Calculation of the K_{l3} Form Factors

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We have determined the K_{l3} form factors $f_{\pm}(t)$ in a model of strong interactions based on the rest-frame group $O(4,2)\otimes SU(3)$. Using the values of the parameters determined from the boson-mass spectrum and pion electromagnetic form factor, we obtain at $t=t_0=(m_K-m_\pi)^2$, $\xi(t_0)\cong(m_K-m_\pi)/(3m_K-m_\pi)=0.27$. We then test the validity of exact $SU(3)$ symmetry for the rest-frame states. The condition $f_+{}^K(t_0)=\frac{1}{2}f_+{}^\pi(t_0)$ yields $\lambda_+=0.0152$, $\lambda_-=0.199$, and $\xi(0)=0.097$, in agreement with the latest experimental values.

I. INTRODUCTION

THE form factors occurring in the reaction $K\rightarrow\pi l\nu$ have been the subject of a large number of investigations, all using the techniques of current algebra or field algebra.[1] The results obtained differ widely. It is therefore of interest to have a calculation based on an entirely different approach of symmetry breaking. This is done here.

A number of hadron properties (mass spectra, form factors, decay rates, diffraction scattering) have been successfully described by the relativistic $O(4,2)$ model of strong interactions.[2] The underlying general hypotheses can be applied to weak interactions of hadrons and thereby further tested. In this model the hadrons are described by the relativistic analog of "wave functions" and the information contained in these wave functions enters into strong, electromagnetic, and weak interactions in the same way.

In a previous paper,[3] the general hypotheses concerning the weak currents (scalar, pseudoscalar, vector, and axial vector) acting on the "wave functions" of the hadrons have been discussed and the form factors in nonleptonic decays have been calculated using scalar and pseudoscalar couplings. In the present paper, we discuss the important case of vector currents.

Let us emphasize that the theory amounts essentially to a direct generalization of the usual weak-interaction currents; instead of writing the currents between Dirac spinors [i.e., simplest $O(4,2)$ spinors] we write them between the relativistic hadron "wave functions," the more general $O(4,2)$ spinors. The rest-frame hadron states are labeled by $|njm\pm,I,I_3Y)$, where the range of ($njm\pm$, n=principal quantum number, (jm)=spin and spin component, $\pm$parity) are given by the $O(4,2)$ representation used, and the range of I,I_3Y by the rest-frame $SU(3)$ representation, for example. The minimal conserved-current elements are the matrix elements of the following operator:

$$j_\mu=\alpha_1\Gamma_\mu+\alpha_2P_\mu+\alpha_3P_\mu S+i\alpha_4L_{\mu\nu}q^\nu+i\alpha_5\tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}L_{\nu\lambda}q_\rho\Pi\,,\quad(1.1)$$

where Γ_μ, S, and $L_{\mu\nu}$ are $O(4,2)$ generators; $P_\mu=(p'+p)_\mu$; $q_\mu=(p'-p)_\mu$; and Π is the simple pseudoscalar operator in $O(4,2)$, taken between the so-called "tilted" states of momentum $p_\mu=(m\cosh\xi,\hat{\xi}m\sinh\xi)$, defined in Sec. II A.

The coefficients α_i in Eq. (1.1) are tensor operators with respect to an algebra of currents, $SU(3)\times SU(3)$, for example. Note the difference between this algebra of currents and the $SU(3)$ algebra which labels the multiplets.[2]

The conserved current (1.1) fixes a mass spectrum depending on the coefficients α_1, α_2, and α_3 (α_4 and α_5 do not contribute to the mass spectrum).

The hypothesis was made that the electromagnetic current, the weak vector current, and even (perhaps) the strong vector current are conserved and proportional to the matter current (1.1).[2,3] In Sec. II, we apply the current (1.1) to derive K_{l3} decay-form factors $f_+(t)$ and $f_-(t)$. We derive general formulas for the complete functional form of $f_\pm(t)$. We then evaluate these functions numerically by using the estimated values of α_i from the mass spectrum. We then test the hypothesis that the (unboosted) rest frame states satisfy the $SU(3)$ symmetry. The theory, as we shall see, contains definite procedures for evaluating the effect of symmetry breaking.

