Validity of the Balázs Method*

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A systematic study of the Balázs method, as applied to the ρ-bootstrap problem in the 2π system, is made in order to determine its validity. Inelastic effects as estimated from Regge theory are included. We find that the method becomes well-behaved as the number of poles used to approximate the left-hand cut is increased, in contrast to results obtained previously by Bond, and that the sensitivity of the solutions to the choice of matching point may be largely eliminated by a procedure proposed by Williamson and Everett. The method predicts a ρ mass of somewhere between 680 and 760 MeV as compared to the experimental value of 750 ± 25 MeV, and a ρ half-width of about 150 MeV as compared with the experimental value of about 50 MeV.

I. INTRODUCTION

IN a series of papers[1–3] (hereafter referred to as BI, BII, and BIII) Balázs developed an approximation procedure for doing bootstrap calculations with partial-wave dispersion relations at low energies. The main features of the method are the following.

(i) The partial-wave amplitude is represented by the N/D method. The numerator function is approximated by a series of pole terms whose positions are determined by the kernel of the integral equation for the numerator function and whose residues are determined by requiring that the partial-wave amplitude and $n-1$ of its derivatives, as given by the N/D equations, be in agreement, at some matching point ν_F, chosen in between the right- and left-hand cuts, with the expression for the partial-wave amplitude and $n-1$ of its derivatives as obtained by projection from a fixed-energy dispersion relation for the amplitude. n is the number of poles approximating the numerator function, and ν is the square of the c.m. momentum. This procedure differs from the usual approximation of the numerator function by a series of poles in that the pole positions and residues are not arbitrary parameters. On the other hand, it has the usual advantage of decoupling the N/D integral equations. Also, some account is taken of the distant parts of the left-hand cut.

(ii) At high energy in the t channel, the absorptive part of the amplitude is obtained by a Regge representation. This avoids the need for a cutoff in the dispersion integral and takes into account the high-energy contributions to the amplitude.

(iii) A method for calculating the inelasticity constant $R_l^I(\nu)$, the ratio of total to elastic partial-wave cross sections, by using the Regge theory is given. This allows the incorporation of high-energy contributions to the amplitude in the N/D equations without making the elastic approximation.

A number of papers, some discussing the validity of the Balázs method and others using it, have appeared.[4–7] Most of the papers discussing the method find a strong sensitivity to the matching point ν_F. After a systematic study of the Balázs method, Bond[6] concluded that the method is totally unreliable, at least in one-channel bootstraps, partly because of its sensitivity to the matching point, but even more because of its extreme sensitivity to n, the number of poles used to approximate the left-hand cut. Williamson and Everett[5] (hereafter referred to as WE) proposed a method for alleviating the sensitivity to ν_F; their calculation employed a simplified version of the method in which inelastic and high-energy effects were ignored. In this paper, a study of the Balázs method, mostly as applied to the problem of the ρ bootstrap, is made, and it is found that the method gives reasonable results when used in conjunction with the WE criterion for choosing ν_F. In particular, we find that the method is not unduly sensitive to the number of poles approximating the left-hand cut. We will first sketch out the Balázs method, then show that solutions which are continuous well-behaved functions of the parameters exist, and finally show that the WE criterion holds not only in the simple two-pole approximation where it was first applied by WE but also for the two-, three-, and four-pole cases in the more elaborate version of the method used in this paper. As a result, it seems reasonable to hope that the criterion will hold in general.

II. BALÁZS METHOD

The threshold and asymptotic behavior of the partial-wave amplitude is factored out of the N/D equations by Balázs, in BIII, Eq. (1), in the following manner:

$$A_l^I(\nu)=\nu^l(\nu-\nu_K)^{1-l}H_l^I(\nu), \tag{1}$$

with

$$H_l^I(\nu)=N_l^I(\nu)/D_l^I(\nu). \tag{2}$$

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[1] L. A. P. Balázs, Phys. Rev. **128**, 1939 (1962).
[2] L. A. P. Balázs, Phys. Rev. **129**, 872 (1963).
[3] L. A. P. Balázs, Phys. Rev. **132**, 867 (1963); **134**, AB1(E) (1964).
[4] M. L. Mehta and P. K. Srivastava, Phys. Rev. **137**, 423 (1965).
[5] M. R. Williamson and A. E. Everett, Phys. Rev. **147**, 1074 (1966).
[6] A. H. Bond, Phys. Rev. **147**, 1058 (1966).
[7] S. K. Bose and M. Dersarkissian, Nuovo Cimento **30**, 878 (1963); **30**, 894 (1963); J. C. Pati, Phys. Rev. **134**, 387 (1964); G. L. Kane, *ibid.* **135**, 843 (1964); S. R. Choudhury and L. K. Pande, *ibid.* **135**, 1027 (1964); V. Singh and B. M. Udgaonkar, *ibid.* **128**, 1820 (1962); **130**, 1177 (1963); T. Kanki and A. Tubis, *ibid.* **136**, B723 (1964); P. Narayanaswamy and L. K. Pande, *ibid.* **136**, B1760 (1964); K. Igi, Phys. Rev. Letters **21**, 184 (1968).

In this paper we will consider only $l=1$. However, for the sake of generality we leave the factor $(\nu-\nu_K)^{1-l}$ in Eq. (1). For $l\neq 1$, this factor corrects the asymptotic behavior at the cost of introducing an artificial singularity at $\nu=\nu_K$.

The integrals for the N/D equations and partial-wave amplitude in the π-π problem are given in BIII, and we refer the reader to them rather than repeat them here. We will only present the final results for general values of l in an n-pole approximation. The numerator function $N_l{}^I(\nu)$ is given by a series of pole terms

$$N_l{}^I(\nu)=\sum_{i=1}^{n}\frac{f_{lI}{}^i}{(\nu+w_i)} \tag{3}$$

and the denominator function can then be written as

$$D_l{}^I(\nu)=1+\sum_{i=1}^{n}C_{lI}{}^i(\nu)f_{lI}{}^i, \tag{4}$$

where, in an approximation discussed below,

$$C_{lI}{}^i(\nu)=-[(\nu-\nu_0)/\pi]\times\{I_l{}^i(\nu,0)+[R_l{}^I(\nu_D)-1]I_l{}^i(\nu,\nu_D)\}, \tag{5}$$

with

$$I_l{}^i(\nu,y)=\int_y^\infty d\nu'\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\times\frac{\nu'^l}{(\nu'-\nu_0)(\nu'-\nu)(\nu'+w_i)(\nu'-\nu_k)^{l-1}}. \tag{6}$$

ν_D is the point of separation between the low- and high-energy regions in the t channel, and ν_0 is the subtraction point. For non-negative integer and half-integer values of l, Eq. (6) can be integrated in closed form; otherwise numerical integration is required.

An estimate of $R_l{}^I(\nu)$, the ratio of total to elastic partial-wave cross sections, can be obtained following the procedure of Balázs in BIII, Appendix A. For integer l it is possible to obtain an analytic solution for $R_l{}^I(\nu)$. The complete expression is somewhat complicated and for reasons of space will be omitted. However, asymptotically we have

$$R_l{}^I(\nu)=2\pi\epsilon_P\ln(2\nu)/\sigma_{\text{tot}}{}^I(\infty), \tag{7}$$

where ϵ_P=slope of the Pomeranchuk trajectory, and $\sigma_{\text{tot}}{}^I(\nu)$ is the total π-π cross section in the I channel. To obtain Eq. (5) we set

$$\begin{aligned}R_l{}^I(\nu)&=1 \quad \text{for } \nu<\nu_D\\ &=R_l{}^I(\nu_D) \quad \text{for } \nu\geq\nu_D.\end{aligned} \tag{8}$$

The partial-wave scattering amplitude as given by projection from a fixed-energy dispersion relation is separated into two parts.

