the Regge-pole terms (both $t$ and $u$ Regge poles) obtained by fitting the high- and the intermediate-energy data, when extrapolated down to the resonance regions, ought to give a qualitative agreement. For example, if the combined $t$- and $u$-channel Regge-pole contribution can produce the resonance-type structures, they should produce the correct peaks and dips. If they cannot produce the resonance-type structures, they at least should give an average value.

It is in this spirit that we study the best-fitted reaction $\pi^- p \to \pi^0 n$. The following assumptions are made:

(a) Only $t$- and $u$-channel Regge poles are introduced.

(b) Four poles, $\rho$, $\rho'$, $\Delta$, and $N$, are used, namely, the minimum number which is required by the phenomenology.

(c) All poles contribute in an additive way.

(d) When extrapolation is needed, straight lines are used.

Under these assumptions, it is immediately recognized that the important problem is the study of the residues. In fact, the $\alpha$ factors, which greatly determine the structure of the residues at small $t$ (in such a region the pure $t$ factors are also important), become of overwhelming importance when the full range of $t$ ($u$) is studied.

We have made the following requirements in performing the $s$-channel partial-wave analysis: (i) to use amplitudes in the regions of $s$, $t$, and $u$ where they are found to fit the experimental data as well as possible; and (ii) to obtain properly circulating and shaped Argand circles. The first requirement is important because the linearity of the trajectories (although the only possibility in order to make practical calculations) does not have a deep theoretical support. Consequently, we prefer to retain a phenomenological spirit and avoid discussions, for example, at very high $s$, where there is a very wide range of $t$, in which the experimental data are not abundant. The easily found complicated patterns in the Argand diagrams for high spin $J$ are, for this reason, not discussed in this study. The second requirement is the logical consequence of the starting postulates, namely, that the amplitude should be represented well by crossed-channel Regge poles alone even in the resonance region. Thus, they should produce well-shaped Argand circles, which we usually associate with resonances.

In addition to the Argand circles of the partial-wave amplitudes, we have observed a peculiar phenomenon. The combination of the $t$- and $u$-channel Regge-pole

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[1] V. Barger and D. Cline, Phys. Rev. **155**, 1792 (1967); R. Logan and L. Sertorio, Phys. Rev. Letters **17**, 834 (1966); V. Barger and M. Olsson, Phys. Rev. **151**, 1123 (1966).

[2] C. Schmid, Phys. Rev. Letters **20**, 689 (1968); P. Collins, R. Johnson, and E. Squires, Phys. Letters **27B**, 23 (1968); V. Alessandrini, E. Squires, P. Freund, and R. Oehme, *ibid.* **27B**, 463 (1968); C. B. Chiu and A. Kotanski, Nucl. Phys. **B7**, 615 (1968).

[3] G. Veneziano, Nuovo Cimento **57A**, 190 (1968); N. N. Khuri, Phys. Rev. **176**, 2026 (1968); M. Suzuki (private communication).

[4] G. Chew and A. Pignotti, Phys. Rev. Letters **20**, 1078 (1968).

[5] So far Veneziano's model has not been generalized to more realistic reactions.

terms can give an oscillitory behavior in the cross sections as $s$ varies for fixed $t$, $u$, or $\cos\theta_s$. The $t$- or $u$-channel Regge poles alone cannot produce such an effect. We find that a specific choice of the sign and the zero-choosing mechanism of the residues of the $\rho$, $\rho'$, $N$, and $\Delta$ can produce a reasonable structure in $d\sigma/dt|_{t=0}(\pi^- p \to \pi^0 n)$ and $\sigma(\pi^- p)-\sigma(\pi^+ p)$. We also find that such a set of choices also produces the most reasonably shaped Argand circles. The set of choices is the following: The residue of the $\Delta$ should be negative and small and the residue of the $N$ should be positive and large. Besides the well-established "choosing no fixed pole" mechanism for the $N$ trajectory at $\alpha_N=-\frac{1}{2}$, $N$ must choose a fixed pole for $\alpha_N>-4.5$. As indicated in the $\pi^\pm p$ backward scattering, the $\Delta$ trajectory chooses a fixed pole at $\alpha_\Delta=-1.5$, but should choose no fixed pole for all $\alpha_\Delta\leq-9.5$, and the $\Delta$ trajectory should at least go down to $\alpha_\Delta=-9.5$. The statement about the $\Delta$ is of less certainty than that about the $N$. Certainly the fit to the oscillations of the data can be made closer than that presented in this paper by more searching of the parameters. Nevertheless, it seems hard to fit the data perfectly.

The $\pi N$ charge-exchange reaction has been analyzed in the same spirit before.[2] Our main contributions are as follows: using exact formulas in the calculations, including the $u$-channel Regge poles as well as the $t$-channel poles, and observing the oscillatory behavior in the cross sections produced by the combined effects of the $t$- and $u$-channel Regge poles.

In Sec. II D, the result of a similar study of elastic $\pi N$ scattering is discussed.

## II. ACTUAL CALCULATIONS

Before beginning to explain the details of our calculations, we would like to remark that the full study of the $\pi N$ system under the mentioned hypotheses should include a treatment of all channels. Namely, we have two channels $\pi N\to\pi N$ and $N\bar{N}\to\pi\pi$. The crossing relation connects the two. More precisely, when we consider the reconstruction of direct-channel Regge recurrences as given by crossed-channel $s$ Regge poles, we should consider two systems:

$$\int_{-1}^{1} A^s(s,t,u)_{1,2}P_l(Z_s)dZ_s$$

$$=\int_{-1}^{1}([\text{Regge pole}]^t+[\text{Regge pole}]^u)_{1,2}P_l(Z_s)dZ_s,$$

where the indices 1 and 2 refer to $\pi N\to\pi N$ and $N\bar{N}\to\pi\pi$. In such a way the self-consistency of the approach would be complete. We study this general formula only in the $\pi N\to\pi N$ case, and more particularly in $\pi N$ charge exchange. The poles considered are $\rho$ and $\rho'$ in the $t$ channel, and $N$ and $\Delta$ in the $u$ channel. The $\rho$ and $\rho'$ poles are from the fit of Ref. 6. Since there are no charge-exchange data in the backward direction, we take the predicted $\Delta$, $N$ contribution to the charge-exchange reaction from the present fit[6] to $\pi^\pm p$ elastic backward-scattering data.[7] All these fits are at fairly high energy and small momentum transfers.

Because our purpose is to consider only Regge-pole contributions to the amplitudes in the unexplored regions of energy and momentum transfer, a few comments about the extrapolation are necessary.

*Large* $|t|$. This means, more precisely, every momentum transfer. The customary Regge procedure explores the small region of $|t|$, especially the difficult point at $t=0$. Consequently, the parametrizations are biased. For example, some trajectory factors and $t$ factors that are zeros near the forward direction become infinite in our study. Also, curved trajectories, which can be helpful in understanding details of the small $|t|$ region, are unacceptable for the full $t$ range. In our calculation we make the simplest extrapolation, i.e., straight-line trajectories and constant residue with the appropriate $s_0$. Hopefully, this problem of extrapolation to large $t$ can be eventually settled by future high-energy large-momentum-transfer experiments.

