Approximate Solution of the Nonrelativistic Lee Model in All Sectors*

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An approximate solution of the nonrelativistic Lee model in all sectors is described. The treatment is valid only in the strong-coupling regime, but the restriction to large source radius, characteristic of North's strong-coupling solution, has been eliminated.

THE Lee model has been extensively studied in the first and second sectors, but relatively little work has been done on the third and higher sectors.[1] One exception is an interesting paper by North[2] which uses the methods of old-fashioned strong-coupling theory to solve the nonrelativistic Lee model provided[3]

$$1 \ll (mg^2)/(mR)^3 \ll (mR)^4. \tag{40N}$$

Here g is the unrenormalized coupling constant, m is the meson mass, and R^{-1} is the momentum cutoff.

The present paper describes the results of an attempt to remove the restriction to large source radius which was required in Ref. 2. The present calculation is actually based on Tomonaga's intermediate-coupling approximation,[4] but in order to simplify the calculations, we minimize a strong-coupling approximation of the lower set of energy eigenvalues. Thus, the results given below are valid only for large values of g.

The basic difficulty with regard to the construction of a strong-coupling theory of the Lee model valid for large g and small R can be stated as follows: The interaction Hamiltonian presumably dominates if g is large. Therefore, one should diagonalize it first and treat the rest of the Hamiltonian as a perturbation. On the other hand, the contributions coming from the free meson Hamiltonian become extremely large as the momentum cutoff R^{-1} becomes large. Therefore, one cannot treat the free meson Hamiltonian as a perturbation. In other words, neither the free meson Hamiltonian nor the interaction Hamiltonian can be treated as small perturbations if g is large and R small.

The Hamiltonian for the nonrelativistic Lee model may be written in the form

$$H = H_{\text{mes}} + H_{\text{int}} + H_{\text{nucl}}, \tag{1}$$

where

$$H_{\text{mes}} = \sum_k \omega_k a_k^\dagger a_k, \tag{2}$$

$$H_{\text{int}} = g\tau_- \sum_k u_k a_k + \text{H.c.}, \tag{3}$$

$$H_{\text{nucl}} = \tfrac{1}{2}(1-\tau_3)\epsilon_0. \tag{4}$$

Here $a_k^\dagger$ and a_k are the creation and annihilation operators of the mesons of momentum $\mathbf{k}$, and

$$u_k = (2\pi)^{-3/2} \int U(\mathbf{r}) e^{i\mathbf{k}\cdot\mathbf{r}} d^3r, \tag{5}$$

where $U(\mathbf{r})$ is the nucleon source function, normalized according to

$$\int U(\mathbf{r}) d^3r = 1. \tag{6}$$

ω_k denotes the total energy of a nonrelativistic meson with momentum $\mathbf{k}$, $\omega_k = \mathbf{k}^2 + \frac{1}{2}$, where energy is expressed in units of $2m$. We also note that the operator of total charge for the meson-nucleon system is given by

$$q = \tfrac{1}{2}(1+\tau_3) - \sum_k a_k^\dagger a_k. \tag{7}$$

It can be shown[5] that Tomonaga's intermediate-coupling approximation is equivalent to the following substitution of reduced-space operators in the Hamiltonian:

$$a_k \rightarrow f_k a, \tag{8a}$$

$$a_k^\dagger \rightarrow f_k a^\dagger. \tag{8b}$$

The trial function f_k will be chosen to minimize the "lower" set of eigenvalues of the reduced-space Hamiltonian, which is given by

$$H_r = w a^\dagger a + gQ(\tau_- a + \tau_+ a^\dagger) + \tfrac{1}{2}(1-\tau_3)\epsilon_0, \tag{9}$$

where

$$w \equiv \sum_k \omega_k f_k^2, \tag{10}$$

$$Q \equiv \sum_k u_k f_k. \tag{11}$$

Furthermore, the reduced-space operator of total charge is given by

$$q_r = \tfrac{1}{2}(1+\tau_3) - a^\dagger a, \tag{12}$$

where the normalization condition,

$$\sum_k f_k^2 = 1, \tag{13}$$

has been taken into consideration.

Following the same mathematical procedure as North,[2] we seek simultaneous eigenfunctions of H_r and q_r in the form

$$\phi_n = (1+c_n^2)^{-1/2} \begin{pmatrix} \psi_n \\ c_n \psi_{n-1} \end{pmatrix}, \tag{14}$$

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[1] Two recent articles which deal with sectors higher than the second are J. B. Bronzan, Phys. Rev. **172**, 1429 (1968); Daniel I. Fivel, University of Maryland, Department of Physics and Astronomy, Technical Report No. 764, 1967 (unpublished).

[2] Gerald R. North, Phys. Rev. **164**, 2056 (1967).

[3] Equation (jN), where j is an integer, refers to Eq. (j) of Ref. 2.

