$\pi\Lambda$ Phase-Shift Information from High-Energy Antineutrino Reactions and from Ξ_{e4} Decays*

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Methods are described to obtain the S-wave $\pi\Lambda$ phase shift from the processes $\Xi \to \Lambda\pi e\bar{\nu}$ and $\bar{\nu}+N \to \Lambda\pi\bar{\mu}$. Each of these processes is fully specified (for vanishing lepton mass) by a set of sixteen seven-dimensional distributions describing the intensity and the various polarization spectra. From each distribution, nine functions of a single variable, the invariant $\pi\Lambda$ mass, are extracted. Ratios of these functions determine the S-wave phase shift, for energies where only S- and P-wave amplitudes are appreciable and where the S phase shift is the only significant one. For the decays there are 36 independent determinations of this kind. For the reactions there are 8, and these are independent of a detailed knowledge of the incoming $\bar{\nu}$ spectrum. In all instances, knowledge of the polarizations of the parent baryon and of the Λ is indispensable. No assumptions on the structure of hadronic form factors are necessary. We do assume that possible deviations from lepton pair locality and from T invariance are insignificant.

I. INTRODUCTION

In this paper we discuss ways to obtain information about the scattering of pions on Λ's. This is but one of a considerable number of two-body scattering problems where access to important dynamical information is hampered by the fact that both particles involved are unstable. Specifically, we shall be concerned with the extraction of S-wave $\pi\Lambda$ scattering phase shifts from an analysis, first, of the $e4$ decay modes of cascade particles:

$$\Xi^0 \to \Lambda+\pi^+ +e^- +\bar{\nu}_e\,, \quad (1.1)$$

$$\Xi^- \to \Lambda+\pi^0 +e^- +\bar{\nu}_e\,, \quad (1.2)$$

and, second, of the reaction

$$\bar{\nu}_\mu + p \to \Lambda+\pi^0+\mu^+. \quad (1.3)$$

For our purpose, the principal interest in the reactions (1.1)–(1.3) lies in the fact that in a partial-wave expansion of the decay or reaction form factors with respect to angular momentum in the $\pi\Lambda$ system, a partial wave with definite total and orbital angular momentum must have the phase of the corresponding $\pi\Lambda$ scattering amplitude. This is strictly true insofar as time-reversal invariance is valid; we shall assume[1] that possible T-violating effects are negligible.

Let us first make the trivial observation that if one measures everything about these various reactions, one also gets everything one can get, and this of course includes $\pi\Lambda$ phase shifts. It is not the exclusive purpose of this paper to note that measuring everything means to map out in a seven-dimensional[2] phase space the intensity spectra, single polarization spectra, as well as polarization correlation distributions. (In what follows, the lepton mass is put equal to zero.) Rather, we shall explore what is the optimal amount of integration over phase-space domains, such that the scattering information sought for can still be extracted without unwarranted *a priori* assumptions on the structure of the form factors involved. Central to the method is the assumption of effectively local coupling of lepton pairs to hadronic currents, an assumption tested by Eqs. (2.27), (2.28), and (3.9) below. The methods employed are closely akin to those recently applied[3] to the problem of extracting $\pi\pi$ scattering information from K_{l4} decays.

Theoretically, the present work is addressed to the problem, one almost of principle, of learning how to read off effectively what constitutes the most direct physical information in reactions of fair complexity. Apart from its methodological aspects, the relative rarity of the processes considered may at some future date perhaps make our approach of practical relevance, as it is attuned to limited statistics situations. Indeed the acquisition of a sufficient number of Ξ_{e4}-decay events clearly belongs to a generation of experiments that is yet to come. Branching ratios $\lesssim 10^{-7}$ are presumably involved.[4] We have no reliable estimates for the reaction (1.3) but, surely, it will not be common either. Even so, we believe that the formidable task of experimentally attacking problems of this sort with weak (or electromagnetic) probes will eventually have to be faced, if it were only because, to our knowledge, there exists no alternative to obtain the scattering

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[1] Of course, this assumption is open to experimental verification.

[2] As seen in Secs. II A and III, two of the seven variables describe the orientation of characteristic momentum vectors of the problem relative to the polarization direction of the parent baryon.

[3] A. Pais and S. B. Treiman, Phys. Rev. **168**, 1858 (1968). In what follows this paper is referred to as I.

[4] Compare estimates for Σ_{e4} decays by Y. Singh [Phys. Rev. **161**, 1497 (1967)].

information under discussion from purely strong-interaction phenomena without uncontrollable approximations. To be sure, one might be able to obtain such scattering data from appropriate reactions, if one were able to justify extrapolations of reaction amplitudes to some unphysical region.[5] However, we shall continue to take the view that for extrapolation problems the more important question really is to find the extent of justification with the help of known scattering data.

We next summarize our findings, first for the decays, then for the $\bar{\nu}$ reaction. We believe that the case of the antineutrino reactions may well turn out to be of relatively greater interest, but it is methodologically helpful to start with the decays.

A. Decay Problem

The general structure of the problem is conveniently represented by a density matrix ρ in the Λ spin space:

$$\rho=I+(\boldsymbol{\sigma}+\boldsymbol{\Sigma})\cdot\mathbf{S}+(\boldsymbol{\sigma}-\boldsymbol{\Sigma})\cdot\mathbf{A}+\sigma_i\Sigma_j(S_{ij}+A_{ij}+\delta_{ij}R). \quad (1.4)$$

$\boldsymbol{\Sigma}$ is the Ξ polarization vector, assumed to be determined separately. $\boldsymbol{\sigma}$ is the Pauli spin vector. $2I$ is the spin-averaged intensity distribution. The vectors $\mathbf{S}$ and $\mathbf{A}$ determine single-polarization distributions. The tensors S_{ij} (symmetric and traceless) and A_{ij} (antisymmetric) and the scalar R describe polarization correlations. It is helpful to define a symbol Z as follows:

$Z=$ any one of the sixteen distributions

$$I,\ \mathbf{S},\ \mathbf{A},\ S_{ij},\ A_{ij},\ R. \quad (1.5)$$

We recall[3] that any Z depends on five dynamical variables. Our choice of these variables is the same as that in I. This will be recapitulated in Sec. II. A, where we shall see that any Z can be decomposed into nine parts:

$$Z=Z^{(1)}+Z^{(2)}\cos2\theta_l+Z^{(3)}\sin^2\theta_l\cos2\phi+Z^{(4)}\sin2\theta_l\cos\phi+Z^{(5)}\sin\theta_l\cos\phi+Z^{(6)}\cos\theta_l+Z^{(7)}\sin\theta_l\sin\phi+Z^{(8)}\sin2\theta_l\sin\phi+Z^{(9)}\sin^2\theta_l\sin2\phi. \quad (1.6)$$

Here θ_l and ϕ are the two "trivial" variables (i.e., the variables that do not appear in the Ξ_{e4} form factors). They are, respectively, the dilepton "decay" angle and the azimuthal angle between the normals to the planes defined by the $\pi\Lambda$ system and by the dilepton system. As a result of the single assumption of lepton pair locality, the $Z^{(i)}$ only depend on the three remaining variables but the (θ_l,ϕ) dependence in Eq. (1.6) is fully explicit. The recurrence of the same ninefoldness as found in I is due, of course, to the fact that this expresses pure lepton pair properties, including both their local pair structure and their parity-mixed structure.