*Low* $s$. The extrapolation of the $t$, $u$ Regge pole to low energies has to be understood in a fundamental way. This is really the heart of the problem. The conventionally used expression $s^\alpha$ cannot be expected to give us a correct answer at low energies, especially near threshold. Since it does not have the correct $s$-channel analyticity and does not carry explictly the $s$-channel unitarity, we have tried the other two simplest alternatives. They are $(s-m^2-\mu^2)^\alpha$ and $[s-(m+\mu)^2]^\alpha$. The second one is intended to bring out some of the structure near threshold. The details of our findings are given in the following subsections.

### A. Formulas

Through crossing, the $s$-channel helicity amplitudes are related to $t$- and $u$-channel contributions,[8]

$$f_{+,+}{}^s(s,t)=f_{+,+}{}^{s,(u)}(s,t)+f_{+,+}{}^{s,(t)}(s,t)\,, \qquad (2.1)$$

$$f_{-,+}{}^s(s,t)=f_{-,+}{}^{s,(u)}(s,t)+f_{-,+}{}^{s,(t)}(s,t)\,, \qquad (2.2)$$

where $f^{s,(u)}$ is the $u$-channel contribution to the $s$ channel and $f^{s,(t)}$ is the $t$-channel contribution to the $s$ channel.

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[6] Our parametrizations are from V. Barger and D. Cline, Phys. Rev. Letters **21**, 392 (1968); Phys. Letters **21**, 312 (1968); M. Toller and L. Sertorio, Phys. Rev. Letters **19**, 1146 (1967).

[7] J. Orear, D. P. Owen, F. C. Peterson, A. L. Read, D. G. Ryan, D. H. White, A. Ashmore, C. J. S. Damerell, W. R. Frisken, and R. Rubinstein, Phys. Rev. Letters **21**, 389 (1968).

[8] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959); T. L. Trueman and G. C. Wick, *ibid.* **26**, 322 (1964); I. J. Muzinichi, J. Math. Phys. **5**, 1481 (1964); M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964), Appendices A and B.

The normalization of our amplitudes is such that the unpolarized differential cross section is

$$\frac{d\sigma}{dt}=\frac{1}{4\pi s p_s^2}(|f_{+,+}^s|^2+|f_{-,+}^s|^2). \tag{2.3}$$

The following amplitudes will be relevant for the partial-wave analysis:

$$\bar{f}_{+,+}^s(\sqrt{s},t)\equiv(\cos\tfrac{1}{2}\theta_s)^{-1}f_{+,+}^s(\sqrt{s},t) \equiv 2\pi s^{1/2}[f_1^s(\sqrt{s},t)+f_2^s(\sqrt{s},t)], \tag{2.4}$$

$$\bar{f}_{-,+}^s(\sqrt{s},t)\equiv(\sin\tfrac{1}{2}\theta_s)^{-1}f_{-,+}^s(\sqrt{s},t) \equiv 2\pi s^{1/2}[f_1^2(\sqrt{s},t)-f_2^s(\sqrt{s},t)], \tag{2.5}$$

where $f_1$ and $f_2$ are the most conventionally used "parity-conserving" helicity amplitudes. They are related by MacDowell symmetry,

$$f_2^s(\sqrt{s},t)=-f_1^s(-\sqrt{s},t). \tag{2.6}$$

The partial-wave projection actually used in our calculation is

$$F_{\frac{1}{2},\frac{1}{2}}^{J\mp}(\sqrt{s})=\sqrt{2}\,2\pi s^{1/2}\int_{-1}^{+1}[f_1^s(\sqrt{s},t)C_{\frac{1}{2},\frac{1}{2}}^{J\pm}(Z_s) +f_2^s(\sqrt{s},t)C_{\frac{1}{2},\frac{1}{2}}^{J\mp}(Z_s)]dZ_s, \tag{2.7}$$

where

$$C_{\frac{1}{2},\frac{1}{2}}^{J+}=\tfrac{1}{2}\sqrt{2}P_{J-\frac{1}{2}},\quad C_{\frac{1}{2},\frac{1}{2}}^{J-}=\tfrac{1}{2}\sqrt{2}P_{J+\frac{1}{2}}.$$

The $F^{J\mp}$ are related to the conventional notation:

$$F_{\frac{1}{2},\frac{1}{2}}^{J-}=f_{l+}\quad\text{and}\quad F_{\frac{1}{2},\frac{1}{2}}^{J+}=f_{l-}. \tag{2.8}$$

First, we give the $u$-channel Regge-pole parametrization:

$$f_1^{s,(u)}(\sqrt{s},u)=\frac{(\sqrt{s}+m)^2-\mu^2}{4s}\sum_{j=N,\Delta}I^jR^j, \tag{2.9}$$

where

$$R^N=\xi_N2\beta_N[1+\delta_N(-\sqrt{s}+2m)](s/s_{0N})^{\alpha_N-\frac{1}{2}}, \tag{2.10}$$

$$R^\Delta=\xi_\Delta2\beta_\Delta[1+\delta_\Delta(-\sqrt{s}+2m)](s/s_{0\Delta})^{\alpha_\Delta-\frac{1}{2}}. \tag{2.11}$$

The $\xi$'s are signature factors multiplied by the $\alpha$ factors:

$$\xi_N=\{\tan[\tfrac{1}{2}\pi(\alpha_N+\tfrac{1}{2})]+i\}(\alpha_N+\tfrac{1}{2})(\alpha_N+\tfrac{3}{2}), \tag{2.12}$$

$$\xi_\Delta=\{\cot[\tfrac{1}{2}\pi(\alpha_\Delta+\tfrac{1}{2})]-i\}(\alpha_\Delta+\tfrac{1}{2})(\alpha_\Delta+\tfrac{3}{2}). \tag{2.13}$$

The $I$'s are isospin factors. For the charge exchange, we have

$$I^\Delta=-I^N=\tfrac{1}{3}\sqrt{2}.$$

The parameters given by Ref. 6 are

$$\begin{aligned} &\alpha_N=-0.38+0.88u, && \alpha_\Delta=0.19+0.87u,\\ &\beta_N=-32\text{ BeV}^{-1}, && \beta_\Delta=0.1\text{ BeV}^{-1},\\ &\delta_N=-1.58/m_N\text{ BeV}^{-1}, && \delta_\Delta=1.54/m_\Delta\text{ BeV}^{-1},\\ &s_{0N}=0.45\text{ BeV}^2, && s_{0\Delta}=2.85\text{ BeV}^2. \end{aligned}$$

Actually, in their fit, they cannot determine the signs of $\beta_N$ and $\beta_\Delta$. Notice that the parametrization of the $\alpha$ factors in Eqs. (2.12) and (2.13) is only good for small $|u|$ where the $\alpha$'s$>-\frac{5}{2}$. Since we extrapolate to large $|u|$, we have to use a more general form, for example, the $\Gamma$ functions or the sine and cosine functions. The zero at $\alpha_N=-\frac{1}{2}$, which is a nonsense wrong-signature point of $\alpha_N$, is quite well established. We call this the "choosing-no-fixed-pole" mechanism. There are strong indications that the zero at $\alpha_\Delta=-\frac{3}{2}$, which is a nonsense wrong-signature point of $\alpha_\Delta$, is not there.[7] We call this the "choosing-fixed-pole" mechanism.[9] We have no experimental indication from the fit of Ref. 6 to Ref. 7 which are the zero-choosing mechanisms for $\alpha_N<-\frac{1}{2}$, $\alpha_\Delta<-\frac{3}{2}$. We try both mechanisms.