$$A_l{}^I(\nu)=A_l{}^{I(L)}(\nu)+A_l{}^{I(H)}(\nu). \tag{9}$$

$A_l{}^{I(L)}(\nu)$ represents the contribution to the amplitude of the t-channel low-energy resonances, and is given by

$$A_l{}^{I(L)}(\nu)=-\frac{4}{\nu}\sum_i\beta_{II_i}(2l_i+1)\Gamma_i\nu_i{}^{l_i}(\nu_i-\nu_K)^{1-l_i}\times P_{l_i}\left(1+\frac{2(\nu+1)}{\nu_i}\right)Q_l\left(1+\frac{2(\nu_i+1)}{\nu}\right). \tag{10}$$

In Eq. (10) each set (ν_i,Γ_i,l_i,I_i) describes the c.m. momentum squared, the reduced half-width, the angular momentum, and the isospin of a low-energy resonance whose contribution to the amplitude is included. In our work, we kept only one term in the above resonance sum, corresponding to the ρ resonance with $(\nu_R,\Gamma_1{}^1,1,1)$. The half-width in the variable ν_i is connected to Γ_i by

$$(\Delta\nu_i)=[\nu_i/(\nu_i+1)]^{1/2}\nu_i{}^{l_i}(\nu_i-\nu_K)^{1-l_i}\Gamma_i.$$

Equation (10) is based on a narrow-width approximation for the resonances, in which we take

$$\text{Im}H_1{}^1(\nu)=\pi\Gamma_1{}^1\delta(\nu-\nu_R).$$

$A_l{}^{I(H)}(\nu)$ represents the contributions to the amplitude of the s-channel Regge trajectories and is given by

$$A_l{}^{I(H)}(\nu)=\sum_j K_j(l)\left(\frac{\beta_i}{\nu_j{}^{l_j}}\right)\times\frac{\nu^l}{l_j-l+\epsilon_j(\nu-\nu_j)}(2\nu_D)^{l_j-l+\epsilon_j(\nu-\nu_j)}, \tag{11}$$

where each set $(\nu_j,\epsilon_j,l_j,\beta_j)$ belongs to a trajectory of isospin I whose contribution to the amplitude is to be included. In Eq. (11),

$$K_j(l)=-\frac{(2l_j+1)\Gamma(l_j+\frac{1}{2})\Gamma(l+1)}{2^{l-l_j+1}\Gamma(l_j+1)\Gamma(l+\frac{3}{2})}. \tag{12}$$

β_j is the residue function, and the trajectory is given by

$$\text{Re}l=l_j+\epsilon_j(\nu-\nu_j). \tag{13}$$

In our present work we kept only one term in Eq. (11) corresponding to the top-lying Regge trajectory in the I channel; which for $I=0$ is the Pomeranchuk trajectory, and for $I=1$ is the ρ trajectory.

As mentioned earlier, the residues $f_{lI}{}^i$ in the numerator function are determined by requiring that Eq. (1) and $n-1$ of its derivatives hold at a point ν_F located in between the right- and left-hand cuts, where in Eq. (1) the left-hand side is understood to be given by Eqs. (9)–(11) and the right-hand side is given by Eqs. (2)–(6). For noninteger l, Eq. (1) and its derivatives are complex in between the cuts due to the factor ν^l since ν is negative there. On the other hand, both $N_l{}^I(\nu)$ and $D_l{}^I(\nu)$ are real in between the cuts. Thus it is convenient to factor ν^l out of Eq. (1) when determining the residues. We define the function $B_l{}^I(\nu)$ by

$$B_l{}^I(\nu)=\nu^{-l}A_l{}^I(\nu). \tag{14}$$

Equation (1) can then be written as

$$B_l^I(\nu)D_l^I(\nu)=(\nu-\nu_K)^{1-l}N_l^I(\nu). \quad (15)$$

Evaluating Eq. (15) at the matching point ν_F and making use of Eqs. (3) and (4), we obtain

$$\sum_{i=1}^{n} S_{lI}{}^i(\nu_F)f_{lI}{}^i=B_l^I(\nu_F), \quad (16)$$

where

$$S_{lI}{}^i(\nu)=(\nu-\nu_K)^{1-l}/(\nu+w_i)-B_l^I(\nu)C_{lI}{}^i(\nu). \quad (17)$$

If we define the n-columns F and B by

$$F_k=f_{lI}{}^k, \quad B_k=\frac{d^{k-1}B_l^I(\nu)}{d\nu^{k-1}}\bigg|_{\nu=\nu_F} \quad (18)$$

and define the $n\times n$ matrix S by

$$S_{ki}=\frac{d^{k-1}S_{lI}{}^i(\nu)}{d\nu^{k-1}}\bigg|_{\nu=\nu_F}, \quad (19)$$

then Eq. (16) and $n-1$ of its derivatives can be summarized by the following matrix equation:

$$SF=B, \quad (20a)$$

with the solution

$$F=S^{-1}B. \quad (20b)$$

The determination of the numerator-function pole positions w_i will be discussed in Sec. IV.

It should perhaps be pointed out that the inclusion of the direct (s) channel ρ-trajectory contribution in Eq. (11) is in no sense an abandonment of the "bootstrap" approach. The basic assumption of this approach is that the exchange forces in the crossed channel generate the pole in the partial-wave amplitude. That is to say, when the partial-wave amplitude is written in the N/D decomposition, the resonance occurs because the denominator function develops a zero without the necessity for the introduction of a CDD (Castillejo-Dalitz-Dyson) pole. When this is true, the resonance parameters can, hopefully, be determined self-consistently, without the introduction of arbitrary parameters, whereas two arbitrary parameters accompany the introduction of a CDD pole. It is the question of whether or not there are arbitrary parameters which is basic in determining whether or not one has a bootstrap. The presence of the s-channel trajectory in Eq. (11) does not, in principle, introduce arbitrary constants; the ρ parameters are still susceptible of self-consistent determination. The assumption of a $t^{\alpha(s)}$ behavior for the $I=1$ amplitude at large t does not automatically force a pole in the $l=1$ partial-wave amplitude at $s=s_1$, where $\alpha(s_1)=1$, and, in fact, this happens only for a single well-defined value of s_1 determined by the self-consistency conditions. The inclusion of the large-t contribution to the fixed-s dispersion integral, as given by Eq. (11), which will be present if there is an s-channel resonance, means that one has a higher degree of self-consistency in the solution, because the input t-channel absorptive part at high t as well as low t is approximately consistent with the s-channel output. One parameter is introduced, namely, the slope of the ρ trajectory in Eq. (11). In principle, one might hope to determine this also by considering the output partial waves for other values of l than $l=1$. In this paper, we do not attempt this elaborate calculation, and confine ourselves to taking a reasonable empirical value for this slope.

III. VALADITY OF BALÁZS METHOD

It has been pointed out that the Balázs method is sensitive to the position of the matching point ν_F.[4-6] A method for alleviating this shortcoming has been suggested by WE, who give a criterion for choosing an optimum matching point, or range of matching points, by a natural extension of the Balázs method. Balázs requires that both forms of the partial-wave scattering amplitude (the one given by the N/D equations, and the one given by partial-wave projection of a fixed-energy dispersion relation for the scattering amplitude) and $n-1$ of their derivatives be equal at some point ν_F in between the right- and left-hand cuts where the expression for both amplitudes is valid. There are no other restrictions on ν_F except that it should not be too near to any branch point. The WE criterion is to choose the ν_F that makes the nth derivatives also equal at ν_F, or equivalently (as would be evident from a Taylor-series expansion of both forms of the partial-wave amplitude) the ν_F that produces the best fit between the two forms of the partial-wave amplitudes over their mutual range of validity $[\nu_L,0]$, where ν_L is the limit of the Lehmann ellipse.[8] Even if a ν_F that forces the nth derivatives to be equal cannot be found, a choice of ν_F can still be made on the basis of the best-fit criterion. WE applied their criterion to a simplified two-pole case of the ρ bootstrap without high-energy effects included and obtained good results.[5] We find the criterion to be also valid in the two-, three-, and four-pole cases in the present calculation. Specifically, we find that in the range where the matching of the partial-wave amplitudes is insensitive to variations in ν_F (thus not allowing a choice of one value of ν_F over another) the bootstrapped values of ν_R and $\Gamma_1{}^1$ are also insensitive to variations in ν_F. On the other hand, for ranges of ν_F where $(\nu_R,\Gamma_1{}^1)$ are sensitive to changes in ν_F, we find that the matching of the partial-wave amplitudes is also sensitive to ν_F. In other words, whenever there is a need to make a choice between values of ν_F, a choice can be made.