Let us now do some counting. For K_{l4} we met[3] four form factors; for $m_l=0$ this number effectively reduces to three. In this massless case, for fixed values of the triple of variables on which these form factors depend, we had, therefore, five real quantities to be fixed: three absolute values and two relative phases of those form factors. For $m_l=0$, all information has to come from $Z=I$. On the other hand, we showed[3] that the nine quantities $I^{(i)}$ can be measured separately. Hence, the intensity spectrum not only determines the K_{e4} form factors, as it has to, but actually overdetermines them.

For Ξ_{e4} decays the situation is different. For Ξ_{l4}, $m_l\neq0$, one finds that there are 16 form factors. For Ξ_{e4}, $m_e=0$, this number effectively reduces to 12. Thus, where five quantities had to be determined for K_{e4}, the corresponding number is 23 for Ξ_{e4}. While also here we can get at the nine quantities $I^{(i)}$ separately, the intensity spectrum alone is therefore insufficient to determine the form factors. Nor is this a great surprise, because now we have of course a variety of hadron polarization distributions at our disposal.

Because of the great complexity of the Ξ_{e4} problem when treated in full generality, and because of the absence of transparent general results, as just noted, there is not much merit in detailing the general answers, i.e., those containing all partial waves of the $\pi\Lambda$ system. This is all the more true because, with the limitations on phase space, it seems ample to consider only the S- and the two P-wave amplitudes. The various distributions obtained in this approximation will be discussed in Sec. II.

Even then the problem is still quite complex. We shall meet four $S_{1/2}$ form factors, four for $P_{1/2}$, six for $P_{3/2}$. Our partial-wave expansion of course makes explicit the dependence on θ, the $\pi\Lambda$ decay angle. The 14 partial-wave form factors just enumerated depend only on two variables, the invariant dilepton mass $\sqrt{s_l}$ and the effective $\pi\Lambda$ mass $\sqrt{s}$. In order to get at the $S_{1/2}$ $\pi\Lambda$ phase shift by the present methods, P-wave form factors had better be there significantly, as we are relying essentially on SP-interference effects. However, it seems reasonable to assume that, for the small $\pi\Lambda$ energies concerned, the $P_{1/2}$ and $P_{3/2}$ $\pi\Lambda$ phase shifts are negligible compared with the $S_{1/2}$ phase shift. In other words, we assume that all P-wave form factors are real; and experimental tests for this assumption will be indicated below [see Eq. (2.44)]. If this last assumption is satisfied (and only then) will it turn out that we will be able to get the $S_{1/2}$-wave phase shift, for fixed s, and *without loss of information* by (1) integrating the distributions over all values of s_l and θ and then (2) taking ratios of certain specific pairs of

[5] Such extrapolations have proved to be most useful to obtain $\Lambda\pi$-scattering information from K^-p scattering, which led to the prediction of $Y_0^*(1405)$. The extrapolation involved extends over an energy region $\sim$35 MeV. The same reaction when invoked for the low-energy S-wave $\Lambda\pi$ scattering would demand an extrapolation over a region $\sim$185 MeV.

the thus averaged $Z^{(i)}$. In all, there are 36 useful pairs [see Eqs. (2.47)–(2.49)].

B. $\bar{\nu}$ Reactions

As we show in Sec. III, it is possible to use large parts of the formalism developed for Ξ_{e4} in such a way that results for the reaction (1.3) can be obtained with relatively little labor. However, this is true only under such circumstances where we may neglect the role of the muon mass. Thus, the results of Sec. III apply to high-energy antineutrino reactions, where the barycentric energy of the $(\bar{\nu},p)$ system is large, while at the same time the invariant $\pi\Lambda$ mass is below the first inelastic threshold, so that the kinetic energy of the muon is large compared with its mass.

By the same reasoning as used for the decays, $\pi\Lambda$ scattering information is contained in the various distributions characteristic for the reaction (1.3). A description in terms of the Λ spin density matrix ρ, Eq. (1.4), is again appropriate. $\mathbf{\Sigma}$ now refers to the polarization of the target nucleon. Z is again defined as in Eq. (1.5).

Let us compare the five variables describing this reaction with the set of five corresponding to the decay:

$$\begin{aligned} &\text{decay:} && s,\, s_l,\, \theta,\, \theta_l,\, \phi\,, \\ &\text{reaction:} && s,\, t,\, \theta,\, w,\, \phi\,. \end{aligned} \tag{1.7}$$

That is to say, s_l is now replaced by t, the invariant lepton momentum transfer. Instead of the final-state decay variable θ_l we now must deal with w, the barycentric energy variable of the reaction. (More detailed definitions are found in Sec. III.) s, θ, and ϕ have essentially the same meaning in both cases.

The reaction form factors depend on only three of the five variables, namely, s, t, and θ, if lepton pair locality is assumed once more. Correspondingly, w and ϕ are the trivial variables in this case. We see in Sec. III that now the Z's, in general, take the following form:

$$\begin{aligned} Z = Z^{(1)} &+ Z^{(2)} \cosh 2\theta_l' + Z^{(3)} \sinh^2\theta_l' \cos 2\phi \\ &+ Z^{(4)} \sinh 2\theta_l' \cos\phi + Z^{(5)} \sinh\theta_l' \cos\phi \\ &+ Z^{(6)} \cosh\theta_l' + Z^{(7)} \sinh\theta_l' \sin\phi \\ &\quad + Z^{(8)} \sinh 2\theta_l' \sin\phi + Z^{(9)} \sinh^2\theta_l' \sin 2\phi\,, \end{aligned} \tag{1.8}$$

where a hyperbolic angle θ_l' has been introduced, defined by[6]

$$\sinh\theta_l' = X^{-1}[M^2 s - w^2(M^2+s+t) + w^4]^{1/2}\,, \tag{1.9}$$

$$\cosh\theta_l' = \tfrac{1}{2}X^{-1}(2w^2 - s - t - M^2)\,, \tag{1.10}$$

$$X = \tfrac{1}{2}[M^4 - 2M^2(s-t) + (s+t)^2]^{1/2}\,, \tag{1.11}$$

where M is the nucleon mass. The $Z^{(i)}$ depend on s, t, and θ only; the w and ϕ dependence is fully explicit from Eqs. (1.8)–(1.11).