(i) Choosing-no-fixed-pole mechanism[10]:

$$\xi_N=(2/\pi)\{\sin^2[\tfrac{1}{2}\pi(\alpha_N+\tfrac{1}{2})]+i\sin[\pi(\alpha_N+\tfrac{1}{2})]\} \times\Gamma(\tfrac{1}{2}-\alpha_N), \tag{2.14}$$

$$\xi_\Delta=(2/\pi)\{\cos^2[\tfrac{1}{2}\pi(\alpha_\Delta+\tfrac{1}{2})]-i\sin[\pi(\alpha_\Delta+\tfrac{1}{2})]\} \times\Gamma(\tfrac{1}{2}-\alpha_\Delta)/(\alpha_\Delta+\tfrac{3}{2}). \tag{2.15}$$

The division factor of $\alpha_\Delta+\frac{3}{2}$ in Eq. (2.15) is to avoid the zero at $\alpha_\Delta=-\frac{3}{2}$.[11] We always normalize the amplitude to be equal to that of Eqs. (2.9) and (2.10) at $u=0$. The replacement of $\xi_N$ and $\xi_\Delta$ in Eqs. (2.12) and (2.13) by those of (2.14) and (2.15) changes the original fit of Ref. 6 insignificantly. Notice that by choosing this mechanism the imaginary part of the Regge poles in Eqs. (2.10) and (2.11) has a simple zero at all nonsense values of $\alpha$, i.e., all negative half-integers, except $\alpha_\Delta=-\frac{3}{2}$; and the real part has a double zero at every nonsense wrong-signature value of $\alpha$, i.e., every two units.

(ii) Choosing-fixed-pole mechanism:

$$\xi_N=(1/\pi)(\alpha_N+\tfrac{1}{2})\{\sin[\tfrac{1}{2}\pi(\alpha_N+\tfrac{1}{2})]+i\cos[\tfrac{1}{2}\pi(\alpha_N+\tfrac{1}{2})]\} \times\Gamma(\tfrac{1}{4}-\tfrac{1}{2}\alpha_N), \tag{2.16}$$

$$\xi_\Delta=(1/\pi)\{\cos[\tfrac{1}{2}\pi(\alpha_\Delta+\tfrac{1}{2})]-i\sin[\tfrac{1}{2}\pi(\alpha_\Delta+\tfrac{1}{2})]\} \times\Gamma(\tfrac{3}{4}-\tfrac{1}{2}\alpha_\Delta), \tag{2.17}$$

Here the imaginary part has simple zeros at nonsense right-signature values of $\alpha$ and the real part has simple zeros at nonsense wrong-signature values of $\alpha$. Notice that the zero at $\alpha_N=-\frac{1}{2}$ is always there.

For the $t$-channel contributions, we use amplitudes which are already known and written as $t$-channel helicity amplitudes. We then obtain the $t$-channel contribution $f_{+,+}^{s(t)}$, $f_{+,-}^{s(t)}$ by

$$f_{++}^{s(t)}=K_{11}f_{++}^t+K_{12}f_{++}^t, \tag{2.18}$$

$$f_{+-}^{s(t)}=-K_{12}f_{++}^t+K_{11}f_{+-}^t, \tag{2.19}$$

[9] C. E. Jones and V. L. Teplitz, Phys. Rev. **159**, 1271 (1967); S. Mendelstam and L. L. Wang, *ibid.* **160**, 1490 (1967).

[10] We have used the formula $[\Gamma(\alpha)]^{-1}=(1/\pi)\sin\pi\alpha\Gamma(1-\alpha)$.

[11] We have also tried the case with the zero at $\alpha_\Delta=-\frac{3}{2}$. The result is not satisfactory.