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[1] The recent literature on K_{l3} form factors is, in fact, so extensive that we do not attempt to list all the papers published on this subject. We refer to reviews in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (Wiley-Interscience Publishers, Inc., New York, 1968); Proceedings of the Fourteenth International Conference on High-Energy Physics, Vienna, 1968 (unpublished).

[2] References can be found in two recent reviews, one in *Lectures in Theoretical Physics* (Gordon and Breach, Science Publishers, Inc., New York, 1968) Vol. XB, the other in Acta Physica Hungarie (to be published).

[3] A. O. Barut and S. Malin, Nucl. Phys. **B9**, 194 (1969).

II. K_{l3} DECAY FORM FACTORS

The basic vector vertex functions in the processes like $K \rightarrow \pi + (l\nu)$ is given by

$$\langle K^i; p_K | V_\mu{}^k | \pi^j; p_\pi \rangle = (4p_K{}^0 p_\pi{}^0)^{-1/2} i[f_{ijk}] \frac{G}{\sqrt{2}} \sin\theta_v \times \{ (p_K + p_\pi)_\mu f_+(t) + (p_K - p_\pi)_\mu f_-(t) \} . \quad (2.1)$$

Here i, j, and k are the $SU(3)$ indices, $[f_{ijk}]$ are the relevant $SU(3)$ reduced matrix elements, $p_K{}^\mu$ and $p_\pi{}^\mu$ are the four-momenta of K and π, and $f_\pm(t)$ are the (invariant) scalar amplitudes.

We now have a dynamical theory to evaluate the *left-hand side* of (2.1). Thus, by comparing it with the right-hand side we can evaluate the complete functional form of the two form factors f_+ and f_-.

A. States

The states $|K,p_\mu\rangle$, $|\pi,p_K\rangle$, $\cdots$ are the tilted, rotated, and boosted $O(4,2) \otimes SU(3)$ states. This means we take the basis vectors $|njm; II_3Y)$ and define the new states

$$|njm; II_3Y; p\rangle_T = \frac{1}{N} e^{i\zeta\cdot\mathbf{M}} e^{i\theta_n(II_3Y)L_{45}} e^{-2i\theta_v\lambda_7} |njm, II_3Y\rangle . \quad (2.2)$$

The reason one defines these new "physical" states is that the current operators have simple transformation properties as elements of the Lie algebra with respect to these new states $|\ \rangle_T$, and not with respect to the old basis vectors $|\ \rangle$. In Eq. (2.2), on the right, the first operation $e^{-2i\theta_v\lambda_7}$ is simply the Cabibbo rotation[4]; the second operation is the noncompact tilting in $O(4,2)$,[2] and the third is a pure Lorentz transformation; N is a normalization factor. The generators of the pure Lorentz transformations are the $L_{i6} = M_i$ elements of the $O(4,2)$ Lie algebra.

B. Current

With respect to the new states (2.2) the current operator (1.1), in the case of bosons, has the simple form

$$j_\mu{}^i = G\lambda^i \otimes (\alpha_1 \Gamma_\mu + \alpha_2 P_\mu + \alpha_3 P_\mu S) = \sum_{s=1}^{3} G\lambda^i \otimes \alpha_s F_\mu{}^s . \quad (2.3)$$

The terms α_4 and α_5 do not contribute in the case of bosons. Here, λ^i are the usual $SU(3)$ generators. Thus, the current operator factorizes. The situation is more complicated for baryons where α_4 and α_5 terms are also present.

[4] N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).

C. Vector Vertex Amplitudes

In our model, the left-hand side of (2.1) is given by

$$\langle K^i; p_K | V_\mu{}^k | \pi^j; p_\pi \rangle = {}_T\langle K^i; P_K | j_\mu{}^k | \pi^j; p_\pi \rangle_T , \quad (2.4)$$

with the states $|\ \rangle_T$ defined in (2.2) and $j_\mu{}^k$ defined in (2.3). Because of the direct-product form assumed in (2.3), we can evaluate the $SU(3)$ part of (2.3) in a standard manner. If the octet of mesons are assigned to the $SU(3)$ tensor $T_j{}^i$, the $SU(3)$ part of (2.3) is

$$a = (G/\sqrt{2}) \sin\theta_v \langle [8]; 1, \tfrac{1}{2}, -\tfrac{1}{2} | T_3{}^1 | [8]; 1,0,0 \rangle . \quad (2.5)$$