[6] Equations (1.9)–(1.11) refer specifically to the case of zero lepton mass.

At this point we note the essential difference between the decay and the reaction problem. As has been stressed in I, the variables θ_l and ϕ in Eq. (1.6) can be treated as statistically discrete; one needs to lump the data in at most four (θ_l,ϕ) domains in order to separate out the nine $Z^{(i)}$. Obviously, taking θ_l discrete has no θ_l' or w counterpart.

Let us then, to start with, disentangle Z-parts by distinct ϕ dependences alone. This yields a fivefold decomposition into $(Z^{(1)}, Z^{(2)}, Z^{(6)})$; $Z^{(3)}$; $(Z^{(4)}, Z^{(5)})$; $(Z^{(7)}, Z^{(8)})$; and $Z^{(9)}$. By $(Z^{(1)}, Z^{(2)}, Z^{(6)})$ we mean the agglomerate of the corresponding three terms in Eq. (1.8) which is ϕ independent. Likewise, $(Z^{(4)}, Z^{(5)}) \propto \cos\phi$ and $(Z^{(7)}, Z^{(8)}) \propto \sin\phi$, while $Z^{(3)} \propto \cos 2\phi$, $Z^{(9)} \propto \sin 2\phi$. To be sure, one can make further (experimental) distinctions within the agglomerates by the recognition of various distinct w dependences. However, the reader will verify that, after having done so, one still will be unable to use integrations over t (as we used integrations over s_l for decays) to obtain averaged ratios of $Z^{(i)}$ which yield the phase shift.

On the other hand, no further decomposition is necessary in the cases of $Z^{(3)}$ and $Z^{(9)}$, each of which are singled out by their ϕ dependence alone. The final step, described in Sec. III, is then to take the $Z^{(3)}$'s and $Z^{(9)}$'s, and to integrate over θ and t. Ratios of the thus averaged $Z^{(i)}$ give the S-wave phase shift, in eight independent ways. Moreover, these particular eight ratios are *independent of* w. Hence, the eight phase-shift determinations can all be made without reference to a detailed knowledge of the incoming $\bar{\nu}$ energy spectrum.

In conclusion, we reiterate that the phase-shift information obtained here applies to situations where the effects of the muon mass may be neglected. This implies in particular that we have not made use of information obtainable from muon polarization at such barycentric energies of the reaction where the muon mass cannot be ignored.

II. Ξ_{e4} DECAYS

A. Preliminaries

Consider either reaction (1.1) or (1.2) and denote by K, k^Λ, k, p, and q the momentum four-vectors of Ξ, Λ, π, e, and $\bar{\nu}_e$, respectively. The masses are $K^2 = -M^2$, $(k^\Lambda)^2 = -m^2$, $k^2 = -\mu^2$, $q^2 = 0$, and we also put $p^2 = 0$ throughout. Define

$$P = k^\Lambda + k\,,\quad Q = k^\Lambda - k\,,\quad L = p + q\,,\quad N = p - q\,. \tag{2.1}$$

The quantities

$$P^2 \equiv -s\,,\quad L^2 \equiv -s_l$$

constitute two of our five variables. The other three are

(i) θ, the angle between the pion three momentum in the $\pi\Lambda$ rest frame and the line of flight of the $\pi\Lambda$ in the frame $\mathbf{K} = 0$;

(ii) θ_l, the angle between the electron three momentum in the dilepton rest frame and the dilepton line of flight in the frame $\mathbf{K}=0$;

(iii) ϕ, the angle between the normals to the planes of the $\pi\Lambda$ system and of the dilepton system, both defined in the frame $\mathbf{K}=0$.

These angles are conveniently exhibited in the $\pi\Lambda$ rest frame, in which we introduce Euclidean coordinates, with the three direction pointing along $\mathbf{L}=\mathbf{K}$, the two direction along $\mathbf{L}\times\mathbf{Q}$, and the one direction along $(\mathbf{L}\times\mathbf{Q})\times\mathbf{L}$. Thus, $\mathbf{Q}$ lies in the 13 plane, and θ is the angle between $\mathbf{Q}$ and the three axis. ϕ is the angle between the projection of $\mathbf{N}$ on the 12 plane and the one axis (going counterclockwise from the one axis to the $\mathbf{N}$ projection). Finally, θ_l appears in this frame in terms of the magnitude of the N_μ components. We find that

$$N_1=s_l^{-1/2}\sin\theta_l\cos\phi\,, \tag{2.2}$$

$$N_2=s_l^{-1/2}\sin\theta_l\sin\phi\,, \tag{2.3}$$

$$N_3=-s^{-1/2}(P\cdot L)\cos\theta_l\,, \tag{2.4}$$

$$N_0=-iN_4=s^{-1/2}X\cos\theta_l\,,\quad X=[(P\cdot L)^2-ss_l]^{1/2}. \tag{2.5}$$

Note that

$$P\cdot L=-\tfrac{1}{2}(M^2-s-s_l)\,, \tag{2.6}$$

$$X=\tfrac{1}{2}[M^4-2M^2(s+s_l)+(s-s_l)^2]^{1/2}. \tag{2.7}$$

The remaining kinematic quantities needed for what follows are

$$|\mathbf{Q}|,\quad |\mathbf{L}|,\quad Q_0=-iQ_4,\quad L_0=-iL_4. \tag{2.8}$$

Again in the $\pi\Lambda$ rest frame

$$|\mathbf{Q}|=s^{-1/2}[s-(m+\mu)^2]^{1/2}[s-(m-\mu)^2]^{1/2}, \tag{2.9}$$

$$L=s^{-1/2}X\,, \tag{2.10}$$

$$Q_0=s^{-1/2}(m^2-\mu^2)\,, \tag{2.11}$$

$$L_0=-s^{-1/2}(P\cdot L)\,. \tag{2.12}$$

The transition amplitudes for Ξ_{e4} decays are given by[7]

$$(G/\sqrt{2})\sin\theta_C\,\mathfrak{M}\,, \tag{2.13}$$

$$\mathfrak{M}=\langle\pi\Lambda|A_\lambda+V_\lambda|\Xi\rangle\epsilon_\lambda\,,$$

where

$$\epsilon_\lambda=\bar{u}(p)\gamma_\lambda(1+\gamma_5)v(q) \tag{2.14}$$

is the lepton current, which satisfies

$$L_\lambda\epsilon_\lambda=0\,. \tag{2.15}$$

Upon reduction to the two-component baryon spinor space, $\mathfrak{M}$ will have the general form (suppressing baryon spinors)

$$\mathfrak{M}=A+\boldsymbol{\sigma}\cdot\mathbf{B}\,. \tag{2.16}$$

[7] G is the Fermi constant. θ_C is the Cabibbo angle. A_λ and V_λ are the strangeness-changing hadronic axial-vector and vector currents, respectively.