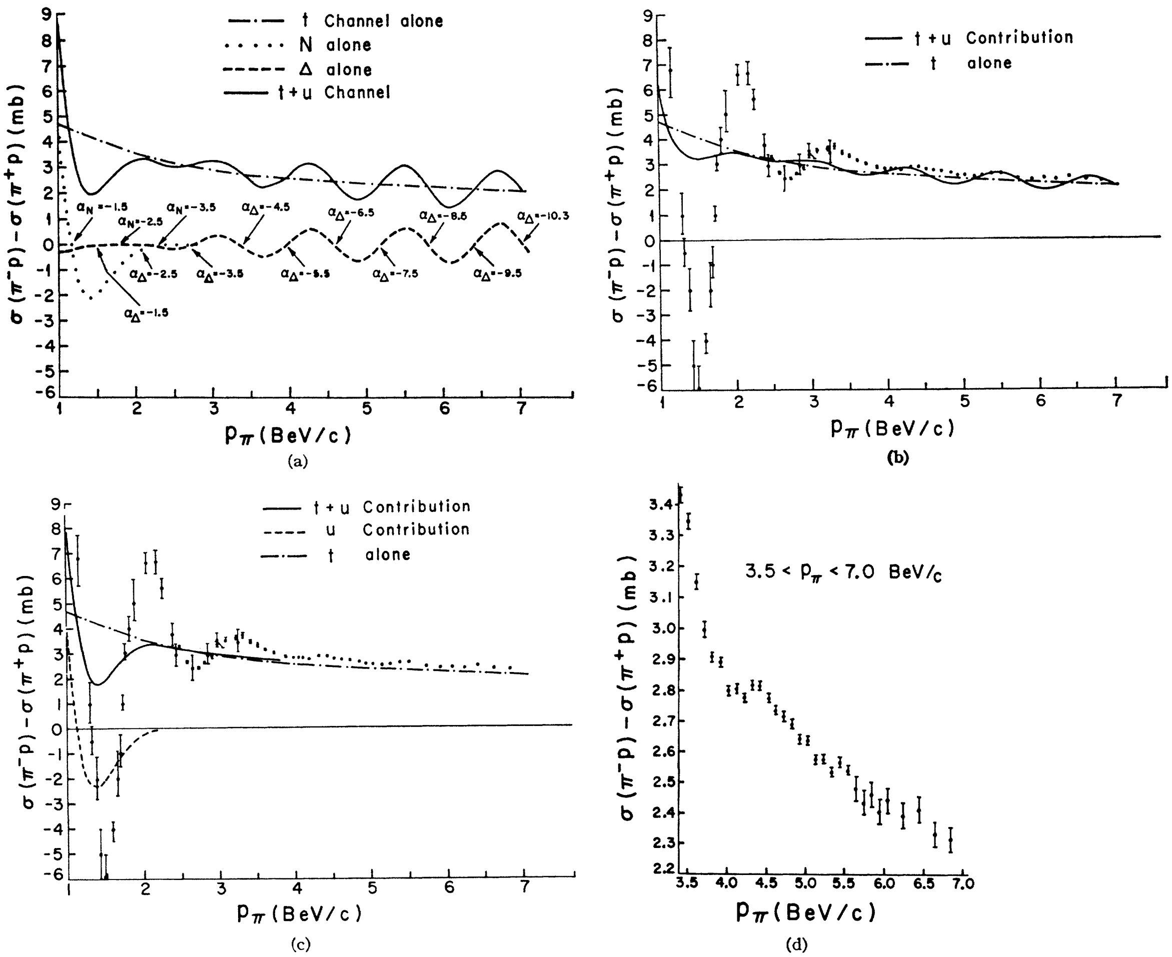



FIG. 1. Calculated $\sigma(\pi^-p)-\sigma(\pi^+p)$ from the $\rho$, $\rho'$, $N$, and $\Delta$ trajectories with $\beta_N=32$, $\beta_\Delta=-0.1$. The data points in Figs. 1 and 2 are taken from Barger and Olsson (Ref. 1). The references to the original papers are given in Ref. 18 and the following: T. Devlin *et al.*, Phys. Rev. Letters **14**, 1031 (1965); A. Diddens *et al.*, *ibid.* **10**, 262 (1963); A. Stirling *et al.*, data quoted by B. Amblard *et al.*, Phys. Letters **10**, 138 (1964); A. Citron *et al.*, Phys. Rev. **144**, 1101 (1966). (a) The $N$ trajectory chooses fixed pole as given in Eq. (2.16) and the $\Delta$ trajectory chooses no fixed pole as given in Eq. (2.15). The energy-dependent form for the $u$-channel is $s-(m+\mu)^2$ as mentioned in Sec. II. (b) Same as (a) except that the energy-dependent form is $s$. (c) Both the $N$ and $\Delta$ trajectories choose fixed poles, and the energy-dependent form is $s-(m+\mu)^2$ for the $u$ channel. (d) Experimental data of $\sigma(\pi^-p)-\sigma(\pi^+p)$ above $p_\pi=3.5$ GeV/$c$.

where

$$K_{11}=2m(st+\mathcal{S}^2)^{1/2}/\mathcal{S}(-t+4m^2)^{1/2},$$

$$K_{12}=(-t)^{1/2}(s+m^2-\mu^2)/(-t+4m^2)^{1/2}\mathcal{S},$$

$$\mathcal{S}=\{[s-(m+\mu)^2][s-(m-\mu^2)]\}^{1/2}.$$

The $t$-channel parametrization[6] of the $I_t=1$ amplitude used is

$$f_{++}{}^t=\left(1-\frac{t}{4m^2}\right)^{1/2}A\tfrac{1}{2}m, \tag{2.20}$$

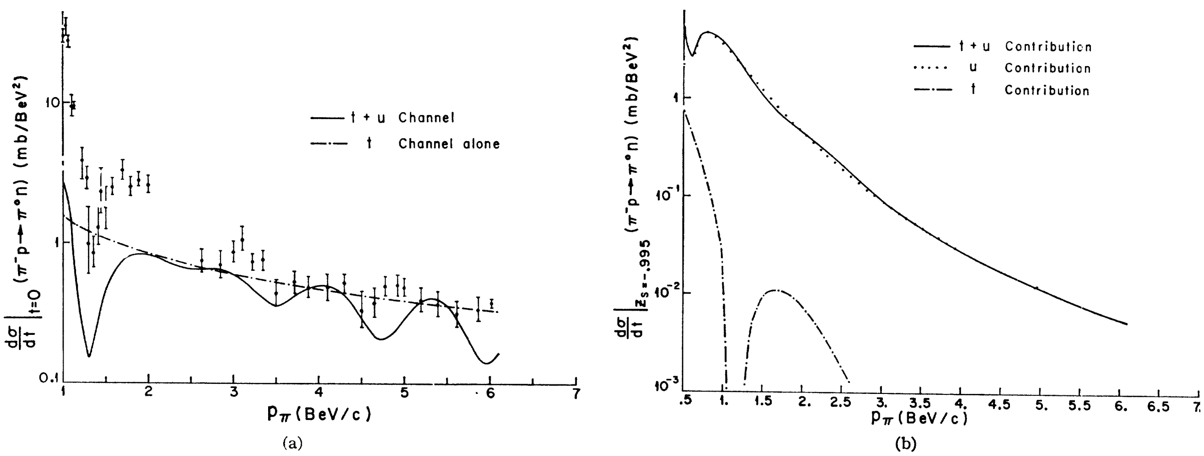
$$f_{+,-}{}^t=\frac{1}{2m}\left(p^2+\frac{st}{4m^2}\right)^{1/2}\left(\frac{-t}{4m^2}\right)^{1/2}\times\left(1-\frac{t}{4m^2}\right)^{1/2}B\tfrac{1}{2}m, \tag{2.21}$$

$$A=C_\rho(\alpha_\rho+1)\xi_\rho(E/E_0)^{\alpha_\rho}[(1+H)\exp(C_\rho t)-H]+C_{\rho'}t(\alpha_{\rho'}+1)\xi_{\rho'}(E/E_0)^{\alpha_{\rho'}}\exp(C_{\rho'}t), \tag{2.22}$$

$$B=D_\rho\alpha_\rho(\alpha_\rho+1)\xi_\rho(E/E_0)^{\alpha_\rho-1}\exp(D_\rho t)+D_{\rho'}(\alpha_{\rho'}+1)\times\xi_{\rho'}(E/E_0)^{\alpha_{\rho'}-1}\exp(D_{\rho'}t), \tag{2.23}$$

where the $\xi$'s are signature factors multiplied by $\alpha$ factors. They will be defined later. Notice that the $\rho'$ is a conspiring, fixed-pole-choosing trajectory and $\rho$ is nonconspiring and sense-choosing. We have also used the parametrization of Ref. 12 with $\rho'$ conspiring and fixed-pole-choosing, and with $\rho$ nonconspiring and nonsense-choosing. We present only the figures that correspond to the first kind, because the extrapolation is easier. As in the $u$ channel, we extrapolate to large $|\alpha|$

[12] G. Fox and L. Sertorio, Phys. Rev. **176**, 1739 (1968).