D. Kinematics

In the rest frame of K, we have for the invariant momentum transfer

$$t = (p_K - p_\pi)^2 = (m_K{}^2 + m_\pi{}^2 - 2m_K E_\pi) \quad (2.6)$$

and for the parameter ζ of Lorentz transformations in (2.2) we have

$$\zeta = \hat{p}_\pi \tanh^{-1}\left(\frac{p_\pi}{E_\pi}\right), \quad \cosh\zeta = \frac{m_K{}^2 + m_\pi{}^2 - t}{2m_K m_\pi}, \quad \tanh\tfrac{1}{2}\zeta = \left[\frac{(m_K - m_\pi)^2 - t}{(m_K + m_\pi)^2 - t}\right]^{1/2} . \quad (2.7)$$

E. Spin Part of the Vertex Amplitudes

In the rest frame of K, we then have from (2.3), (2.4), and (2.5)

$$\langle K^+ | (V_\mu)_3{}^1 | \pi^0; p_\pi \rangle = a\, {}_T\langle n=1, j=0, m=0 | \alpha_s F_\mu{}^s | n=1, j=0, m=0; p_\pi \rangle_T = a g_\mu(\zeta) , \quad (2.8)$$

where, in the original basis $|njm)$,

$$g_\mu(\zeta) = (100 | e^{-i\theta_K L_{45}} \alpha_s F_\mu{}^s e^{-i\zeta\cdot\mathbf{M}} e^{i\theta_\pi L_{45}} | 100) . \quad (2.9)$$

The quantities $g_\mu(\zeta)$ are the actual $O(4,2)$ parts of the amplitudes. In the general case these are analytic functions of the external spin j, principal quantum number n, and ζ given by (2.7). In (2.9), θ_K and θ_π are the appropriate tilting parameters for the K and the π states which we shall determine.

The evaluation of (2.9) is carried out in the Appendix. The final result is given by Eqs. (A9) and (A10), which can be written approximately as in Eqs. (A11) and (A12).

F. Determination of the Parameters

Apart from an over-all coupling constant G, the conserved current (2.3) contains three parameters α_1, α_2, and α_3, which are common to both the π tower and the K tower of states (that is states with $j^P = 0^-, 1^+, 2^+, \cdots$, with the same internal quantum numbers as the π or the K). Now the condition of current conservation expresses the mass as a function of α_1, α_2, α_3 and two new parameters β and γ.[5] This can also be seen by noting that the current conservation is equivalent to writing down an equation of the form

$$(j_\mu{}^i P^\mu+\beta^i S+\gamma^i)\psi(p)=0 \tag{2.10}$$

with $j_\mu{}^i$ given by (2.3). Here, $S=L_{46}$ is the Lorentz scalar element of the Lie algebra of $O(4,2)$. The parameters β and γ are, in contrast to α_1, α_2, α_3, different for π and K tower because of the $K-\pi$ mass differences which cannot be neglected. Our formalism takes care of these mass differences via the values of ζ and via the tilting angles θ_K and θ_π in Eq. (2.9). These angles are, however, not new parameters; they are related to α_i and to β and γ by[2]

$$\sinh\theta_n=n(\beta-\alpha_3 M_n{}^2)/(\gamma-\alpha_2 M_n{}^2)\,, \tag{2.11}$$

where n is the principal quantum number, $n=1$ for both π and K.