In addition to the problem of sensitivity to ν_F, Bond[6] failed to find any solutions for the ρ-bootstrap problem for the three-pole case, and concluded that the

[8] J. Lehmann, Nuovo Cimento **10**, 579 (1958).

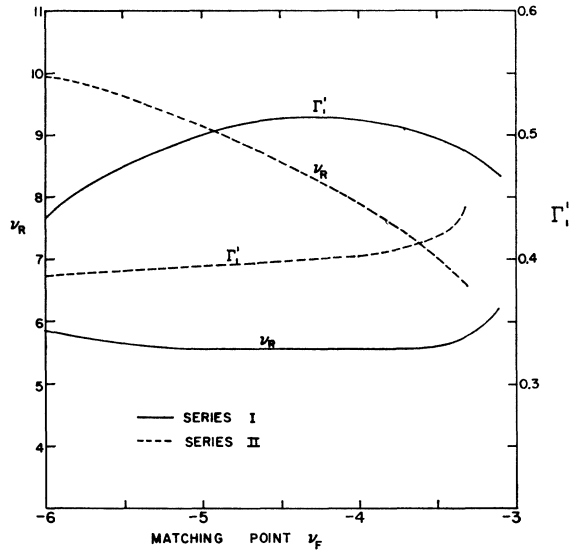

FIG. 1. The two series of solutions for the three-pole approximation with $\{w_i\}=(7.3, 13, 60)$, $\epsilon_\rho=0.05$, $R_1{}^1=3$, $\nu_D=20$, $\nu_0=-2$, $I=1$, and $l=1$.

solution obtained in the two-pole case is coincidental and therefore meaningless. We have already shown how the WE criterion solves the problem of the sensitivity to ν_F. As for the sensitivity to n, we disagree with Bond's conclusion, and have been able to find solutions for all values of n tried, i.e., for $n=2, 3, 4$.

For the values of n considered we have found, in fact, that there are $n-1$ solutions for the case of an n-pole approximation, i.e., $n-1$ pairs of values $(\nu_R,\Gamma_1{}^1)$ which, when used as input, give rise to an output ρ with very nearly the same mass and width. These group themselves into what we call series of solutions. That is to say, it turns out that one of the solutions in both the three- and four-pole cases leads to self-consistent values ν_R and $\Gamma_1{}^1$ which are similar to the values obtained in the two-pole case. We refer to this set of solutions, which are slowly varying with n, as series I. There is another pair of solutions, consisting of the second solution for $n=3$ and one of the remaining $n=4$ solutions, which are quite similar and which we refer to as series II, while it is natural to conjecture that the third solution for $n=4$ is the first member of a third series. The self-consistent values of ν_R and $\Gamma_1{}^1$ for the three series of solutions are given, as functions of ν_F, in Figs. 1 and 2 and in Table I. In seeking self-consistency, the output value of ν_R is defined by

$$\mathrm{Re}D_1{}^1(\nu_R)=0. \tag{21}$$

The output width is obtained from

$$(\Gamma_1{}^1)_{\mathrm{out}}=-\frac{N_1{}^1(\nu_R)}{\partial\,\mathrm{Re}D_1{}^1(\nu)/\partial\nu\,|_{\nu=\nu_R}}, \tag{22}$$

where, for simplicity, we make a linear approximation for the denominator function

$$\mathrm{Re}D_1{}^1(\nu)\approx(\nu-\nu_R)/(\nu_0-\nu_R). \tag{23}$$

For completeness, we show in Fig. 3 plots of the output total cross section for the $I=1$, $l=1$ partial wave obtained from Eqs. (1)–(6) for the two-pole solutions

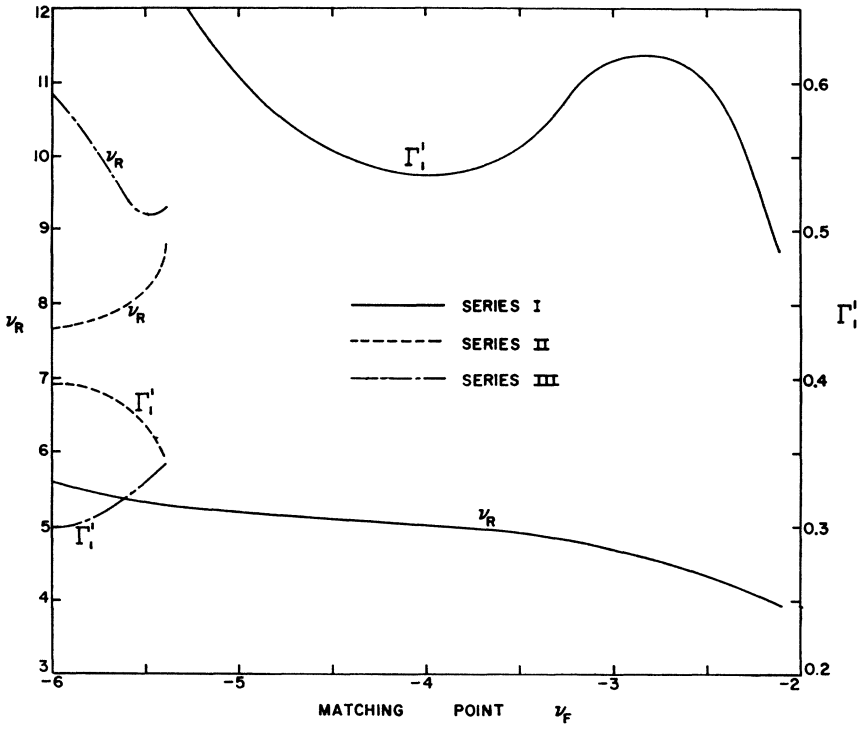

FIG. 2. The three series of solutions for the four-pole approximation with $\{w_i\}=(6.8, 9.5, 19, 105)$, $\epsilon_\rho=0.05$, $R_1{}^1=3$, $\nu_D=20$, $\nu_0=-2$, $I=1$, and $l=1$.

TABLE I. The $n-1$ series of solutions of (ν_R,Γ_1^1) versus ν_F for an n-pole approximation; $n=2, 3, 4$. The parameters are $I=1$, $l=1$, $\epsilon_\rho=0.05$, $R_1^1=3.0$, $\nu_D=20$, $\nu_0=-2.0$, $(n,\{w_i\})=(2,\{8,35\})$, $(3,\{7.3,13,60\})$, and $(4,\{6.8,9.5,19,105\})$.

	$\nu_F=-6.0$		$\nu_F=-5.0$		$\nu_F=-3.5$		$\nu_F=-2.0$		Series
n	ν_R	Γ_1^1	ν_R	Γ_1^1	ν_R	Γ_1^1	ν_R	Γ_1^1	No.
2	6.40[a]	0.518[a]	6.41[a]	0.517[a]	6.32	0.515	5.81	0.510	I
3	5.85	0.432	5.60[a]	0.500[a]	5.63[a]	0.500[a]	...	...	I
4	5.60	1.371	5.41	0.605	4.93[a]	0.555[a]	...	...	I
2	...	...	...	...	...	...	...	...	II
3	9.95	0.385	9.15	0.394	7.00	0.417	...	...	II
4	7.70	0.395	...	...	...	...	...	...	II
2	...	...	...	...	...	...	...	...	III
3	...	...	...	...	...	...	...	...	III
4	10.83	0.300	...	...	...	...	...	...	III

[a] These are the values of ν_R and Γ_1^1 for optimum values of ν_F, judged by the WE criterion as discussed in the text, for the corresponding value of n.

with three different values of ν_F. The output values of ν_R and Γ_1^1 obtained from these curves are very close to those obtained from Eqs. (21)–(23).