A and $\mathbf{B}$ contain the lepton variables. We make the latter dependence explicit by putting[8]

$$A=a_\alpha\xi_\alpha\,,\quad B_k=b_{k\alpha}\xi_\alpha\,, \tag{2.17}$$

where the ξ_i are defined by

$$\boldsymbol{\varepsilon}=\xi_i\mathbf{e}_i\,. \tag{2.18}$$

Here $\boldsymbol{\varepsilon}$ denotes the space components[9] of ϵ_λ, Eq. (2.14), while $\mathbf{e}_i$ is the unit vector along the ith direction of the momentum coordinate system, defined above, in the $\pi\Delta$ rest frame.

We now compute the decay probability for a Ξ considered prepared in a polarized state, with a polarization vector denoted by $\boldsymbol{\Sigma}$, as in Sec. I. Thus, we sum over the lepton spins, but wish to retain a spin density matrix structure in the spin space of the Λ. The decay distributions can then be written as follows:

$$d^7w=\frac{1}{(4\pi)^7}(G^2\sin^2\theta_C)\frac{mX|\mathbf{Q}|}{M^2\sqrt{s}}\rho ds ds_l d\cos\theta \times d\cos\theta_l d\phi d\cos\vartheta d\psi\,. \tag{2.19}$$

Here ϑ denotes the polar angle, in the frame $\mathbf{K}=0$, between $\boldsymbol{\Sigma}$ and $\mathbf{P}$. ψ is defined in the same frame as the azimuthal angle between the normal to the plane defined by $\boldsymbol{\Sigma}$ and $\mathbf{P}$ and the plane defined by the $\pi\Lambda$ decay. ρ is the Λ spin density distribution defined in Eq. (1.4), but where now the following connection with the quantities a_α and $b_{k\alpha}$ of Eq. (2.17) can be established.

$$I=(a_\alpha^*a_\beta+b_{k\alpha}^*b_{k\beta})\tau_{\alpha\beta}\,, \tag{2.20}$$

$$S_k=(a_\alpha^*b_{k\beta}+a_\beta b_{k\alpha}^*)\tau_{\alpha\beta}\,, \tag{2.21}$$

$$A_k=-i\epsilon_{klm}b_{l\alpha}^*b_{m\beta}\tau_{\alpha\beta}\,, \tag{2.22}$$

$$S_{ij}=(b_{i\alpha}^*b_{j\beta}+b_{j\alpha}^*b_{i\beta}-\tfrac{2}{3}\delta_{ij}b_{k\alpha}^*b_{k\beta})\tau_{\alpha\beta}\,, \tag{2.23}$$

$$A_{ij}=-i\epsilon_{ijk}(a_\alpha^*b_{k\beta}-a_\beta b_{k\alpha}^*)\tau_{\alpha\beta}\,, \tag{2.24}$$

$$R=(a_\alpha^*a_\beta+\tfrac{1}{3}b_{k\alpha}^*b_{k\beta})\tau_{\alpha\beta}\,. \tag{2.25}$$

The symbol $\tau_{\alpha\beta}$ is defined by

$$\tau_{\alpha\beta}=L_\alpha L_\beta-N_\alpha N_\beta+s_l\delta_{\alpha\beta}+i\epsilon_{\alpha\beta\gamma}[N_\gamma L_0-L_\gamma N_0]. \tag{2.26}$$

The explicit form of the $\tau_{\alpha\beta}$ in terms of our dynamical variables follows from Eqs. (2.2)–(2.12). For easy reference, the $\tau_{\alpha\beta}$ are tabulated in the Appendix [Eqs. (A1)–(A10)].

Let us locate the variable dependences in the expressions (2.20)–(2.25). We have

$$a_\alpha=a_\alpha(s,s_l,\theta)\,,\quad b_{k\alpha}=b_{k\alpha}(s,s_l,\theta)\,.$$

All (θ_l,ϕ) dependence resides in the $\tau_{\alpha\beta}$. The $\tau_{\alpha\beta}$ are specified in terms of the nine independent real expres-

[8] A summation from one to three over doubly occurring indices is implied.

[9] Because of Eq. (2.15), the time component of ϵ_λ can be eliminated from the description.

sions given in Eqs. (A1)–(A9). Evidently, the following properties of Z, defined in Eq. (1.5) are generally true:

(i) Every Z has a ninefold decomposition of the form given in Eq. (1.6).

(ii) Consider the dependence of any Z on θ_l, all other variables being integrated over. Then,

$$dZ/d\cos\theta_l = a + b\cos\theta_l + c\cos^2\theta_l. \tag{2.27}$$

(iii) Likewise, integrating over all variables but ϕ,

$$dZ/d\phi = \alpha + \beta\cos\phi + \gamma\sin\phi + \delta\cos2\phi + \epsilon\sin2\phi. \tag{2.28}$$

Equations (2.27) and (2.28) are the simplest direct consequences of lepton pair locality and provide tests for this assumption.

Thus far all results are general. For the reasons explained in Sec. I, we now turn to the approximation where orbital angular momentum values 0 and 1 in the $\pi\Lambda$ system are retained only. Then $\mathfrak{M}$ defined in Eq. (2.13) can be written as follows:

$$\mathfrak{M} = S_{1/2}{}^A(\alpha_1,\alpha_2) + S_{1/2}{}^V(\bar\alpha_1,\bar\alpha_2) + P_{1/2}{}^A(\beta_1,\beta_2) + P_{1/2}{}^V(\bar\beta_1,\bar\beta_2) + P_{3/2}{}^A(\gamma_1,\gamma_2,\gamma_3) + P_{3/2}{}^V(\bar\gamma_1,\bar\gamma_2,\bar\gamma_3). \tag{2.29}$$

Here $S_{1/2}$, $P_{1/2}$, $P_{3/2}$ refer to the contributions which stem from the $\Lambda\pi$ system in the corresponding L_J state. The superscripts A and V distinguish contributions from the axial-vector and the vector current, respectively. The α's, β's, and γ's, barred and unbarred, represent the set of 14 form factors, functions of s and s_l only, which contain all dynamical information. Explicit representations for the various S and P quantities are given in the Appendix [Eqs. (A11)–(A16)].