From the mass spectrum, the normalization condition and pion electromagnetic form factor we determine,[6]

$$\theta_\pi\cong 0\,,\quad \beta_\pi\cong\gamma_\pi\cong 0 \tag{2.12}$$

and, consequently,

$$\alpha_1\cong -1\,,\quad \alpha_2\cong 1/m_\pi\,,\quad \alpha_3\cong 0. \tag{2.13}$$

With these values, Eqs. (A11) and (A12) become

$$\begin{aligned}g_0(\zeta)\cong&\cosh^{-2}(\tfrac{1}{2}\theta_K)\cosh^{-2}(\tfrac{1}{2}\zeta)\\ &\times\{\alpha_1[1-3\tanh^2(\tfrac{1}{2}\theta_K)\tanh^{-2}(\tfrac{1}{2}\zeta)]\\ &+\alpha_2(m_K+E_\pi)[1-\tanh^2(\tfrac{1}{2}\theta_K)\tanh^2(\tfrac{1}{2}\zeta)]\\ &+\alpha_3(m_K+E_\pi)\tanh(\tfrac{1}{2}\theta_K)\cosh^{-2}(\tfrac{1}{2}\zeta)\\ &\quad\times[1-2\tanh^2(\tfrac{1}{2}\theta_K)\tanh^2(\tfrac{1}{2}\zeta)]\}\,,\end{aligned} \tag{2.14}$$

$$\begin{aligned}g_3(\zeta)\cong&\cosh^{-2}(\tfrac{1}{2}\theta_K)\cosh^{-4}(\tfrac{1}{2}\zeta)(p_\pi)_3\\ &\times\Big\{\frac{\alpha_1}{m_\pi}[1-3\tanh^2(\tfrac{1}{2}\theta_K)\tanh^2(\tfrac{1}{2}\zeta)\\ &+\alpha_2\cosh^2(\tfrac{1}{2}\zeta)[1-\tanh^2(\tfrac{1}{2}\theta_K)\tanh^2(\tfrac{1}{2}\zeta)]\\ &+\alpha_3\tanh(\tfrac{1}{2}\theta_K)\\ &\quad\times[1-2\tanh^2(\tfrac{1}{2}\theta_K)\tanh^2(\tfrac{1}{2}\zeta)]\Big\}\,.\end{aligned}$$

Equation (2.14) inserted into (2.8) together with (2.5) and (2.1) completes the evaluation of the vector-vertex amplitudes.

G. Determination of the Form Factors

Having determined the left-hand side of (2.1) we can now determine $f_\pm(t)$ occurring on the right-hand side. In the rest frame of K, and taking $p_\pi{}^\mu=(E_\pi,0,0,p_\pi)$, we have

$$\begin{aligned}ag_0(\zeta)=\langle K^+|(V_0)_3{}^1|\pi^0;p_\pi\rangle=&(4m_K E_\pi)^{-1/2}\sqrt{2}a\\ &\times[(m_K+E_\pi)f_+(t)+(m_K-E_\pi)f_-(t)]\,,\end{aligned} \tag{2.15}$$

$$\begin{aligned}ag_3(\zeta)=\langle K^+|(V_3)_3{}^1|\pi^0;p_\pi\rangle=&(4m_K E_\pi)^{-1/2}\sqrt{2}a\\ &\times p_\pi[f_+(t)-f_-(t)]\,;\end{aligned} \tag{2.16}$$

the components V_1 and V_2 vanish.

1. π_{l3} Form Factors

Current conservation implies that $f_-(t)\equiv 0$, and we immediately obtain from (2.14) and either (2.18) or (2.19)

$$\begin{aligned}f_+(t)=&\frac{(4m_\pi E_\pi)^{-1/2}}{(m_\pi+E_\pi)}\frac{\sqrt{2}}{\cosh^2(\frac{1}{2}\zeta)\cosh^2(\frac{1}{2}\theta_\pi)}\\ &\times\{[\alpha_1+\alpha_2(m_\pi+E_\pi)]\\ &-[3\alpha_1+\alpha_2(m_\pi+E_\pi)]\tanh^2(\tfrac{1}{2}\theta_\pi)\tanh^2(\tfrac{1}{2}\zeta)\\ &+\alpha_3(m_\pi+E_\pi)\tanh(\tfrac{1}{2}\theta_\pi)\cosh^{-2}(\tfrac{1}{2}\zeta)\\ &\quad\times[1-2\tanh^2(\tfrac{1}{2}\theta_\pi)\tanh^2(\tfrac{1}{2}\zeta)]+\cdots\}\,,\end{aligned} \tag{2.17}$$

or, with our value $\theta_\pi\approx 0$,

$$\begin{aligned}f_+(t)=&\sqrt{2}\frac{(\cosh\zeta)^{-1/2}}{\cosh^4(\frac{1}{2}\zeta)}[\alpha_1+2\alpha_2 m_\pi\cosh^2(\tfrac{1}{2}\zeta)]\\ =&\sqrt{2}\frac{(\cosh\zeta)^{1/2}}{\cosh^4(\frac{1}{2}\zeta)}\,.\end{aligned} \tag{2.18}$$

Hence,

$$f_+(\zeta=0)=\sqrt{2} \tag{2.19}$$

($\zeta=0$ is $t=0$ for π_{l3}).