[4] S. Tomonaga, Progr. Theoret. Phys. Japan **2**, 6 (1947).

[5] T. D. Lee and D. Pines, Phys. Rev. **92**, 883 (1953).

TABLE I. Values of R, x, and $\tilde{E}^- \equiv (27/2)\pi^2(1+x^2)(\frac{1}{2}R^{-2}-x^2)/R$.

R	x	$\tilde{E}^-$
0.01	7.9×10^{-5}	6.662×10^{7}
0.02	3.14×10^{-4}	8.327×10^{6}
0.03	7.06×10^{-4}	2.467×10^{6}
0.04	1.25×10^{-3}	1.041×10^{6}
0.05	1.96×10^{-3}	5.330×10^{5}
0.06	2.82×10^{-3}	3.084×10^{5}
0.07	3.83×10^{-3}	1.942×10^{5}
0.08	5.00×10^{-3}	1.301×10^{5}
0.09	6.31×10^{-3}	9.139×10^{4}
0.1	7.78×10^{-3}	6.662×10^{4}
0.2	3.02×10^{-2}	8.334×10^{3}
0.3	6.51×10^{-2}	2.476×10^{3}
0.4	0.1096	1049
0.5	0.1609	539.7
0.6	0.2168	312.0
0.7	0.2756	193.4
0.8	0.3363	123.9
0.9	0.3980	78.69
1.0	0.4603	46.53
2.0	1.078	−149.4
3.0	1.677	−467.2
4.0	2.267	-1.045×10^{3}
5.0	2.853	-1.977×10^{3}
6.0	3.436	-3.352×10^{3}
7.0	4.017	-5.260×10^{3}
8.0	4.597	-7.789×10^{3}
9.0	5.177	-1.103×10^{4}
10.0	5.756	-1.507×10^{4}

where the harmonic oscillator functions ψ_n are defined according to

$$a\psi_n = 0, \tag{15a}$$

$$\psi_n = (n!)^{-1/2}(a^\dagger)^n\psi_0, \tag{15b}$$

$$a^\dagger a\psi_n = n\psi_n. \tag{15c}$$

The desired eigenfunctions satisfy

$$H_r\phi_n = E_n\phi_n, \tag{16}$$

$$q_r\phi_n = -(n-1)\phi_n. \tag{17}$$

The matrix equation (16) is equivalent to two simultaneous equations:

$$nw + c_n gQ\sqrt{n} = E_n, \tag{18a}$$

$$gQ\sqrt{n} + c_n(n-1)w + c_n\epsilon_0 = c_nE_n. \tag{18b}$$

Eliminating c_n, one finds that the energy eigenvalues are given by

$$E_n^{\pm} = (n-\tfrac{1}{2})w + \tfrac{1}{2}g^2Q^2w^{-1} \pm \tfrac{1}{2}\Delta_n, \tag{19a}$$

$$\Delta_n \equiv +[(g^2Q^2/w)^2 + 2(2n-1)g^2Q^2 + w^2]^{1/2}. \tag{19b}$$

Here ϵ_0 has been set equal to $g^2Q^2w^{-1}$ in order to make E_1^- vanish (i.e., in order to renormalize the physical neutron mass to zero).

At this point we introduce the ansatz

$$(a\omega_k + b)f_k = u_k, \tag{20}$$

where the parameters a and b are to be chosen to minimize E_n^- and simultaneously satisfy the normalization condition (13). Note that Eq. (20) implies

$$aw + b = Q. \tag{21}$$

In order to simplify the algebra, we replace expression (19) for E_n^- by the approximate formula[6]

$$E_n^- \cong n(n-1)w^3(gQ)^{-2}. \tag{22}$$

Using Eq. (21) one can easily verify that expression (22) has a minimum value of $(9/4)n(n-1)w(ga)^{-2}$ when $b = -\frac{1}{3}aw$. We, therefore take

$$f_k = u_k/a(\omega_k - \mu), \tag{23}$$

where

$$\mu \equiv \tfrac{1}{3}w. \tag{24}$$

We now use Eqs. (10) and (13) to determine the values of a and μ:

$$3\mu = \sum_k \omega_k f_k^2 = \sum_k k^2 f_k^2 + \tfrac{1}{2}, \tag{10'}$$

$$\sum_k f_k^2 = 1. \tag{13}$$

Assuming a square momentum cutoff at $k=1/R$ and going to the continuum limit, Eqs. (10′) and (13) become

$$3\mu - \tfrac{1}{2} = \frac{1}{2(\pi a)^2}\int_0^{1/R} \frac{k^4 dk}{(k^2+\frac{1}{2}-\mu)^2} = \frac{1}{2(\pi a)^2} \times\left[\frac{1}{R} + \frac{b^2}{2R(R^{-2}+b^2)} - \tfrac{3}{2}b\tan^{-1}(1/bR)\right], \tag{10''}$$

$$b \equiv (\tfrac{1}{2}-\mu)^{1/2}, \tag{25}$$

$$2(\pi a)^2 = \int_0^{1/R} \frac{k^2 dk}{(k^2+\frac{1}{2}-\mu)^2} = -\frac{1}{2R(R^{-2}+b^2)} + \frac{1}{2b}\tan^{-1}(1/bR). \tag{13'}$$