One would, of course, like to have some criterion for choosing between the different series. In this case, the solutions which correspond most closely to the physical ρ are those of series I, the series which includes solutions for all values of n; it is not clear, however, that this need be true in general. In this case, however, we can again make the choice on the basis of the WE criterion. From Fig. 4 it is seen that in the three-pole case the matching of the third derivatives of the two forms of the partial-wave amplitude for series I is superior to that of series II for all values of ν_F for which a solution exists. Similarly, in the four-pole case, one finds the matching of the fourth derivatives for series I is again much better than for series II and far superior to that for series III. These results are closely related to the fact that the solutions for series II and III exist for only a rather limited range of ν_F. Hence the WE criterion does give one a basis for choosing the solutions that correspond most closely with the physical situation.

Experiments give a value of 6.2 ± 0.4 for ν_R (in units with $\hbar=c=m_{\pi\pm}=1$) and we see from Table I, series I, that for $n=2$ the theoretical value of ν_R is somewhat nearer to the experimental value than for $n=4$. On the other hand, the agreement of all three values ($n=2, 3, 4$) with experiment to within about 10% in mass is certainly satisfactory, considering the approximations in the method.

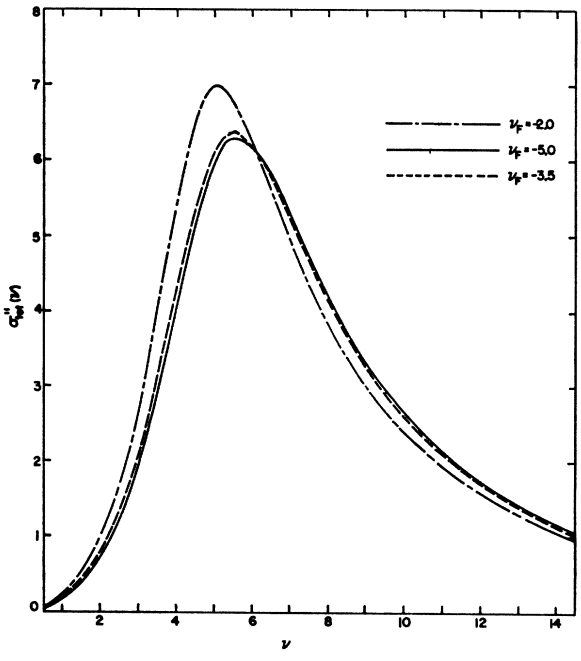

FIG. 3. Plot of the total partial-wave cross section for $I=1$ and $l=1$, $n=2$, $w_1=8$, $w_2=35$, $\epsilon_\rho=0.05$, $R_1^1=3$, $\nu_D=20$, and $\nu_0=-2$. The output values of (ν_R,Γ_1^1) obtained above are very near to the input values as given by Table III. The cross section is given by Eq. (34) with $A_l^I(\nu)$ given by Eqs. (1)–(6).

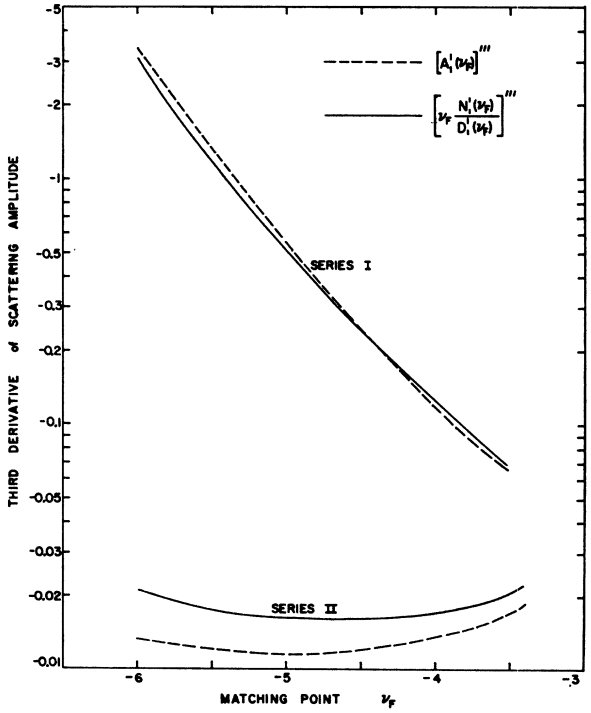

FIG. 4. Third derivatives of the two forms of the partial-wave scattering amplitude at the matching point. The other parameters are the same as in Fig. 1.

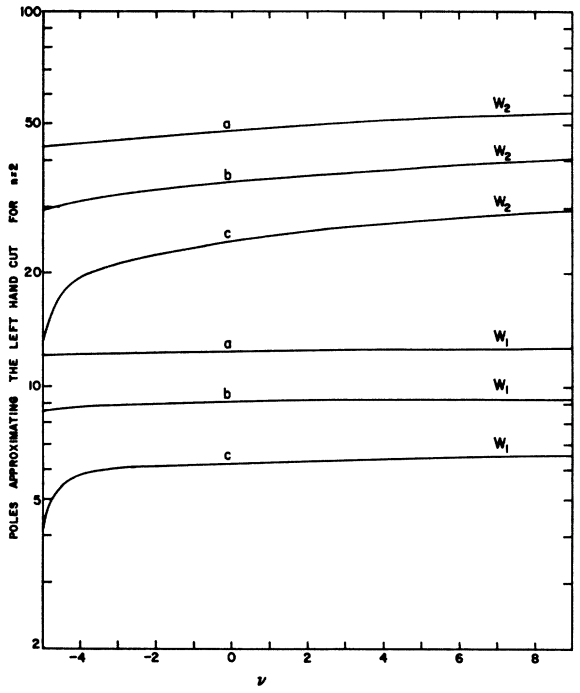

FIG. 5. The position of the poles approximating the left-hand cut for $n=2$ as a function of ν. (a) $\nu_R=9$; (b) $\nu_R=6.2$, which is the average experimental value; and (c) $\nu_R=4$.

There are two reasons that may account for Bond's[6] having failed to obtain the solutions for the case $n=3$. The first one is that as n increases the output value of ν_R becomes extremely sensitive to the input value of ν_R, especially in solutions of series I. This necessitates a very fine search in ν_R in order not to miss the solution. At first sight this may seem to be a shortcoming of the method, but since the value of ν_R varies smoothly with n, one can first obtain solutions for the case $n=2$, which are very easy to find, and then move by values of $\Delta n=1$ to any value of n desired. At each step the range in which ν_R need be varied is thus considerably restricted. On the other hand, sensitivity to the input value of ν_R is an advantage because it makes the predictions of the method clear cut and sharp.

A second reason why no solution for the case $n=3$ was found in Ref. 6 may be a modification introduced there into the Balázs method. Instead of requiring the two forms of the partial-wave amplitude and $n-1$ of their derivatives to be equal at a matching point ν_F, the two forms of the partial-wave amplitude are required to be equal at n points ν_{F_i}, $i=1,\cdots,n$. This simplifies the algebra considerably, because it does away with the necessity for taking derivatives. But, on the other hand, it makes any attempt to study the sensitivity of the solution to variations in $\{\nu_{F_i}\}$ a very complicated problem because now there are n values of ν_{F_i} to be varied rather than one. It seems to us that the variations of $\{\nu_{F_i}\}$ tried in Ref. 6 correspond to a small variation in ν_F, when the method of matching derivatives at one point is used, about the point $\nu_F=-2$. From our results, Figs. 1 and 2, we find that for $n=3$ there is no solution for $\nu_F>-3$, while for $n=4$ a solution exists for $\nu_F=-2$, accounting for the fact that Bond found a solution for $n=4$, but not for $n=3$.