In our S and P approximation, the θ dependence of the a_α and the $b_{k\alpha}$ can now be made explicit. We have

$$a_\alpha = \rho_\alpha \sin\theta, \quad \alpha = 1, 2 \tag{2.30}$$

$$a_3 = \omega_3 + \rho_3\cos\theta, \tag{2.31}$$

$$b_{k\alpha} = \rho_{k\alpha}\sin\theta, \quad (k\alpha) = (13), (31), (23), (32) \qquad = \omega_{k\alpha} + \rho_{k\alpha}\cos\theta, \text{ otherwise}. \tag{2.32}$$

All ρ and ω quantities, generally functions of s and s_l, are linear combinations of the α, β and γ form factors. These connections are recorded in the Appendix [Eqs. (A17)–(A30)]. The reader will readily observe the existence of numerous θ tests, [similar to the θ_l and ϕ tests given in Eqs. (2.27) and (2.28)] which provide checks on the validity of the S and P approximation.

We are now prepared to discuss the ways in which $\pi\Lambda$ phase-shift information can be extracted from Ξ_{e4} decays.

B. πΛ Phase-Shift Extraction

Let δ, δ', δ'' denote the $S_{1/2}$, $P_{1/2}$, $P_{3/2}$ $\pi\Lambda$ scattering phase shifts, respectively. These phase shifts enter into our analysis through the relations

$$\alpha_i \text{ or } \bar\alpha_i = (|\alpha_i| \text{ or } |\bar\alpha_i|)e^{i\delta}, \quad i=1,2 \tag{2.34}$$

$$\beta_i \text{ or } \bar\beta_i = (|\beta_i| \text{ or } |\bar\beta_i|)e^{i\delta'}, \quad i=1,2 \tag{2.35}$$

$$\gamma_i \text{ or } \bar\gamma_i = (|\gamma_i| \text{ or } |\bar\gamma_i|)e^{i\delta''}, \quad i=1,2,3. \tag{2.36}$$

Correspondingly, these phases appear in a specific way in the ρ and ω symbols given in Eqs. (A17)–(A30), and hence, via Eqs. (2.30)–(2.33) they appear in Eqs. (2.20)–(2.25).

As has been stressed in Sec. I, we are only concerned with such extractions of phase shifts which can be obtained by optimal integrations over all variables that are extraneous to the determination of the s dependence of the phase shifts. Accordingly, instead of Eq. (2.19), we consider the reduced distributions

$$d^5w = \frac{1}{(4\pi)^7}G^2\sin^2\theta_C \frac{m|\mathbf{Q}|}{M^2\sqrt{s}} W\langle\rho\rangle ds \times d\cos\theta_l d\phi d\cos\vartheta d\psi, \tag{2.37}$$

where

$$W = \int\int X ds_l d\cos\theta = 2\int X ds_l, \tag{2.38}$$

$$W\langle\rho\rangle = \int\int X\rho ds_l d\cos\theta. \tag{2.39}$$

$\langle\rho\rangle$ is the same function of the 16 quantities $\langle Z\rangle$ as ρ is of the Z, where

$$\langle Z\rangle = \int\int XZ ds_l d\cos\theta. \tag{2.40}$$

The $\langle Z\rangle$'s satisfy the same ninefold decomposition,

$$\langle Z\rangle = \langle Z^{(1)}\rangle + \langle Z^{(2)}\rangle\cos2\theta_l + \cdots, \tag{2.41}$$

as do the Z's, where

$$\langle Z^{(i)}\rangle = \int\int XZ^{(i)} ds_l d\cos\theta. \tag{2.42}$$

The $\langle Z^{(i)}\rangle$, which depend on s only, can be determined individually (for any Z) by making large cuts in the θ_l and ϕ distributions along the lines detailed in I. The question before us is then whether the average quantities $\langle Z^{(i)}\rangle$ can give us tangents of phase-shift differences by taking appropriate $\langle Z^{(i)}\rangle$ ratios. This was possible, we recall, for the $\pi\pi$ phase shifts in the instances of the K_{e4} intensity spectrum and the $K_{\mu4}$ polarization spectrum.[3] A careful inspection of the present situation reveals first of all the following two general facts:

(1) The method of taking ratios does not work in the general presence of P-wave phase shifts. As discussed in Sec. I, this led us to consider the presumably

reasonable approximation

$$\delta'=\delta''=0. \tag{2.43}$$

The experimental validity of Eq. (2.43) can itself be checked with the observations on the $\langle Z^{(i)}\rangle$. Thus, for example, Eq. (2.43) implies that

$$\langle I^{(9)}\rangle-3\langle R^{(9)}\rangle=0. \tag{2.44}$$

(2) Even in this approximation, the set $\langle I^{(i)}\rangle$ by itself does not suffice for our purposes. That is, the ninefold decomposition of the intensity spectrum alone does not carry enough information. It is indispensable that polarization distributions be considered as well. We give next a complete list of determinations of $\tan\delta$ in terms of $\langle Z^{(i)}\rangle$ ratios.

I combined with R yields two of these:

$$\begin{aligned}\tan\delta&=-\tfrac{1}{2}\langle I^{(7)}-3R^{(7)}\rangle/\langle I^{(4)}-3R^{(4)}\rangle\\&=-2\langle I^{(8)}-3R^{(8)}\rangle/\langle I^{(5)}-3R^{(5)}\rangle,\end{aligned} \tag{2.45}$$

which have perhaps the "experimental" advantage that they do not depend on the orientations of momentum three-vectors of the individual decay event. All the others do. They are

$$\begin{aligned}\tan\delta&=-\tfrac{1}{2}\langle S_3{}^{(7)}\rangle/\langle S_3{}^{(4)}\rangle=-2\langle S_3{}^{(8)}\rangle/\langle S_3{}^{(5)}\rangle\\&=2\langle A_{12}{}^{(4)}\rangle/\langle A_{12}{}^{(7)}\rangle=\tfrac{1}{2}\langle A_{12}{}^{(5)}\rangle/\langle A_{12}{}^{(8)}\rangle\\&=-2\langle S_{12}{}^{(4)}\rangle/\langle S_{12}{}^{(7)}\rangle=-\tfrac{1}{2}\langle S_{12}{}^{(5)}\rangle/\langle S_{12}{}^{(8)}\rangle\\&=\tfrac{1}{2}\langle A_3{}^{(7)}\rangle/\langle A_3{}^{(4)}\rangle=2\langle A_3{}^{(8)}\rangle/\langle A_3{}^{(5)}\rangle\\&=\tfrac{1}{2}\langle T_1{}^{(7)}\rangle/\langle T_1{}^{(4)}\rangle=\tfrac{1}{2}\langle T_2{}^{(7)}\rangle/\langle T_2{}^{(4)}\rangle\\&=-\tfrac{1}{2}\langle T_3{}^{(7)}\rangle/\langle T_3{}^{(4)}\rangle=2\langle T_1{}^{(8)}\rangle/\langle T_1{}^{(5)}\rangle\\&=2\langle T_2{}^{(8)}\rangle/\langle T_2{}^{(5)}\rangle=-2\langle T_3{}^{(8)}\rangle/\langle T_3{}^{(5)}\rangle.\end{aligned} \tag{2.46}$$