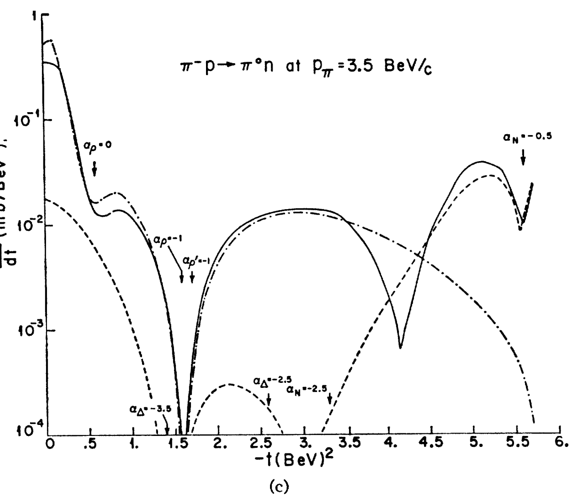




FIG. 2. Calculated $d\sigma/dt(\pi^- p \to \pi^0 n)$ from the $\rho$, $\rho'$, $N$, and $\Delta$ trajectories. All the $u$-channel energy-dependent form is $s-(m+\mu)^2$. (a) $d\sigma/dt|_{t=0}$. The data are from I. Mannelli *et al.*, Phys. Rev. Letters **14**, 408 (1965); A. V. Sterling *et al.*, *ibid.* **14**, 763 (1965); L. Guerriero, Proc. Roy. Soc. (London) **A289**, 471 (1966); C. Chiu, University of California Radiation Laboratory Report No. UCRL-16209, 1966 (unpublished); P. Borgeaud *et al.*, Phys. Letters **10**, 134 (1964); P. Falk-Vairant *et al.*, whose data are quoted in G. Hohler *et al.*, *ibid.* **21**, 223 (1966); M. A. Azimov *et al.*, Zh. Eksperim. i Teor. Fiz. Pis'ma v Redaktsiyu **3**, 336 (1966) [English transl.: Soviet Phys.—JETP Letters **3**, 216 (1966)]. The specification is exactly the same as in Fig. 1(a). (b) $d\sigma/dt$ at $Z_s=-0.995$. The specification for the $N$ and $\Delta$ is the same as in Fig. 1(a). The $\rho$ and $\rho'$ choose fixed poles as given in Eq. (2.24). (c) $d\sigma/dt$ from $Z_s=+1$ to $-0.9735$ for $p_\pi=3.5$ BeV/$c$, with the specification the same as in (b).

in two ways, with $\rho'$ always fixed-pole-choosing at $\alpha_{\rho'}=0$.

Choosing fixed pole:

$$\xi=(1/\pi)[i\cos(\tfrac{1}{2}\pi\alpha)+\sin(\tfrac{1}{2}\pi\alpha)]\Gamma[\tfrac{1}{2}(1-\alpha)]. \quad (2.24)$$

Choosing no fixed pole:

$$\xi=(1/\pi)(1/\alpha)[i\sin\pi\alpha+(1-\cos\pi\alpha)]\Gamma(1-\alpha). \quad (2.25)$$

Substituting the $t$- and $u$-channel contribution into Eqs. (2.1) and (2.2), one can easily calculate the cross sections; and through Eqs. (2.4), (2.5), and (2.7), one can obtain the projected $s$-channel partial-wave amplitudes.

### B. Structures in the Cross Sections

We present in Figs. 1 and 2 the calculated cross sections at $\cos\theta_s=+1$ and $-0.995$ down to a pion laboratory momentum of 1 BeV/$c$.[13] In Fig. 2(c) we show the cross section $d\sigma/dt$ in the full range of $Z_s=+1$ to $-0.9735$ from $t$- and $u$-channel Regge-pole contributions. In Table I we give the relevant values of $p_\pi$, $t$, $u$, $\alpha_N$, and $\alpha_\Delta$ at $Z_s=+1$ for discussing the structures in the cross sections. To see more clearly the structure in $\sigma(\pi^- p)-\sigma(\pi^+ p)$ above $p_\pi=3.5$ GeV/$c$, we show enlarged data points in Fig. 1(d).

We summarize several interesting points here. Since we did not do a complete fit, we only emphasize our qualitative results.

(a) For fixed $t$, the amplitude from any $t$-channel Regge-pole term is smooth in $s$, but for fixed $u$ or fixed $\cos\theta_s$, $t=-2q_s^2(1-\cos\theta_s)=2m^2-2\mu^2-u-s$. The $t$ varies as $s$ varies. Therefore, the amplitude from a $t$-channel Regge pole oscillates according to the zeros by the signature factor and the $\alpha(t)$ factor. Thus, the cross sections given by $t$-channel Regge poles oscillate as $s$ increases and $u$ or $\cos\theta_s$ is kept fixed—with one exception, i.e., the case of an amplitude containing *only one* crossed-channel pole, which chooses the fixed-pole mechanism for *all* nonsense value of $\alpha$. In that case the differential cross section is smooth for fixed $u$ or $\cos\theta_s$ as

[13] The $p_\pi$ in all figures is the pion laboratory momentum. The form of energy dependence of the $t$-channel contribution is always $E$ as given in Eqs. (2.22) and (2.23).

$s$ varies.[14] A similar statement is true for the $u$-channel Regge-pole terms. For fixed $u$, the amplitude given by $u$-channel poles is smooth, and for fixed $t$ or $\cos\theta_s$, it is oscillatory.

(b) The fit for $d\sigma/du$ of $\pi^+p$ elastic backward scattering is quite well determined[6] for $|u|<2$ BeV$^2$. Therefore we see from Table I that at $Z_s=+1$ for the pion laboratory momentum $p_\pi<1.5$ BeV/$c$ the $u$-dependent $u$-channel contribution is fixed. However, we still have the freedom in extrapolating the high-energy fit to low energies. We can change the $u$-channel contribution for small $p_\pi$ by changing $(s/s_0)^{\alpha-\frac{1}{2}}$ in Eqs. (2.10) and (2.11) to $[(s-m^2-\mu^2)/s_0]^{\alpha-\frac{1}{2}}$ or to $\{[s-(m+\mu)^2]/s_0\}^{\alpha-\frac{1}{2}}$. Compare Figs. 1(a) and 1(b). The $u$-channel contribution for $p_\pi>1.5$ BeV/$c$ depends more on the large $|u|$ extrapolation of Eq. (2.9). A similar discussion holds for the $t$-channel contribution at $Z_s=-0.995$ as given in Fig. 2(b).