We may remark in passing that it does not seem to us to be disturbing that a solution in the three-pole case cannot be found for all values of ν_F. Because of the approximate nature of the method, there is, in fact, no reason for demanding exact equality of the two forms of the amplitude and their derivatives at any point. What can be required is that the two forms of the amplitude should be quite similar to one another in their common range of validity. One convenient way of achieving this may be by demanding exact equality at one point, but this requirement is no more than a convenient device for obtaining solutions in which the two forms of the amplitude are reasonable fits to one another. Since, by imposing an exact matching condition at $\nu_F<-3$ we are able to obtain solutions in which the two forms of the amplitude are in reasonable agreement throughout their common region of validity, including the range $\nu>-3$, we do not believe that the failure to find solutions in which they are exactly matched at some $\nu_F>-3$ represents a failure of the method.

IV. SENSTIVITY TO PARAMETERS

There are many parameters in the Balázs method, but all of them can be at least partially determined. The number of poles, n, approximating the left-hand

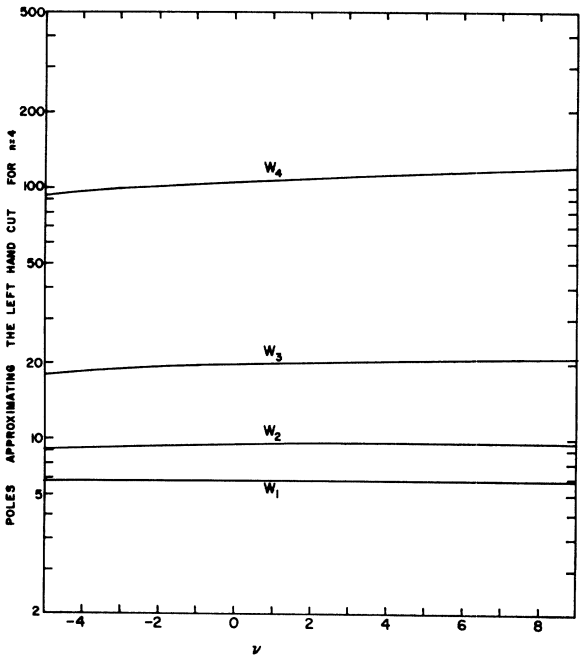

FIG. 6. The position of the poles approximating the left-hand cut for $n=4$ and $\nu_R=5.5$ as a function of ν.

cut should *a priori* be taken as large as is convenient. We have already considered the optimization of the matching point ν_F and consider it further below, together with the other parameters in the $I=1$, $l=1$ problem, i.e., the ρ bootstrap. A central set of parameters about which variations are considered is $n=2$, $w_1=8$, $w_2=35$, $\nu_D=20$, $R_l^I(\nu_D)=3$, $\nu_0=-2$, $\epsilon_\rho=0.05$, and $\nu_F=-3.5$, all in units of $\hbar=c=m_{\pi^\pm}=1$.

A. Kernel Approximation

The main step in the Balázs method that decouples the integral equations for $N_l^I(\nu)$ and $D_l^I(\nu)$ is approximating the kernel in the equation for the numerator function by an interpolation formula such as

$$\frac{1}{1+x\nu}\approx\sum_{i=1}^{n}\frac{G_i^n(x)}{1+x_i\nu} \tag{24}$$

in the interval $0\leqslant x\leqslant(1+\nu_R)^{-1}$, where it can be shown that

$$G_i^n(x_j)=\delta_{ij}. \tag{25}$$

A convenient choice for $G_i^n(x)$ is

$$G_i^n(x)=\prod_{j=1,j\neq i}^{n}\frac{(x-x_j)}{(x_i-x_j)}. \tag{26}$$

This is equivalent to approximating $(1+x\nu)^{-1}$ by a polynomial $T_{n-1}(x)$ of degree $n-1$; the x_i, $i=1,\cdots,n$, are then the n roots of the polynomial of degree n given by

$$T_{n-1}(x)(1+x\nu)-1=0. \tag{27}$$

This approximation leads to the following form of the numerator function:

$$N_l^I(\nu)=\sum_{i=1}^{n}\frac{f_{li}}{(\nu+w_i)}, \tag{28}$$

with

$$w_i=x_i^{-1}. \tag{29}$$

From the above discussion it is clear that the pole positions w_i are functions of three parameters, n, ν, and the range of matching $[0,(1+\nu_R)^{-1}]$. Hence to be explicit we should write instead of w_i, $w_i(n,\nu,\nu_R)$. The variations of w_i with ν, ν_R, and n are shown in Figs. 5 and 6. In order to produce a completely self-consistent solution, as the input ν_R is varied in search of a self-consistent ν_R the pole positions w_i must be varied correspondingly. The variation of w_i with ν is not too large as can be seen from Figs. 5 and 6, but there is no consistent way of handling it within the Balázs method as it now stands. w_i is usually evaluated for ν around ν_F. In the present paper we have used $w_i(n,-2,5.5)$. This leads in the two-pole case to $w_1=8$ and $w_2=35$. Balázs uses $w_1=6.25$ and $w_2=50$, which are values corresponding to $\nu=-2$ and $\nu_R=3.5$. Table II shows the variations of (ν_R,Γ_1^1) with (w_1,w_2). The table is a little misleading because, as Figs. 5 and 6 show, for a given ν_R the pole positions are allowed small variations which are not nearly as large as those shown in Table II.

TABLE II. Variation of (ν_R,Γ_1^1) with (w_1,w_2) in the two-pole approximation. The parameters are $n=2$, $\epsilon_\rho=0.05$, $R_1^1=3.0$, $\nu_D=20$, $\nu_0=-2.0$, $\nu_F=-3.5$, $I=1$, and $l=1$.

	$w_1=5.0$		$w_1=6.25$		$w_1=8.0$		$w_1=15.0$	
w_2	ν_R	Γ_1^1	ν_R	Γ_1^1	ν_R	Γ_1^1	ν_R	Γ_1^1
20	3.74	0.671	4.71	0.659	5.65	0.641	7.62	0.550
35	4.02	0.546	5.27	0.541	6.32	0.514	8.63	0.452
50	4.12	0.473	5.83	0.480	6.57	0.444	9.23	0.398
100	4.39	0.374	5.88	0.375	7.22	0.358	10.00	0.312

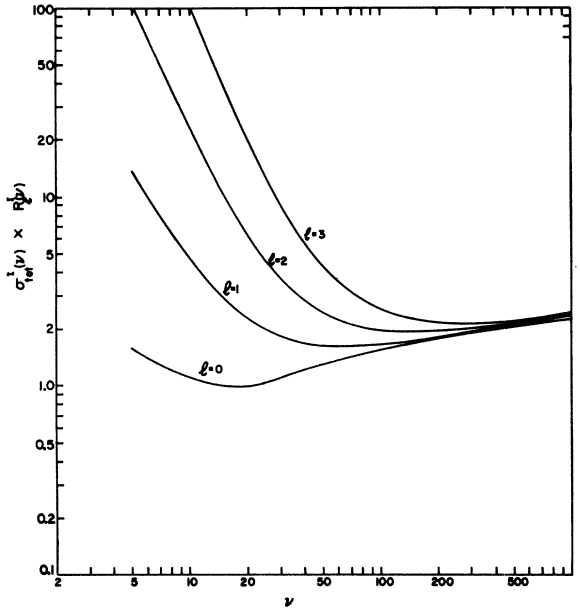

FIG. 7. $R_l^I(\nu)\sigma_{tot}^I(\nu)$ as a function of ν for several values of l. $R_l^I(\nu)$ is the ratio of total to elastic partial-wave cross section. Pomeranchuk slope used is $\epsilon_P=0.05$.

B. Inelasticity

Inelastic effects are introduced into the problem through two parameters: ν_D, which is the point of separation between the high- and low-energy regions, and $R_l^I(\nu)$, which is the ratio of total to elastic partial-wave cross sections. For $\nu<\nu_D$ the amplitude is approximated by low-energy resonance contributions, and for $\nu>\nu_D$ by a Regge representation, thus including high-energy contributions to the amplitude. $R_l^I(\nu)$ appears in the integrand of the denominator function. It is possible to include the full functional form of $R_l^I(\nu)$ as given by the Regge representation in the integrand and evaluate $D_l^I(\nu)$ numerically, but for simplicity we set $R_l^I(\nu)=1$ for $\nu<\nu_D$ and $R_l^I(\nu)=R_l^I(\nu_D)$ for $\nu\gtrsim\nu_D$.