Here T_i is defined as

$$T_i=S_{ii}+2(I-R),\quad i=1,2,3. \tag{2.47}$$

All ratios in Eq. (2.46) refer to different parts of one given Z. In addition there are 20 ratios between corresponding parts of distinct Z's. They are

$$\begin{aligned}\tan\delta&=\langle A_{23}{}^{(6)}\rangle/\langle S_1{}^{(6)}\rangle=-\langle S_2{}^{(6)}\rangle/\langle A_{31}{}^{(6)}\rangle=\langle S_{23}{}^{(6)}\rangle/\langle A_1{}^{(6)}\rangle=\langle A_2{}^{(6)}\rangle/\langle S_{31}{}^{(6)}\rangle=\langle A_{23}{}^{(1)}+A_{23}{}^{(2)}\rangle/\langle S_1{}^{(1)}+S_1{}^{(2)}\rangle\\&=-\langle A_{23}{}^{(1)}-3A_{23}{}^{(2)}\rangle/\langle S_1{}^{(1)}-3S_1{}^{(2)}\rangle=-\langle S_2{}^{(1)}+S_2{}^{(2)}\rangle/\langle A_{31}{}^{(1)}+A_{31}{}^{(2)}\rangle=\langle S_2{}^{(1)}-3S_2{}^{(2)}\rangle/\langle A_{31}{}^{(1)}-3A_{31}{}^{(2)}\rangle\\&=\langle A_2{}^{(1)}+A_2{}^{(2)}\rangle/\langle S_{31}{}^{(1)}+S_{31}{}^{(2)}\rangle=-\langle A_2{}^{(1)}-3A_2{}^{(2)}\rangle/\langle S_{31}{}^{(1)}-3S_{31}{}^{(2)}\rangle=\langle S_{23}{}^{(1)}+S_{23}{}^{(2)}\rangle/\langle A_1{}^{(1)}+A_1{}^{(2)}\rangle\\&=-\langle S_{23}{}^{(1)}-3S_{23}{}^{(2)}\rangle/\langle A_1{}^{(1)}-3A_1{}^{(2)}\rangle,\end{aligned} \tag{2.48}$$

and finally

$$\begin{aligned}\tan\delta&=-\langle S_1{}^{(9)}\rangle/\langle A_{23}{}^{(9)}\rangle=\langle A_{31}{}^{(9)}\rangle/\langle S_2{}^{(9)}\rangle=\langle A_{23}{}^{(3)}\rangle/\langle S_1{}^{(3)}\rangle=-\langle S_2{}^{(3)}\rangle/\langle A_{31}{}^{(3)}\rangle\\&=-\langle A_1{}^{(9)}\rangle/\langle S_{23}{}^{(9)}\rangle=-\langle S_{31}{}^{(9)}\rangle/\langle A_2{}^{(9)}\rangle=\langle A_2{}^{(3)}\rangle/\langle S_{31}{}^{(3)}\rangle=\langle S_{23}{}^{(3)}\rangle/\langle A_1{}^{(3)}\rangle.\end{aligned} \tag{2.49}$$

To recapitulate: Eq. (1.4) defines a total of sixteen independent spectra, describing the spin-averaged decay distribution and various spin-dependent effects. Each spectrum has a ninefold decomposition with respect to the variables θ_l, ϕ, according to Eq. (1.6), the coefficients depending on the variables s, s_l, and θ. Integrating over s_l and θ, the coefficients become functions of the single variable s, and the results recorded immediately above describe the various ratios which serve to determine the Λ-π S-wave phase shift as a function of the variable s. We have not discussed here the geometrical arrangements required experimentally to isolate in turn the various elements of the density matrix, e.g., the correlation functions S_{ij}, A_{ij}, etc. What is required is obvious, and experimentally formidable.

III. ANTINEUTRINO REACTIONS

We turn now to a discussion of the reaction (1.3). For the sake of brevity, the notations are chosen so that large parts of Sec. II can be taken over for the present purposes without much rewriting. This necessitates the use of a number of common symbols for the decay and the reaction processes, but where some symbols have a different meaning in either case. The reader is warned wherever this happens.

Denote by K, k^Λ, k, p, and q the momentum four-vectors of the nucleon, Λ, π, μ, $\bar{\nu}$, respectively. The masses are $K^2=-M^2$, $k^{\Lambda 2}=-m^2$, $k^2=-\mu^2$, $p^2=-m_\mu{}^2$, and $q^2=0$. Define

$$P=k^\Lambda+k,\quad Q=k^\Lambda-k,\quad L=p-q,\quad N=p+q. \tag{3.1}$$

Please note the replacement $q\to -q$ in the definitions of L and N in Eq. (3.1) as compared with Eq. (2.1).

The quantities

$$P^2\equiv -s,\quad L^2\equiv t,\quad (q+K)^2\equiv -w^2 \tag{3.2}$$

constitute three of the five independent variables of the problem. The remaining two are

(1) θ, the angle between the pion three momentum in the $\pi\Lambda$ rest frame and the line of flight of the $\pi\Lambda$ in the laboratory frame, $\mathbf{K}=0$.

(2) ϕ, the angle between the normals to the planes of the $\pi\Lambda$ system and of the dilepton system, both defined in the frame $\mathbf{K}=0$.

We confine ourselves to configurations where w is so large and s is so small that the muon energy is large compared to m_μ. Accordingly we put $m_\mu=0$, so $p^2=0$.

Next, we compute the same kinematic quantities as were given in Eqs. (2.2)–(2.12) for the decay, and use the same coordinate system in the $\pi\Lambda$-rest frame as in Sec. II, with reference to the (redefined) $\mathbf{L}$, and to Q. We obtain

$$N_1=t^{1/2}\sinh\theta_l'\cos\phi, \tag{3.3}$$

$$N_2=t^{1/2}\sinh\theta_l'\sin\phi, \quad (3.4)$$

$$N_3=-s^{-1/2}(P\cdot L)\cosh\theta_l', \quad (3.5)$$

$$N_0=s^{-1/2}X\cosh\theta_l', \quad X=[(P\cdot L)^2+st]^{1/2}. \quad (3.6)$$

The hyperbolic angle θ_l' was defined in Eqs. (1.9)–(1.11). $|\mathbf{Q}|$, Q_0, $|\mathbf{L}|$, and L_0 again take the form (2.9)–(2.12) but where now X is as in Eq. (3.6), while

$$P\cdot L=-\tfrac{1}{2}(M^2-s+t). \quad (3.7)$$

The transition matrix element has again the structure given by Eqs. (2.16)–(2.18). By a reasoning similar to the one used to obtain Eq. (2.19) we get the following differential cross-section expression:

$$d^6\sigma=\frac{1}{2(4\pi)^5}G^2\sin^2\theta_C\frac{mMw^2|\mathbf{Q}|}{(w^2-M^2)^2\sqrt{s}}\rho ds dt \times d\cos\theta d\phi d\cos\vartheta d\psi, \quad (3.8)$$