The form factor $f_+(t)$ falls off much faster with t than does the electromagnetic form factor.[6]

2. K_{l3} Form Factors

The two Eqs. (2.15) and (2.16) are now

$$\begin{aligned}(m_K+E_\pi)f_+(t)+(m_K-E_\pi)f_-(t)=&2(m_K E_\pi)^{1/2}\frac{1}{\sqrt{2}}\Big\{\frac{\alpha_1}{\cosh^2(\frac{1}{2}\theta_K)\cosh^2(\frac{1}{2}\zeta)}[1-3\tanh^2(\tfrac{1}{2}\theta_K)\tanh^2(\tfrac{1}{2}\zeta)]\\ &+\frac{\alpha_2(m_K+E_\pi)}{\cosh^2(\frac{1}{2}\theta_K)\cosh^2(\frac{1}{2}\zeta)}[1-\tanh^2(\tfrac{1}{2}\theta_K)\tanh^2(\tfrac{1}{2}\zeta)]+\frac{\alpha_3(m_K+E_\pi)\sinh(\frac{1}{2}\theta_K)}{\cosh^4(\frac{1}{2}\zeta)\cosh^3(\frac{1}{2}\theta_K)}[1-2\tanh^2(\tfrac{1}{2}\theta_K)\tanh^2(\tfrac{1}{2}\zeta)]\Big\}\end{aligned} \tag{2.20}$$

[5] A. O. Barut, D. Corrigan, and H. Kleinert, Phys. Rev. Letters **20**, 167 (1968); Phys. Rev. **167**, 1527 (1968).
[6] A. O. Barut, Nucl. Phys. **B4**, 455 (1968).

and

$$f_+(t)-f_-(t)=\frac{1}{\sqrt{2}}\,\frac{2(m_KE_\pi)^{1/2}}{1}\left\{\frac{\alpha_1}{m_\pi}\,\frac{1}{2\cosh^2(\frac{1}{2}\theta_K)\cosh^4(\frac{1}{2}\zeta)}\left[1-3\tanh^2(\tfrac{1}{2}\theta_K)\tanh^2(\tfrac{1}{2}\zeta)\right]\right.$$
$$+\alpha_2\frac{1}{\cosh^2(\frac{1}{2}\theta_K)\cosh^2(\frac{1}{2}\zeta)}\left[1-\tanh^2(\tfrac{1}{2}\theta_K)\tanh^2(\tfrac{1}{2}\zeta)\right]$$
$$\left.+\alpha_3\frac{\sinh(\frac{1}{2}\theta_K)}{\cosh^3(\frac{1}{2}\theta_K)\cosh^4(\frac{1}{2}\zeta)}\left[1-2\tanh^2(\tfrac{1}{2}\theta_K)\tanh(\tfrac{1}{2}\zeta)\right]\right\}. \quad (2.21)$$

From these two equations we obtain, using the parameters (2.12) and (2.13), $\alpha_1\cong-1$, $\alpha_2\cong 1/m_\pi$, $\alpha_3\cong 0$

$$f_+(t)=\left(\frac{E_\pi}{2m_K}\right)^{1/2}\frac{1}{\cosh^2(\frac{1}{2}\theta_K)\cosh^2(\frac{1}{2}\zeta)}\left[\frac{2m_K}{m_\pi}\left[1-\tanh^2(\tfrac{1}{2}\theta_K)\tanh^2(\tfrac{1}{2}\zeta)\right]-\left[1-3\tanh^2(\tfrac{1}{2}\theta_K)\tanh^2(\tfrac{1}{2}\zeta)\right]\right.$$
$$\left.\times\left(1+\frac{m_K-E_\pi}{2m_\pi\cosh^2(\frac{1}{2}\zeta)}\right)\right], \quad (2.22)$$