The variation of $R_l^I(\nu)$ with ν as given by the Regge representation is shown in Fig. 7; for completeness, we show its behavior for several values of l. For $l=1$ and $\nu>20$, $R_l^I(\nu)$ is sufficiently slowly varying that Eq. (8)

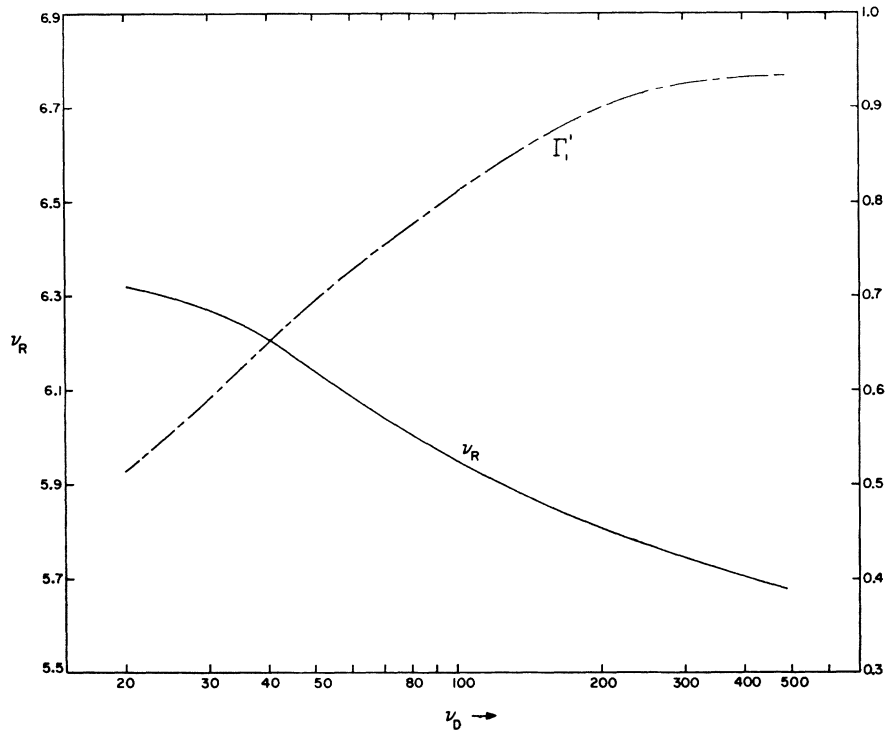

FIG. 8. Variation of the bootstrapped values of (ν_R,Γ_1^1) with the point of separation of low- and high-energy regions. The parameters are the same as in Fig. 3.

should be reasonable. Asymptotically, $R_l^I(\nu)$ is independent of l and proportional to ϵ_P, the slope of the Pomeranchuk trajectory, and increases logarithmically with ν as can be seen from Eq. (7). For all $\nu > \nu_D$, $R_l^I(\nu) \propto 1/\sigma_{\text{tot}}^I(\nu)$. Since $\sigma_{\text{tot}}^I(\nu)$ is not known with any degree of accuracy, it was found convenient in Fig. 7 to graph $\sigma_{\text{tot}}^I(\nu)R_l^I(\nu)$ rather than $R_l^I(\nu)$. The units are $\hbar=c=m_{\pi\pm}=1$, so that $\sigma=1$ corresponds to 20 mb. In our calculations we have set $\sigma_{\text{tot}}^I(\nu)=\sigma_{\text{tot}}^I(\infty)$ for $\nu > \nu_D$, and we have used a value of 0.75 for $\sigma_{\text{tot}}^I(\infty)$ which corresponds to 15 mb. $\sigma_{\text{tot}}^I(\infty)$ may be shown to have approximately this value on the basis of the factorization theorem. For ϵ_P we used a value of 0.05 which Balázs finds to be self-consistent. The Pomeranchuk trajectory seems to have in fact a much flatter slope than that.[9] But our use of a sharply rising trajectory should compensate in part for the smooth functional form we assumed for the Pomeranchuk residue function where we set $\beta(s)=\text{const}\times\nu^{\alpha(s)}$, while in fact $\beta(s)$ seems to be much more rapidly changing than that.[9]

TABLE III. Variation of (ν_R,Γ_1^1) with (ν_F,R_1^1). The parameters are $I=1$, $l=1$, $n=2$, $w_1=8$, $w_2=35$, $\epsilon_p=0.05$, $\nu_D=20$, and $\nu_0=-2.0$.

	$\nu_F=-5.0$		$\nu_F=-3.5$		$\nu_F=-2.0$	
R_1^1	ν_R	Γ_1^1	ν_R	Γ_1^1	ν_R	Γ_1^1
5.28	6.32	0.318	6.26	0.317	5.80	0.314
3.00	6.41	0.517	6.32	0.515	5.81	0.510
1.00	6.38	1.13	6.28	1.12	5.80	1.08

[9] W. Rarita, R. Riddell, Jr., C. Chiu, and R. Phillips, Phys. Rev. **165**, 1615 (1968).

Figure 8 shows the dependence of (ν_R,Γ_1^1) on ν_D. As expected, we find that as the effect of inelasticities is reduced (ν_D increased) the reduced half-width Γ_1^1 is increased, and the ρ mass is decreased slightly, but is quite insensitive to the change.

Decreasing $R_l^I(\nu)$ has, of course, the same qualitative effect on (ν_R,Γ_1^1) as increasing ν_D. This is shown in Table III, where (ν_R,Γ_1^1) are given as functions of $(R_l^I(\nu),\nu_F)$ in the two-pole case. We note that Γ_1^1 is almost independent of ν_F and increases with decreasing R_l^I, while ν_R is almost independent of R_l^I and increases with increasing $|\nu_F|$. Bond[6] also finds Γ_1^1 to be insensitive to ν_F in the special case of neglecting inelastic effects. He finds $\Gamma_1^1\approx 1$, which is essentially in agreement with our results for $\nu_D\to\infty$, as seen from Fig. 8. The small discrepancy is due to the difference in the numerical values of our parameters and those of Bond. The three values of R_1^1 tried in Table III are $R_1^1=5.28$, which is the value used by Balázs in BIII, $R_1^1=3.0$, which is the value we use throughout this paper, and $R_1^1=1.0$, which excludes inelastic effects in $D_1^1(\nu)$.

The discrepancy between our value of $R_1^1(\nu_D)$ and that of Balázs is due to a missing factor of $2/\pi$ in the derivation of Appendix A of BIII. It seems to us that Balázs evaluated $R_1^1(\nu_D)$ for $\sigma_{\text{tot}}^I=0.75$ and $\nu_D=18$. If he had used $\nu_D=20$, which is the value we use, he would have obtained $R_1^1=4.85$, which, when multiplied by the missing factor of $2/\pi$, gives $R_1^1=3.0$. Of course the value of R_1^1 is, in any event, uncertain because

of lack of knowledge of the parameters ϵ_P and $\sigma_{\text{tot}}^I(\infty)$. In his Erratum to BIII, Balázs mentions that the value of R_1^1 should be multiplied by a factor of $4/\pi$ because of an extra factor of $\frac{1}{2}$ introduced in the Pomeranchuk residue function as given in the Erratum,