(c) Let us now discuss Fig. 1(a) of $\sigma(\pi^-p)-\sigma(\pi^+p)$ in detail to see how the particular choices of the signs of the residues and the zero-choosing mechanism for $N$ and $\Delta$ are able to give results which better resemble the data.[15] For $1<p_\pi<392.5$ GeV/$c$, i.e., $0.995<|\mu|<3.8$ GeV$^2$, the $u$-channel contribution is totally from $N$. In order to produce the dip at $p_\pi=1.45$ and the bump at $p_\pi=2.0$, $\alpha_N$ has to choose fixed pole at $\alpha_N=-2.5$ and $\beta_N$ must be positive. (Remember that the sign of the $\rho$ pole is determined already.) For $p_\pi>2.5$ GeV/$c$, i.e., $|u|>3.8$ GeV$^2$, the $\Delta$ trajectory totally takes over. The period of the oscillation is about 1 BeV/$c$ of $p_\pi$. This requires that the $\Delta$ contribution vanishes at all $\alpha_\Delta\leq-2.5$ nonsense values for the given slope, i.e., choosing the no-fixed-pole mechanism, and the $\Delta$ trajectory should at least go down to $\alpha_\Delta\leq-9.5$. The positions of the bumps and dips require $\beta_\Delta$ to be negative. We shall see that these choices are also consistent with the result of the Argand circles. We ought to make it clear that the statement about the $\Delta$ is not so clear cut as for the $N$.

(d) The same discussion [as in (c)] can be given about $d\sigma/dt|_{t=0}(\pi^-p\to\pi^0n)$, but it is more complicated. It turned out that $d\sigma/dt|_{t=0}$ also consistently chooses the same result as in (c). The result is shown in Fig. 2(a). The prediction for $d\sigma/dt(\pi^-p\to\pi^0n)$ at $Z_s=-0.9735$ is given in Fig. 2(b).

(e) If the dip-bump structures, which so far are understood as due to the $s$-channel resonances in the cross sections, are truly associated with the signature factors and $\alpha$ factors of $t$ and $u$ channels, it would mean that for infinitely rising $s$-channel trajectories, the $t$- and $u$-channel Regge trajectories must be also infinitely extended. This is quite consistent with the bootstrap idea. In Fig. 2(c) we show the full range $d\sigma/dt$ from $Z=+1$ to $-0.9735$. Notice the association of the dips to the $\alpha$ values of different trajectories.

TABLE I. Some values of $p_\pi$ and $u$ at $Z_s=+1$ and $\alpha_N$ or $\alpha_\Delta$ near negative half-integers.

| $p_\pi$ (BeV/$c$) | $u$ (BeV$^2$) | $\alpha_N$ | $\alpha_\Delta$ |
|---|---|---|---|
| 0.5 | −0.736 | −0.445 | 0.126 |
| 1.15 | −1.27 | −1.50 | −0.919 |
| 1.5 | −1.93 | −2.08 | −1.49 |
| 1.75 | −2.40 | −2.49 | −1.89 |
| 2.10 | −3.05 | −3.07 | −2.47 |
| 2.35 | −3.52 | −3.49 | −2.87 |
| 2.75 | −4.27 | −4.14 | −3.53 |
| 3.35 | −5.40 | −5.13 | −4.50 |
| 3.95 | −6.52 | −6.12 | −5.48 |

### C. Argand Circles

Here we present only our calculation of $f_{3+}{}^s$, which is the amplitude that contains the $\tau p=-$resonances, e.g., the $\Delta_\delta$. We want to emphasize only the qualitative properties of the Argand circle. There is little point in showing all the $f_{l\pm}{}^s$. The detailed structure of the Argand circle, especially for the high-$l$ ones, depends too much on the detailed behavior of the Regge-pole terms, which have not been well determined.[16]

(a) The signs of the Regge residues determine the turning directions of the circles. The sign of the $\rho$ residue is determined by $\sigma(\pi^-p)-\sigma(\pi^+p)$. In Figs. 3(a) and 3(b) we show the result of the $t$-channel contribution alone. The signs determined produce the properly rising, counterclockwise turning of the circle.

(b) The speed of the turning of the circles depends upon the zero-choosing mechanism. The no-fixed-pole-choosing mechanism (i.e., more zeros in the amplitude) gives greater speed than the fixed-pole-choosing mechanism. Compare Fig. 3(a) with Fig. 3(b).

(c) The choice of $\beta_N=-32$ gives a downward turning circle from the $u$-channel alone. See Figs. 3(d), 3(f), and 3(g). $\beta_\Delta=+0.1$ or $-0.1$ determines the details of the circle. Unlike in Sec. II B, here there is no clear-cut way to choose between the two. Compare Figs. (3a) and 3(g). Both have the top at $p_\pi=1.7$ BeV/$c$, $\sqrt{s}=2.049$ BeV. However, all the results here are consistent with the conclusion that the signs and the zero-choosing mechanism are those given in Sec. II B.

### D. $\pi^\pm$ Elastic Scattering

Similar studies have been attempted to $\pi^\pm p$ elastic scattering including the $P$, $P'$, $\rho$, $N$, and $\Delta$ trajectories.

[14] For example, the contribution of a single trajectory with slope 1 to the differential cross section is

$$\frac{d\sigma}{dt}(Z=-1)=\frac{F(m,\mu)}{4q^2}\exp\left\{-8q^2\left[1+l_n\left(\frac{s}{4q^2}\right)\right]\right\}$$

for choosing fixed pole, and

$$\frac{d\sigma}{dt}(Z=-1)=\frac{F(m,\mu)}{4q^2}\exp\left\{-8q^2\left[1+l_n\left(\frac{s}{4q^2}\right)\right]\right\}(1-\cos 4\pi q^2)$$

for choosing no fixed pole. The first one is smooth and the second is oscillatory.

[15] The sharp minimum at $p_\pi=0.9$ GeV/$c$ cannot be successfully produced. Also notice from Figs. 1(b) and 2(a) that both the magnitude and the frequency of the oscillation indicate that $\alpha_\Delta$ ought to have a steeper slope for large $|u|$.

[16] Notice that many circles have radius greater than 1. Due to our normalization of the partial-wave amplitude in Eq. (2.7), this does not necessarily mean that unitarity is violated.