ρ is again given by Eq. (1.4), where $\mathbf{\Sigma}$ now denotes the nucleon polarization. The phase space has six dimensions, as compared with seven in Eq. (2.19) because now one of the variables, w, refers to the initial state. ϑ and ψ have the same meaning here as for the decay. The Eqs. (2.20)–(2.26) also apply to the reaction case.[10] We further note that all Z's have the decomposition given in Eq. (1.8) and that [in the same sense as for Eq. (2.28)] the relations

$$dZ/d\phi=\alpha+\beta\cos\phi+\gamma\sin\phi+\delta\cos2\phi+\epsilon\sin2\phi \quad (3.9)$$

provide a large number of tests for the lepton pair locality assumption.

Also for the reaction case we employ the S and P approximation, so that Eqs. (2.29)–(2.36) and (A11)–(A30) may be used once more.

In the reaction case, the reduced distributions are

$$d^4\sigma=\frac{1}{2(4\pi)^5}G^2\sin^2\theta_C\frac{mMw|\mathbf{Q}|}{(w-M^2)^2\sqrt{s}}\langle\rho\rangle ds d\phi \times d\cos\vartheta d\psi, \quad (3.10)$$

where

$$\langle\rho\rangle=\int\int\rho dt d\cos\theta \quad (3.11)$$

is expressible in terms of

$$\langle Z^{(i)}\rangle=\int\int Z^{(i)}dt d\cos\theta, \quad (3.12)$$

where the $\langle Z^{(i)}\rangle$ depend on s only. We again use Eq. (2.43). One will verify that of the 36 expressions for $\tan\delta$, in the case of decay, only the eight relations (2.49) survive as useful ratios in the case of reactions.

[10] In Eq. (2.26) it is of course necessary to replace s_l with $-t$.

We record these here once more:

$$\begin{aligned}\tan\delta&=-\langle S_1^{(9)}\rangle/\langle A_{23}^{(9)}\rangle=\langle A_{31}^{(9)}\rangle/\langle S_2^{(9)}\rangle\\&=\langle A_{23}^{(3)}\rangle/\langle S_1^{(3)}\rangle=-\langle S_2^{(3)}\rangle/\langle A_{31}^{(3)}\rangle\\&=-\langle A_1^{(9)}\rangle/\langle S_{23}^{(9)}\rangle=-\langle S_{31}^{(9)}\rangle/\langle A_2^{(9)}\rangle\\&=\langle A_2^{(3)}\rangle/\langle S_{31}^{(3)}\rangle=\langle S_{23}^{(3)}\rangle/\langle A_1^{(3)}\rangle. \end{aligned} \quad (3.13)$$

We repeat that these are ratios of functions of one single variable, the invariant $\pi\Lambda$ mass. We have integrated over all values of the lepton momentum transfer t and of the $\pi\Lambda$ decay angle θ. Moreover, the ratios are independent of w. Hence, the phase-shift determinations Eq. (3.13) are independent of the spectrum of the incoming $\bar{\nu}$ energies.

APPENDIX

The quantities $\tau_{\alpha\beta}$ defined in Eq. (2.26) are fully specified as follows:

$$\tau_{11}=s_l(1-\sin^2\theta_l\cos^2\phi), \quad (A1)$$

$$\tau_{22}=s_l(1-\sin^2\theta_l\sin^2\phi), \quad (A2)$$

$$\tau_{33}=s^{-1}(P\cdot L)^2\sin^2\theta_l, \quad (A3)$$

$$\mathrm{Re}\tau_{12}=-\tfrac{1}{2}s_l\sin^2\theta_l\sin2\phi, \quad (A4)$$

$$\mathrm{Im}\tau_{12}=s_l\cos\theta_l, \quad (A5)$$

$$\mathrm{Re}\tau_{13}=\tfrac{1}{2}\gamma\sin2\theta_l\cos\phi, \quad (A6)$$

$$\mathrm{Im}\tau_{13}=\gamma\sin\theta_l\sin\phi, \quad (A7)$$

$$\mathrm{Re}\tau_{23}=\tfrac{1}{2}\gamma\sin2\theta_l\sin\phi, \quad (A8)$$

$$\mathrm{Im}\tau_{23}=-\gamma\sin\theta_l\cos\phi, \quad (A9)$$

where

$$\gamma=s_l^{1/2}s^{-1/2}(P\cdot L). \quad (A10)$$

We have used the following explicit representation in baryon spin space of the S and P quantities which appear on the right-hand side of Eq. (2.29).

$$S_{1/2}^A=\alpha_1\hat{L}\cdot\boldsymbol{\varepsilon}+i\alpha_2\boldsymbol{\sigma}\cdot(\hat{L}\times\boldsymbol{\varepsilon}), \quad (A11)$$

$$S_{1/2}^V=\bar{\alpha}_1(\boldsymbol{\sigma}\cdot\hat{L})(\hat{L}\cdot\boldsymbol{\varepsilon})+\bar{\alpha}_2(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}), \quad (A12)$$

$$P_{1/2}^A=\beta_1[\hat{Q}\cdot\boldsymbol{\varepsilon}-i\boldsymbol{\sigma}\cdot(\hat{Q}\times\boldsymbol{\varepsilon})]+(\beta_2-\beta_1)(\hat{L}\cdot\boldsymbol{\varepsilon})[\hat{Q}\cdot\hat{L}-i\boldsymbol{\sigma}\cdot(\hat{Q}\times\hat{L})], \quad (A13)$$

$$\begin{aligned}P_{1/2}^V=\bar{\beta}_1[&(\hat{Q}\cdot\boldsymbol{\varepsilon})(\boldsymbol{\sigma}\cdot\hat{L})-i\hat{L}\cdot(\hat{Q}\times\boldsymbol{\varepsilon})\\&-(\boldsymbol{\sigma}\cdot\hat{Q})(\hat{L}\cdot\boldsymbol{\varepsilon})+(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon})(\hat{Q}\cdot\hat{L})]\\&+(\bar{\beta}_2-\bar{\beta}_1)(\hat{L}\cdot\boldsymbol{\varepsilon})[2(\boldsymbol{\sigma}\cdot\hat{L})(\hat{Q}\cdot\hat{L})-(\boldsymbol{\sigma}\cdot\hat{Q})], \end{aligned} \quad (A14)$$