$$f_-(t)=-\left(\frac{E_\pi}{2m_K}\right)^{1/2}\frac{1}{\cosh^2(\frac{1}{2}\theta_K)\cosh^2(\frac{1}{2}\zeta)}\left(1-\frac{m_K+E_\pi}{2m_\pi\cosh^2(\frac{1}{2}\zeta)}\right)\left[1-3\tanh^2(\tfrac{1}{2}\theta_K)\tanh^2(\tfrac{1}{2}\zeta)\right].$$

[Note again that $f_-(t)\equiv 0$ for $m_K=m_\pi$.]

There is still one undetermined parameter left in these expressions, θ_K, the tilting angle of the K tower. But at pion momentum zero, i.e.,

$$\zeta=0,\quad \text{i.e., } t=(m_K-m_\pi)^2, \quad (2.23)$$

we can determine the ratio of the form factors independent of θ_K. We have

$$\xi(t=(m_K-m_\pi)^2)\equiv\frac{f_-(t=(m_K-m_\pi)^2)}{f_+(t=(m_K-m_\pi)^2)}=\frac{(m_K-m_\pi)}{(3m_K-m_\pi)}\cong 0.27. \quad (2.24)$$

In order to determine f_- and f_+ separately, we now impose at $\zeta=0$, the $SU(3)$ symmetry condition

$$f_+^K(\zeta=0)=\tfrac{1}{2}f_+^\pi(\zeta=0)=\tfrac{1}{2}\sqrt{2}=1/\sqrt{2}. \quad (2.25)$$

This requirement is in accordance with our general point of view that the $SU(3)$ symmetry holds exactly for rest-frame states, i.e., $(\zeta=0)$. The factor $1/\sqrt{2}$ comes from $SU(3)$ Clebsch-Gordan coefficients; f_- has also to be normalized by multiplying it by $1/\sqrt{2}$. The right-hand side of (2.25) has been determined above in Eq. (2.19). Consequently, we obtain

$$\cosh^2(\tfrac{1}{2}\theta_K)=\tfrac{1}{2}(3m_K-m_\pi)/(m_Km_\pi)^{1/2}\cong 2.6. \quad (2.26)$$

Now we have the complete values of the form factors

$$f_+(t)\cong 0.7(1+0.015t/m_\pi^2), \quad (2.27)$$

$$f_-(t)\cong 0.068(1+0.199t/m_\pi^2). \quad (2.28)$$

We predict therefore, subject to small uncertainties in the parameters α_1, α_2, α_3, that we have chosen

$$\lambda_+=0.0152,\quad \lambda_-=0.199,$$
$$\xi(0)\equiv f_-(t=0)/f_+(t=0)=0.097. \quad (2.29)$$

The form factor $f_+(t)$ is a very slowly varying function in the range $t=0$ to $t=(m_K-m_\pi)^2$. We could equally well write the symmetry condition (2.25) at $t=0$. The other form factor, f_-, which is the result of the symmetry breaking, is, although more than an order of magnitude smaller, a rapidly varying function. These results are in agreement with the present experimental values within the errors

$\lambda_+(K^+)=0.023\pm0.008$, (Ref. 7)
$\xi(t=4m_\pi^2)=-0.06\pm0.2$, (Ref. 8)
$\xi(t=0)=-0.13\pm0.25$ (Ref. 9)
(assuming $\lambda_-=0.1$).

Whether the theoretical requirement (2.25) is true is not clear. In fact, more accurate experiments would test this symmetry hypothesis. Another possibility of determining the parameter θ_K, instead of (2.26), would be to use the mass spectrum and form factor of the K tower as we did for the π tower, Eqs. (2.12) and (2.13).

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[7] W. J. Willis, in *Proceedings of the Heidelberg International Conference on Elementary Particles*, edited by H. Filthuth (Wiley-Interscience Publishers, Inc., New York, 1968).
[8] T. Eichen *et al.*, Phys. Letters **27B**, 586 (1968).
[9] D. R. Botterill *et al.*, Phys. Rev. Letters **21**, 766 (1968).