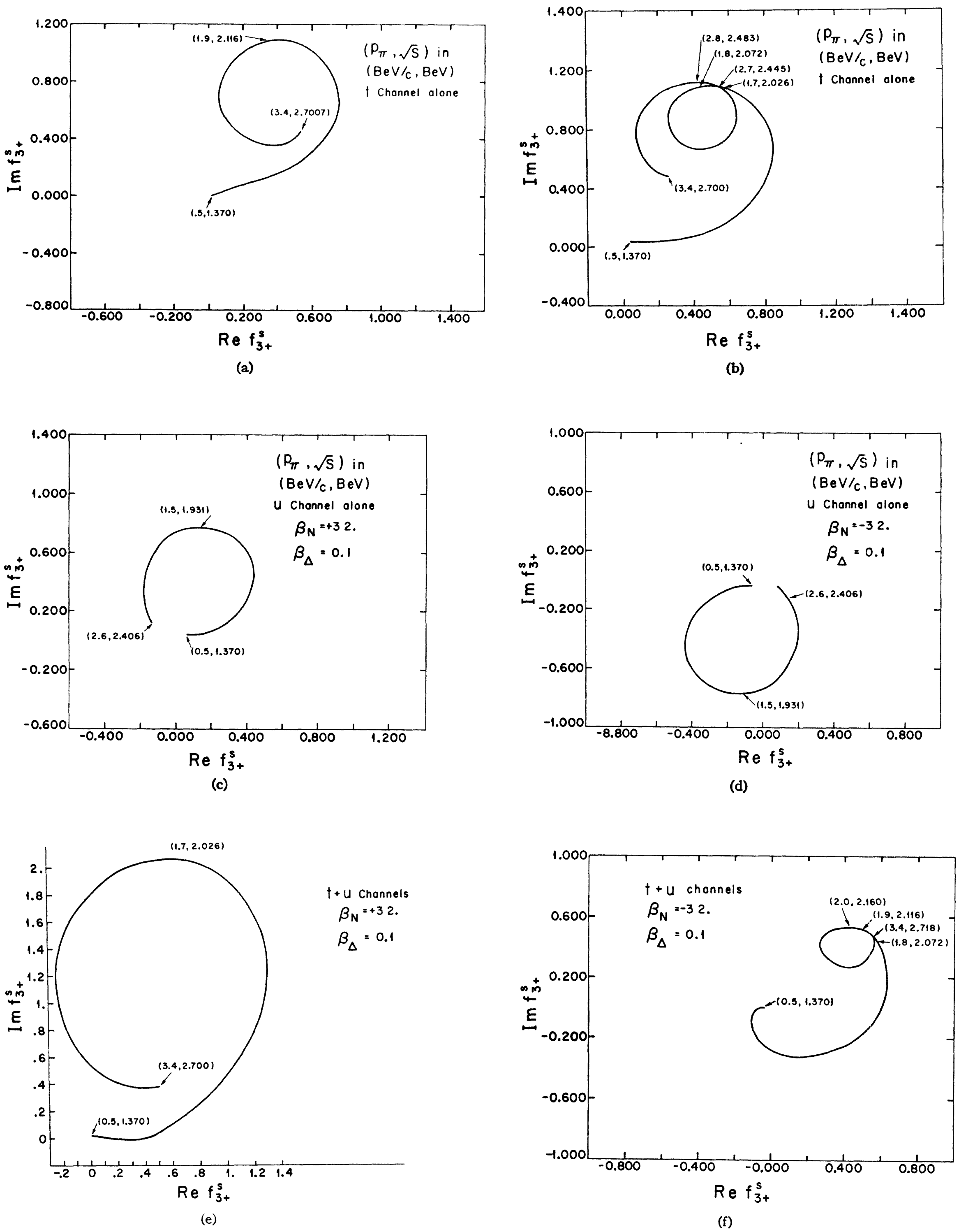



FIG. 3. Argand graphs for the $s$-channel partial-wave amplitudes $f_{3+}^s$ projected from the $t$- and $u$-channel Regge-pole terms, i.e., $\rho$, $\rho'$, $N$, and $\Delta$. The $u$-channel energy-dependent form is $s-(m+\mu)^2$ for all figures except (c) and (d), which have in form of $s$. (a) $t$-channel contribution alone. Both $\rho$ and $\rho'$ choose a fixed pole as given in Eq. (2.24). (b) $t$-channel alone. Both $\rho$ and $\rho'$ choose no fixed pole as given in Eq. (2.25). (c) $u$-channel alone; $\beta_N=32$, $\beta_\Delta=-0.1$. Both $N$ and $\Delta$ choose a fixed pole as given in Eqs. (2.16) and (2.17). (d) Same as (c), except $\beta_N=-32$, $\beta_\Delta=+0.1$. (e) Combined result of (a) and (b). (f) Combined result of (a) and (d). (g) $u$-channel alone. The $N$ chooses a fixed pole as given in Eq. (2.15); the $\Delta$ chooses no fixed pole as given in Eq. (2.15). $\beta=-32$, $\beta_\Delta=0.1$. (h) Same as (g), except $\beta_N=32$, $\beta_\Delta=0.1$. (i) Combined result of (a) and (h). (j) The result corresponding to the specification of Fig. 2(b).