$$\beta(0) = -\sigma_{\text{tot}}^I/(8\pi^2).$$

We disagree with this result and find

$$\beta(0) = -\sigma_{\text{tot}}^I/(4\pi^2). \tag{30}$$

C. Sensitivity to ν_F and ν_0

We studied the sensitivity to ν_F and ν_0 in the $I=1$, $l=1$ channel of the π-π system, that is, the ρ bootstrap. (For $l=1$ there is, of course, no dependence on ν_K.) In the two-pole case we find the matching of the two forms of the partial-wave amplitude is good and is insensitive to ν_F in the range $-6 \leqslant \nu_F \leqslant -4$. In the same region (ν_R, Γ_1^1) is also insensitive to ν_F, while for $\nu_F > -4$ there is a sensitivity to ν_F in both the matching of the partial-wave amplitudes and the value of (ν_R, Γ_1^1). From Fig. 9 we see that the partial-wave amplitudes are better matched for $\nu_F = -3.5$ than for $\nu_F = -2.0$. From graphs similar to these for $\nu_F = -5.0$ and $\nu_F = -6.0$ we come to the conclusion that in the two-pole case ν_F should be taken less than -4, giving a value of $\nu_R = 6.4$ and $\Gamma_1^1 = 0.52$. A similar study for the three-pole case shows that $\nu_F = -5.0$ is far superior to $\nu_F = -6.0$ but almost indistinguishable from $\nu_F = -3.5$ as far as matching partial-wave amplitudes is concerned. From Table I we find that, for $n=3$, (ν_R, Γ_1^1) are the same for $\nu_F = -3.5$ and $\nu_F = -5.0$ but have a slightly different value for $\nu_F = -6.0$. Thus for the three-pole case we choose $\nu_R = 5.6$ and $\Gamma_1^1 = 0.5$. The matching of the partial-wave amplitudes for the four-pole case is shown in Fig. 10, and it is clear that $\nu_F = -3.5$ is to be preferred over $\nu_F = -5.0$ and $\nu_F = -6.0$, thus giving a value of $\nu_R = 4.9$ and $\Gamma_1^1 = 0.55$.

The alternative form of the WE criterion where the optimum ν_F is that which forces the nth derivatives of the two forms of the partial-wave amplitude to be equal gives the same results as above for the case $n=2$ and 3. For the two-pole case the second derivatives are exactly equal for $\nu_F = -4.5$ and are well matched for the range $-6 < \nu_F < -4$, while for $\nu_F > -4$ they are not as well matched. The results are shown in Fig. 11. For $n=3$ the third derivatives are equal for $\nu_F = -4.6$ and are well matched for the range $-3.5 > \nu_F > -5.0$, while for $-5 > \nu_F > -6$ they are not as well matched. The results are shown in Fig. 4. For the four-pole case the fourth derivatives are not equal at any one point but are well matched over the entire range $-6 < \nu_F < -3.5$ making a choice of ν_F based on this form of the WE criterion impossible, and more information is needed before a choice can be made. In fact, from Table IV we see that even matching of the fourth derivatives over the range $-5.5 < \nu < -0.5$ for each ν_F does not provide a criterion for choosing an optimum ν_F. Thus we have to resort to the criterion of matching the two forms of the partial-wave amplitude discussed earlier which gave strong preference to $\nu_F = -3.5$ over $\nu_F = -5.0$ and $\nu_F = -6.0$.

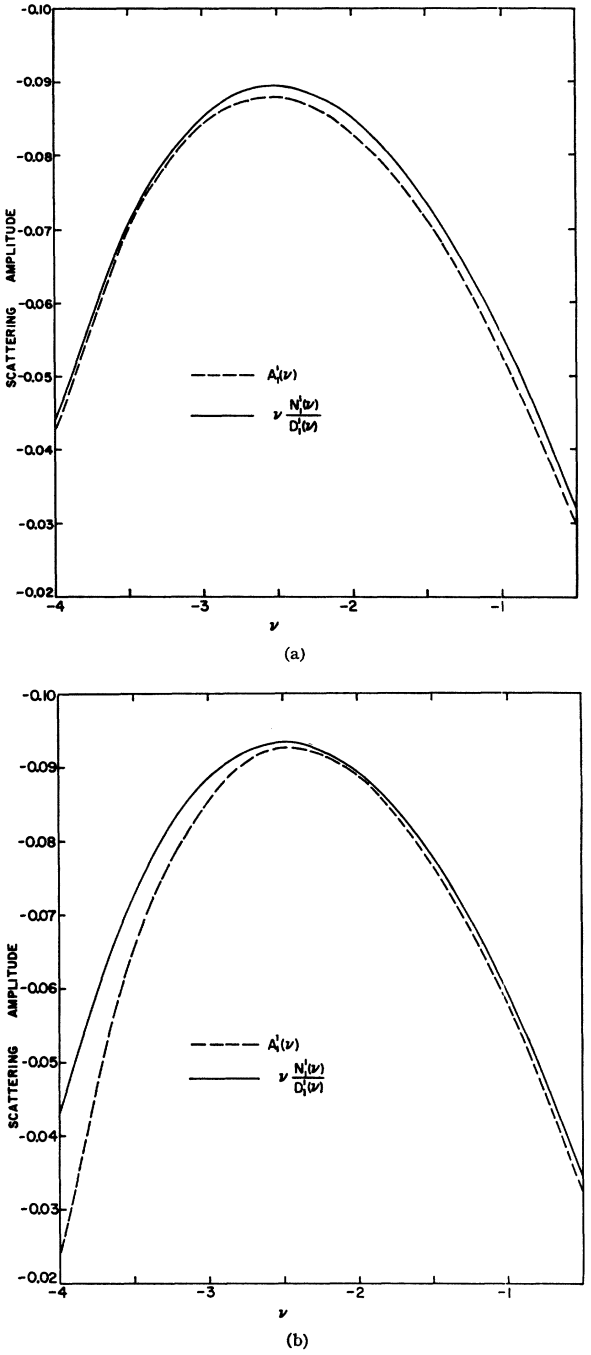

FIG. 9. Matching of the two forms of the partial-wave scattering amplitude. (a) $\nu_F = -3.5$; (b) $\nu_F = -2$. The parameters are the same as in Fig. 3.

Bond[6] gives a general proof showing that the bootstrap equations are independent of the subtraction

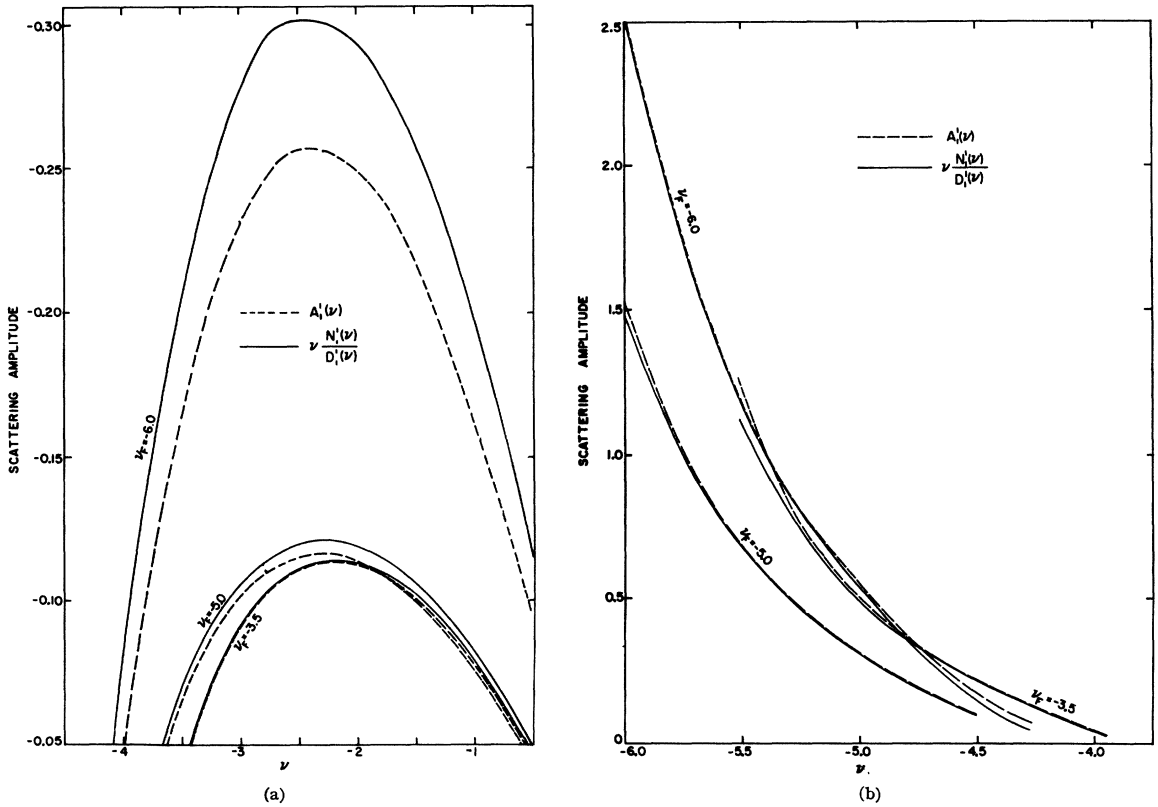

FIG. 10. Matching of the two forms of the partial-wave amplitude for $n=4$ and the same parameters as in Fig. 2. (a) $-4.5\leqslant\nu\leqslant-1.5$; (b) $-6.0\leqslant\nu\leqslant-4.5$.