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APPENDIX: EVALUATION OF $g_\mu(\zeta)$

From (2.9),

$$g_\mu(\zeta)=(100|e^{-i(\theta_K-\theta_\pi)L_{45}}e^{-i\theta_\pi L_{45}}\alpha_s F_\mu{}^s e^{-i\zeta\cdot\mathbf{M}}e^{i\theta_\pi L_{45}}|100)$$

$$=\sum_n (100|e^{-i(\theta_K-\theta_\pi)L_{45}}|n00)\times(n00|\alpha_s F_\mu{}'^s f(\zeta,\theta_\pi)|100)\,, \quad \text{(A1)}$$

where

$$\alpha_s F_\mu{}'^s=e^{-i\theta_\pi L_{45}}\alpha_s F_\mu{}^s e^{i\theta_\pi L_{45}}$$
$$=\alpha_1 T_\mu+\alpha_2(p_K+p_\pi)_\mu+\alpha_3(p_K+p_\pi)_\mu(\cosh\theta_\pi\, S+\sinh\theta_\pi\, L_{56})\,,$$

and the vector T_μ has the components

$$\begin{aligned} T_0&=\cosh\theta_\pi\,\Gamma_0+\sinh\theta_\pi\, S\,,\\ T_1&=\Gamma_1=L_{16}\,,\\ T_2&=\Gamma_2=L_{26}\,,\\ T_3&=\Gamma_3=L_{36}\,. \end{aligned} \quad \text{(A2)}$$

Further,

$$f(\zeta,\theta_\pi)=e^{-i\theta_\pi L_{45}}e^{-i\zeta L_{35}}e^{i\theta_\pi L_{45}}=e^{-i\alpha L_{34}}e^{-i\beta L_{45}}e^{-i\gamma L_{34}}\,, \quad \text{(A3)}$$

with

$$\sinh\tfrac{1}{2}\beta=\cosh\theta_\pi\,\sinh\tfrac{1}{2}\zeta\,,$$
$$\sin\alpha=-\cosh\tfrac{1}{2}\zeta/\cosh\tfrac{1}{2}\beta\,,$$
$$\gamma=\alpha+\pi\,.$$

Thus, we need the matrix elements of the form

$$\sum_n (100|e^{-i(\theta_K-\theta_\pi)L_{45}}|n00)(n00|\Gamma_0 f(\zeta)|100)$$
$$=\sum_n n(100|e^{-i(\theta_K-\theta_\pi)L_{45}}|n00)(n00|f(\zeta)|100) \quad \text{(A4)}$$

and

$$\sum_n (100|e^{-i(\theta_K-\theta_\pi)L_{45}}|n00)(n00|Xf(\zeta)|100)\,,$$
$$X=L_{46} \text{ and } L_{56}\,.$$

These matrix elements can easily be evaluated in the parabolic basis $|n_1n_2m)$ and transformed back to $|nlm)$ by means of the relation

$$|nlm)=(-1)^m(2l+1)^{1/2}\times\begin{pmatrix}\frac{1}{2}(n-1) & \frac{1}{2}(n-1) & l\\ \frac{1}{2}(m-n_1+n_2) & \frac{1}{2}(m+n_1-n_2) & -m\end{pmatrix}\times|n_1n_2m)\,. \quad \text{(A5)}$$

We also have

$$\begin{aligned} L_{46}&=\tfrac{1}{2}(N_1{}^++N_1{}^-+N_2{}^++N_2{}^-)\equiv N_1{}^{(1)}+N_2{}^{(1)}\,,\\ L_{36}&=N_2{}^2-N_1{}^2=-\tfrac{1}{2}i(N_2{}^+-N_2{}^--N_1{}^++N_1{}^-)\,, \end{aligned} \quad \text{(A6)}$$

with

$$\begin{aligned} N_1{}^+|n_1n_2m)&=-[(n_1+1)(n_1+m_1+1)]^{1/2}|n_1+1,n_2m),\\ N_1{}^-|n_1n_2m)&=-[n_1(n_1+m)]^{1/2}|n_1-1,n_2m),\\ N_2{}^+|n_1n_2m)&=[(n_2+1)(n_2+m+1)]^{1/2}|n_1,n_2+1,m),\\ N_2{}^-|n_1n_2m)&=[n_2(n_2+m)]^{1/2}|n_1,n_2-1,m). \end{aligned} \quad \text{(A7)}$$