$$\begin{aligned}P_{3/2}^A=\gamma_1[&2\hat{Q}\cdot\boldsymbol{\varepsilon}+i\boldsymbol{\sigma}\cdot(\hat{Q}\times\boldsymbol{\varepsilon})]\\&+(\gamma_2-\gamma_1)(\hat{L}\cdot\boldsymbol{\varepsilon})[2\hat{Q}\cdot\hat{L}+i\sigma(\hat{Q}\times\hat{L})]\\&+\gamma_3[(\hat{Q}\cdot\hat{L})(\hat{L}\cdot\boldsymbol{\varepsilon})-(\hat{Q}\cdot\boldsymbol{\varepsilon})\\&+i(\hat{Q}\cdot\hat{L})\{\boldsymbol{\sigma}\cdot(\hat{L}\times\boldsymbol{\varepsilon})\}], \end{aligned} \quad (A15)$$

$$\begin{aligned}P_{3/2}^V=\bar{\gamma}_1[&2(\hat{Q}\cdot\boldsymbol{\varepsilon})(\boldsymbol{\sigma}\cdot\hat{L})+(\boldsymbol{\sigma}\cdot\hat{Q})(\hat{L}\cdot\boldsymbol{\varepsilon})\\&-(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon})(\hat{Q}\cdot\hat{L})+i\hat{L}\cdot(\hat{Q}\times\boldsymbol{\varepsilon})]\\&+(\bar{\gamma}_2-\bar{\gamma}_1)(\hat{L}\cdot\boldsymbol{\varepsilon})[(\boldsymbol{\sigma}\cdot\hat{L})(\hat{Q}\cdot\hat{L})+(\boldsymbol{\sigma}\cdot\hat{Q})]\\&+\bar{\gamma}_3[2(\boldsymbol{\sigma}\cdot\hat{L})(\hat{Q}\cdot\hat{L})(\hat{L}\cdot\boldsymbol{\varepsilon})-(\boldsymbol{\sigma}\cdot\hat{L})(\hat{Q}\cdot\boldsymbol{\varepsilon})\\&-(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon})(\hat{Q}\cdot\hat{L})]. \end{aligned} \quad (A16)$$

Baryon spinor symbols have been dropped. $\hat{L}$ and $\hat{Q}$ denote unit vectors along **L** and **Q**, respectively. ϵ is the three-vector part of ϵ_λ; see Eq. (2.14).

Correspondingly, the ω and ρ symbols defined in Eqs. (2.30)-(2.33) are given by

$$\rho_1=\beta_1+2\gamma_1-\gamma_3\,, \tag{A17}$$

$$i\rho_2=\bar{\beta}_1-\bar{\gamma}_1\,, \tag{A18}$$

$$\rho_3=\beta_2+2\gamma_2\,, \tag{A19}$$

$$\omega_3=\alpha_1\,, \tag{A20}$$

$$\rho_{11}=\rho_{22}=\bar{\beta}_1-\bar{\gamma}_1-\bar{\gamma}_3\,, \tag{A21}$$

$$\rho_{33}=\bar{\beta}_2+2\bar{\gamma}_2\,, \tag{A22}$$

$$i\rho_{12}=-i\rho_{21}=-\beta_1+\gamma_1+\gamma_3\,, \tag{A23}$$

$$\rho_{13}=-\bar{\beta}_2+\bar{\gamma}_2\,, \tag{A24}$$

$$\rho_{31}=\bar{\beta}_1+2\bar{\gamma}_1-\bar{\gamma}_3\,, \tag{A25}$$

$$-i\rho_{23}=\beta_2-\gamma_2\,, \tag{A26}$$

$$i\rho_{32}=\beta_1-\gamma_1\,, \tag{A27}$$

$$\omega_{11}=\omega_{22}=\bar{\alpha}_2\,, \tag{A28}$$

$$\omega_{33}=\bar{\alpha}_1+\bar{\alpha}_2\,, \tag{A29}$$

$$-\omega_{12}=\omega_{21}=i\alpha_2\,. \tag{A30}$$

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Crossing-Symmetric Regge Amplitude, Complex Trajectory Functions, and Phase Contours*

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The behavior of a crossing-symmetric Regge model is discussed in the complex t plane at fixed s. The trajectory function is allowed to take on an arbitrary imaginary part. Phase contour maps are drawn.

1. INTRODUCTION

THE purpose of this paper is to add to the *descriptive* literature on the phase-contour model.[1–3] In Ref. 2, a crossing-symmetric Regge model is developed with the drawback that the trajectory function is allowed no more than a small imaginary part. We first of all, in Sec. 2, investigate the phase and modulus contours of a crossing-symmetric Regge amplitude in the complex α plane (α is the trajectory function). A linear parametrization of α is used to transfer to the t plane at fixed s (see Sec. 3). It is shown that zeros deduced in Ref. 2 are very convenient for the drawing of phase contours in a simple way. In Sec. 4, the effect of secondary trajectories is considered and modifications made to phase contour maps.

Throughout this paper we shall be concerned with an amplitude that is even under crossing and describes the scattering of equal-mass spinless bosons of mass m. The kinematic invariants s, t, and u are those used in Refs. 1–3.

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[1] C. B. Chiu, R. J. Eden, and Chung-I Tan, Phys. Rev. **170**, 1490 (1968).

[2] R. J. Eden and Chung-I Tan, Phys. Rev. **170**, 1516 (1968).

[3] R. J. Eden and Chung-I Tan, Phys. Rev. **172**, 1583 (1968).

2. PHASE CONTOURS OF SINGLE REGGE AMPLITUDE IN COMPLEX α PLANE

In Ref. 2, a discussion is held about the phase contours of a Regge amplitude that is even under crossing. For example, when s (the square of the c.m. energy) is large and t (the square of the four-momentum transfer) is small, the following amplitude is assumed to dominate

$$F(s,t)=\frac{s^{\alpha(t)}\exp\{i\pi[1-\frac{1}{2}\alpha(t)]\}}{\Gamma(\alpha(t))\sin[\frac{1}{2}\pi\alpha(t)]}\,. \tag{2.1}$$

This amplitude is valid when we approach the physical sheet in the limit $s+i0$ from complex values of s. The Γ function is introduced as a convenient parametrization of the existence of zeros of the Regge residue in the physical region. The sine function introduces poles at positive even-integer values of α and cancels some of the zeros produced by the Γ function; the remaining zeros occur at negative odd-integer values of α. We now fix s real, positive, and in the limit $+i0$, and investigate the phase contours and modulus contours of amplitude (2.1) in the complex α plane. It is straightforward, but tedious, to do this and the results are presented in Fig. 1 for $s=300$. To transfer to the complex t plane, we invent a parametrization for $\alpha(t)$ and read phases directly off Fig. 1. For example, a linear parametrization of $\alpha(t)$ would mean that phase and modulus contours in