Eliminating $\tan^{-1}(1/bR)$ from Eqs. (10″) and (13′), one finds that

$$2(\pi a)^2 = 1/R - b^2/R(b^2+R^{-2}) \tag{26a}$$

or

$$\mu = \tfrac{1}{2} + 1/R^2 - 1/2(\pi a)^2R^3. \tag{26b}$$

Substituting (26b) into Eq. (13′) one finds

$$(1+x^2)R\tan^{-1}(1/x) = x(R+2/R), \tag{27}$$

where

$$x \equiv bR. \tag{28}$$

For small values of R, the solution of Eq. (27) is given by

$$x \cong (\tfrac{1}{4}\pi)R^2. \tag{29}$$

On the other hand, for large sources the solution of Eq. (27) is given by

$$x \cong R/\sqrt{3}. \tag{30}$$

The numerical values of x for various values of R are given in Table I.[7]

[6] The approximate formula (22) is obtained from (19) by expanding Δ_n, assuming $g^2Q^2 \gg (2n-1)w^2$.

[7] In connection with the values of R listed in Table I, it is essential to remember that energies have been expressed in units of $2m$. Thus, $R=0.01$ actually means $2mR=0.01$ or $mR=0.005$.

Using Eq. (26b) to express $1/a^2$ in terms of x and R, one finds

$$E_n^- = \frac{9}{4} \frac{n(n-1)}{g^2} \frac{w}{a^2}$$
$$= \frac{27}{2}\pi^2 \frac{n(n-1)}{g^2} \frac{(1+x^2)(\frac{1}{2}R^{-2}-x^2)}{R}, \quad (31)$$

which reduces to the following expressions in the two limiting cases of large and small R:

$$E_n^- \cong -\tfrac{3}{2}\pi^2 n(n-1)R^3/g^2 \quad \text{for } R \gg 1, \quad (32a)$$

$$E_n^- \cong (27/4)\pi^2 n(n-1)R^{-3}g^{-2} \quad \text{for } R \ll 1. \quad (32b)$$

Numerical values of

$$\tilde{E}^- \equiv [g^2/n(n-1)]E_n^- = (27/2)\pi^2(1+x^2)(\tfrac{1}{2}R^{-2}-x^2)R^{-1}$$

for various values of R between 0.01 and 10 are also given in Table I.

It is of interest to compare these results to the "large-source approximation" results given in Ref. 2, namely,

$$E_n^- \cong n(n-1)\omega^3/f^2, \quad (22\text{N})$$

where

$$\omega \equiv \lambda^{-2} \textstyle\sum_k \omega_k u_k^2,$$
$$f \equiv g\lambda,$$
$$\lambda^2 \equiv \textstyle\sum_k u_k^2.$$

Again assuming a square momentum cutoff at $k=1/R$ and going to the continuum limit, one can easily verify that

$$\lambda^2 = (6\pi^2 R^3)^{-1},$$
$$\omega = \tfrac{1}{2} + (\tfrac{3}{5})R^{-2},$$
$$E_n^- \cong \frac{3\pi^2 n(n-1)}{4g^2} R^3(1+1.2R^{-2})^3. \quad (22\text{N}')$$

For reasons which are discussed in detail in Ref. 2, North's treatment is only valid in the large source limit, in which case

$$E_n^- \cong \frac{3\pi^2 n(n-1)}{4g^2} R^3. \quad (22\text{N}'')$$

Note that the present treatment leads to substantially lower eigenvalues in the same large source limit; in particular, the eigenvalues given by Eq. (32a) actually turn out to be negative. In this connection it should also be noted that the reduced-space Hamiltonian (9) becomes identical to the bound-state Hamiltonian derived by North, provided we make the ansatz (20) with $a=0$. Then the normalization condition (13) requires us to take $b=\lambda$. In other words, the static approximation obtained by North may be regarded as a special case of the treatment described above.

If we had chosen a and b in the ansatz (20) to minimize expression (19) for E_n^-, then a and b would turn out to also depend on the isobar state (i.e., on the value of n) and on the value of the coupling constant g. Instead we have chosen a and b to minimize the approximate expression (22) for E_n^-, and the resulting calculation is much easier to carry out because a and b then depend on only the value of the single parameter R. However, we wish to emphasize that in principle the same procedure can be used to minimize expression (19) for E_n^-.