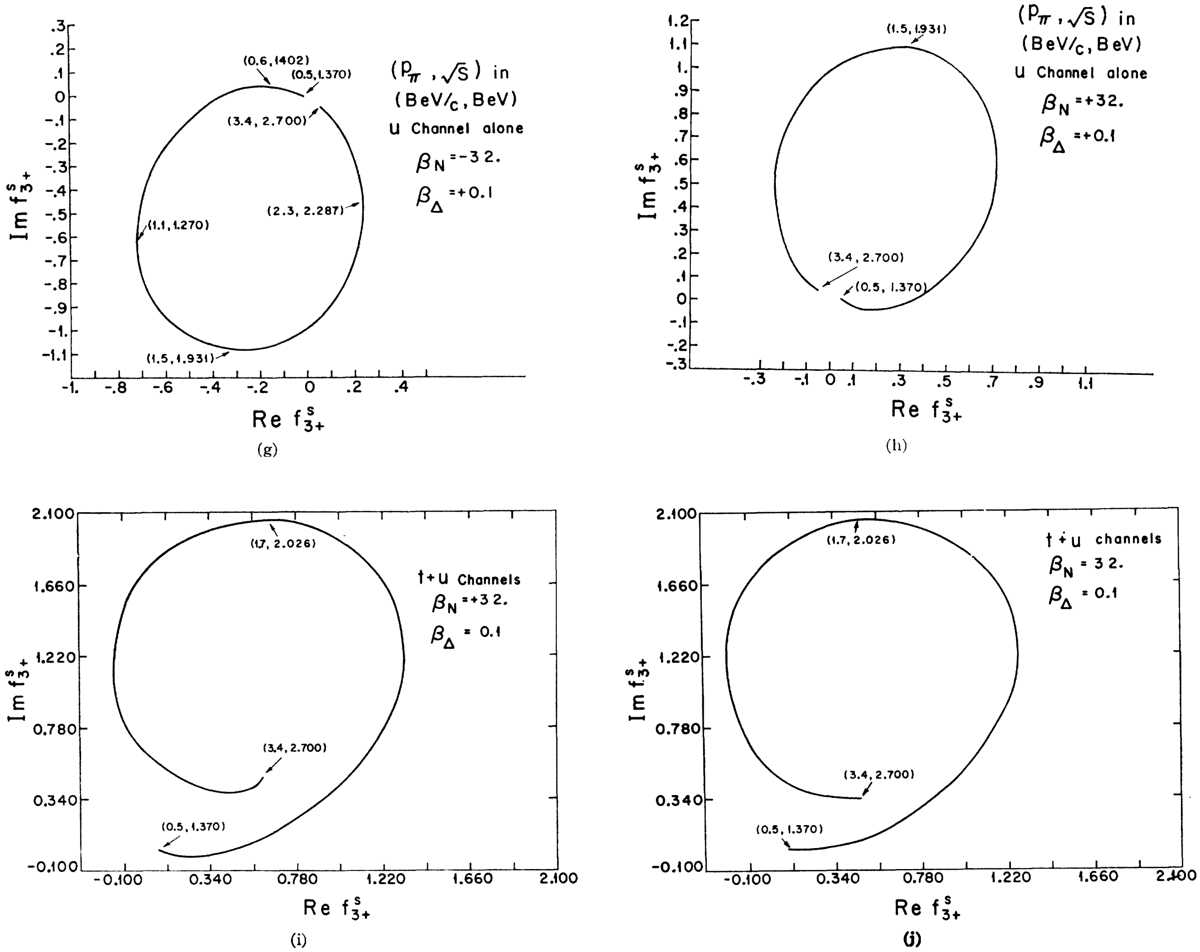



FIG. 3 (*Continued.*)

We take the parametrization of $P$, $P'$, and $\rho$ from Ref. 17, which gives a fit to the $\pi^{\pm}p$ elastic scattering down to $p_\pi = 2.5$ BeV/$c$ and $|t| \leq 2$ GeV$^2$, but the extrapolation is very hard. Many factors become very large at large $t$. The Argand graphs of the partial-wave amplitudes can hardly make a circle. Aside from extrapolation difficulties, we think that this finding is consistent with the observation made by Harari[18] that $P$ should not be related to direct-channel resonances. Since $P$ is very flat, it may never cross any nonsense points. In our calculation, those nonsense zeros are crucial to make the Argand circles. So we expect that $P$ cannot produce circles; thus it should not be related to the direct-channel resonances.

## III. SUMMARY

We have investigated the combined effects of the $t$-channel and $u$-channel Regge poles at low energy. We studied the Argand circles of the $s$-channel partial-wave projections and the $\sigma(\pi^- p) - \sigma(\pi^+ p)$ and $d\sigma/dt \times (\pi^- p \to \pi^0 n)$ in the forward and backward directions. We find that the Argand circles can be produced by $\rho$, $\rho'$; $N$, $\Delta$; and $\rho$, $\rho'$, plus $N$, $\Delta$, respectively. The circles are produced by the signature factor. However, the top and the shape of the various circles depend on the detailed zero-choosing mechanism of the Regge residues and the relative signs of the residues of these poles. In addition, we find that for fixed $t$ or $u$ or $Z_s$, the combined $t$-channel and $u$-channel Regge poles do produce bumps and dips in the cross sections as the energy varies. Again this structure comes from the signature factor, the zeros of the Regge-pole terms, and the signs of the residues. The magnitudes of the bumps and the dips depend upon the way in which the Regge poles are extrapolated down to low energies. We also find that in order to produce qualitatively correct oscillatory structure in the cross sections $d\sigma/dt|_{t=0}(\pi^- p \to \pi^0 n)$ and $\sigma(\pi^- p) - \sigma(\pi^+ p)$, only one set of signs of the residues and the zero-choosing mechanism of $N$ and $\Delta$ is acceptable. The $N$ residue should be positive and large and the

[17] C. B. Chiu, S. Y. Chu, and L. L. Wang, Phys. Rev. **161**, 1563 (1967).

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$\Delta$ residue should be negative and small. Besides the well-established choosing-no-fixed-pole mechanism for the $N$ trajectory at $\alpha_N=-\frac{1}{2}$, $N$ must choose a fixed pole for $\alpha_N>-4.5$. The $\Delta$ trajectory chooses a fixed pole at $\alpha_\Delta=-1.5$, but should choose no fixed pole for all $\alpha_\Delta \leq -9.5$, and the $\Delta$ trajectory at least should go down to $\alpha_\Delta=-9.5$. (The statement about the $\Delta$ is of less certainty than that about the $N$.)

Unfortunately, so far there is no complete theory which gives good interpolation between high energy and low energy for any realistic reactions. Therefore, at this stage, we naively take the present Regge representation and extrapolate it to low energies in a simple way. In doing this we would like to make the following point clear. In the particular reaction that we studied—$\pi^- p \to \pi^0 n$—it happens that the combined $t$- and $u$-channel Regge poles can produce the resonance-type structure. However, there are reactions in which such phenomena do not happen. Thus, at this stage of phenomenological study, we take the following attitude: Whenever the present form of Regge representation predicts something in the low-energy region (like the dip-bump structure given by the $s^{\alpha(u)}$ term for fixed $t$), it should be qualitatively correct. So even if we do not expect $s^{\alpha(t)}+s^{\alpha(u)}$ to represent the whole amplitude, it is still interesting to notice that the $s^{\alpha(u)}$ term at fixed $t$ gives qualitative agreement. When the crossed-channel Regge poles cannot produce any structures in low $s$, we should be happy enough if they give the average.

Our study here is an attempt to see how far it is possible to go with the assumptions made. The lack of full agreement with the data seems very interesting because it is a measure (although a semiquantitative measure) of what is missed in the model.

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# Zero-Mass Bosons in $S$-Matrix Theory*

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We describe the soft coupling of zero-mass bosons to other particles, by considering the limit of a theory with a massive boson. With the standard $S$-matrix assumptions of analyticity and crossing for four-body helicity amplitudes, we demonstrate generally that in the limit of zero mass, a vector boson ($1^-$) couples to a conserved charge and a $2^+$ boson couples to the inertial mass. Bosons of other spin-parity combinations (with the exception of zero spin) have no zero-mass soft coupling. With this technique, we not only give a pedagogically interesting solution to gauge invariance and the kinematics of zero-mass particles, but suggest new applications to small-mass integral-spin systems. We speculate on the application of this technique to such problems as $\rho$ universality, the Adler-Weisberger relation, and the universality of leptonic couplings in a vector or axial-vector state.

## I. INTRODUCTION

RECENTLY, several authors[1] have studied the question of gauge invariance and zero-mass particles in $S$-matrix theory, and the related subject of small-mass mesons has also attracted some attention.[2] There exist two essentially distinct methods for the examination of the $S$-matrix theory of massless particles. One approach uses zero-mass particles from the beginning and entails the construction of certain amplitudes with the aid of the polarization four-vector of the zero-mass particle. The assumptions of Lorentz invariance, analyticity, and crossing are then introduced for these amplitudes. In this approach, the principle of gauge invariance—invariance under the addition of the light-like momentum vector to the polarization four-vector—is explicitly utilized. However, since it has been shown by Weinberg and Zwanziger that gauge invariance is a consequence of Lorentz invariance for zero-mass particles,[3] no new principle has in fact been introduced. This method has further been used by Weinberg to prove certain properties of the couplings of zero-mass particles, such as conservation of charge and the equivalence principle.

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