point ν_0. This is certainly true if one takes the output width to be

$$(\Gamma_1^1)_{\text{out}}=-N_1^1(\nu_R)\Big/\frac{\partial\,\mathrm{Re}D_1^1(\nu)}{\partial\nu}\Big|_{\nu=\nu_R} \tag{31}$$

and requires this to be equal to the input width. But, as pointed out by Bond, if the simplifying assumption

$$\mathrm{Re}D_1^1(\nu)=(\nu-\nu_R)/(\nu_0-\nu_R) \tag{32}$$

is used, giving

$$(\Gamma_1^1)_{\text{out}}=(\nu_R-\nu_0)N_1^1(\nu_R)\,, \tag{33}$$

then a dependence on ν_0 is created due to the above approximation, Eq. (32). We find that for ν_0 varying from -5.0 to -1.5, ν_R varies by 4% and Γ_1^1 by 6% of their value at $\nu_0=-2$.

TABLE IV. Fourth derivatives of the two forms of the partial-wave scattering amplitude versus (ν,ν_F) in the four-pole approximation with $I=1$, $l=1$, $\{w_i\}=\{6.8,9.5,19,105\}$, $\epsilon_P=0.05$, $R_1^1=3.0$, $\nu_D=20$, and $\nu_0=-2.0$.

	$\nu_F=-6.0$		$\nu_F=-5.0$		$\nu_F=-3.5$	
ν	$\left(\nu\frac{N_1^1(\nu)}{D_1^1(\nu)}\right)^{(4)}$	$(A_1^1(\nu))^{(4)}$	$\left(\nu\frac{N_1^1(\nu)}{D_1^1(\nu)}\right)^{(4)}$	$(A_1^1(\nu))^{(4)}$	$\left(\nu\frac{N_1^1(\nu)}{D_1^1(\nu)}\right)^{(4)}$	$(A_1^1(\nu))^{(4)}$
-5.5	1.38×10^{1}	1.36×10^{1}	1.07×10^{1}	1.29×10^{1}	2.22×10^{1}	2.41×10^{2}
-5.0	2.90×10^{0}	2.97×10^{0}	2.13×10^{0}	2.19×10^{0}	4.33×10^{0}	1.07×10^{1}
-4.5	9.30×10^{-1}	9.76×10^{-1}	6.41×10^{-1}	6.32×10^{-1}	1.26×10^{0}	1.86×10^{0}
-4.0	3.87×10^{-1}	4.05×10^{-1}	2.47×10^{-1}	2.43×10^{-1}	4.65×10^{-1}	5.44×10^{-1}
-3.5	1.92×10^{-1}	1.97×10^{-1}	1.13×10^{-1}	1.12×10^{-1}	2.02×10^{-1}	2.11×10^{-1}
-3.0	1.08×10^{-1}	1.07×10^{-1}	5.88×10^{-2}	5.86×10^{-2}	9.91×10^{-2}	9.79×10^{-2}
-2.5	6.92×10^{-2}	6.46×10^{-2}	3.46×10^{-2}	3.43×10^{-2}	5.41×10^{-2}	5.22×10^{-2}
-2.0	5.65×10^{-2}	4.34×10^{-2}	3.71×10^{-2}	2.24×10^{-2}	3.96×10^{-2}	3.17×10^{-2}
-1.5	5.36×10^{-2}	3.32×10^{-2}	2.02×10^{-2}	1.68×10^{-2}	2.52×10^{-2}	2.23×10^{-2}
-1.0	6.82×10^{-2}	2.98×10^{-2}	2.01×10^{-2}	1.49×10^{-2}	2.24×10^{-2}	1.91×10^{-2}
-0.5	2.61×10^{-1}	3.19×10^{-2}	5.36×10^{-2}	1.60×10^{-2}	5.22×10^{-2}	2.05×10^{-2}

D. Sensitivity to Degree of Self-Consistency Required

In searching for self-consistent solutions, one must make an arbitrary decision concerning how close an agreement one will demand between input and output values. In view of the approximate nature of the method, there are no grounds for demanding exact self-consistency even if it could be obtained. Corresponding to the range of output values one is willing to accept for given input values, there will be a range of values of ν_R and Γ_1^1 which one will consider as giving acceptable bootstraps. In our calculations we considered the pair (ν_R,Γ_1^1) to be a solution if the input and output values were equal to within about 2%. Relaxing this restriction to about 5% has negligible effect on the possible values of ν_R because of the sensitivity of the bootstrap calculation to the input value of ν_R mentioned earlier, but allows one to choose Γ_1^1 anywhere within a range of about $\pm 10\%$ around the values given in Table I. Hence these values should be considered to have uncertainties of roughly 10%.

V. CONCLUSION

From the above discussion we see that the Balázs method is a very convenient scheme for doing partial-wave bootstrap calculations at low energies. It includes high-energy contributions to the amplitude, and avoids the need for a cutoff in the integral equations. Also, it does not require the solution of simultaneous integral equations or the introduction of free parameters. At the same time it gives reasonable results. In the ρ-bootstrap problem we find that as n, the number of poles approximating the left-hand cut, varies from 2 to 4, the value of ν_R for optimum matching point ν_F varies from 6.4 to 4.9, corresponding to a variation of the ρ mass from 760 to 680 MeV, as compared with the experimental value of 750 ± 25 MeV corresponding to $\nu_R=6.2$ ± 0.4. For the same variation in n, the reduced half-width in ν varies from 0.51 to 0.55, corresponding to a variation in the ρ half-width in energy from 155 to 141 MeV as compared with the experimental value of about 50 MeV. Since most bootstrap calculations give too large a value for the width, the results obtained above do not represent a shortcoming specific to the Balázs method.

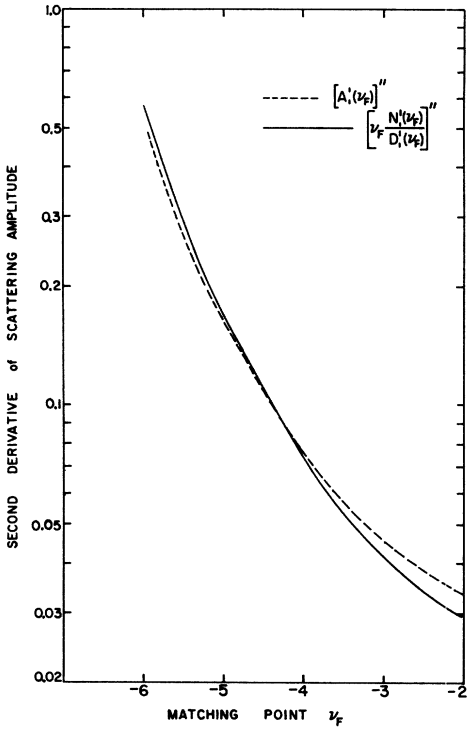

FIG. 11. Second derivatives of the two forms of the partial-wave amplitude in the two-pole approximation with the parameters of Fig. 3.