In the basis $|n_1n_2m)$ we have, because $L_{45}=N_1{}^{(2)}+N_2{}^{(2)}$,

$$(n_1'n_2'm'|e^{-i\vartheta L_{45}}|n_1n_2m)$$
$$=V_{n_1'+(m+1)/2,\,n_1+(m+1)/2}{}^{(m+1)/2}(\sinh\tfrac{1}{2}\vartheta)\times V_{n_2'+(m+1)/2,\,n_2+(m+1)/2}{}^{(m+1)/2}(-\sinh\tfrac{1}{2}\vartheta)\delta_{m'm}, \quad \text{(A8)}$$

where $V_{mn}{}^\phi(a)$ are the matrix elements of $O(2,1)$ used extensively before.[2,5,6]

Thus, collecting terms together, we find, finally, in the rest frame of K,

$$g_0(\zeta)=\sum_n\sum_{n_1n_2}\begin{pmatrix}\frac{1}{2}(n-1) & \frac{1}{2}(n-1) & 0\\ \frac{1}{2}(-n_1+n_2) & \frac{1}{2}(n_1-n_2) & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2}(n-1) & \frac{1}{2}(n-1) & 0\\ \frac{1}{2}(-n_1+n_2) & \frac{1}{2}(n_1-n_2) & 0\end{pmatrix}$$
$$\times V_{\frac12,n_1+\frac12}{}^{\frac12}(\sinh(\tfrac12\theta_K-\tfrac12\theta_\pi))V_{\frac12,n_2+\frac12}{}^{\frac12}(-\sinh(\tfrac12\theta_K-\tfrac12\theta_\pi))\{[n[\alpha_1\cosh\theta_\pi+\alpha_3(m_K+E_\pi)\sinh\theta_\pi]+\alpha_2(m_K+E_\pi)]$$
$$\times V_{n_1+\frac12,\frac12}{}^{\frac12}(\sinh\tfrac12\theta_\pi)V_{n_2+\frac12,\frac12}{}^{\frac12}(-\sinh\tfrac12\theta_\pi)e^{-i\alpha(n_1-n_2)}+[\alpha_1\sinh\theta_\pi+\alpha_3(m_K+E_\pi)\cosh\theta_\pi]$$
$$\times[-\tfrac12(n_1+1)e^{-i\alpha(n_1-n_2+1)}V_{n_1+\frac32,\frac12}{}^{\frac12}(\sinh\tfrac12\beta)V_{n_2+\frac12,\frac12}{}^{\frac12}(-\sinh\tfrac12\beta)e^{-i\alpha(n_1-n_2-1)}V_{n_1-\frac12,\frac12}{}^{\frac12}(\sinh\tfrac12\beta)V_{n_2+\frac12,\frac12}{}^{\frac12}(-\sinh\tfrac12\beta)$$
$$+\tfrac12(n_2+1)e^{-i\alpha(n_1-n_2-1)}V_{n_1+\frac12,\frac12}{}^{\frac12}(\sinh\tfrac12\beta)V_{n_2+\frac32,\frac12}{}^{\frac12}(-\sinh\tfrac12\beta)+\tfrac12 n_2e^{-i\alpha(n_1-n_2+1)}$$
$$\times V_{n_1+\frac12,\frac12}{}^{\frac12}(\sinh\tfrac12\beta)V_{n_2-\frac12,\frac12}{}^{\frac12}(-\sinh\tfrac12\beta)]\}\,, \quad \text{(A9)}$$

and similarly,

$$g_3(\zeta)=\sum_n\sum_{n_1n_2}\begin{pmatrix}\frac{1}{2}(n-1) & \frac{1}{2}(n-1) & 0\\ \frac{1}{2}(-n_1+n_2) & \frac{1}{2}(n_1-n_2) & 0\end{pmatrix}\begin{pmatrix}\frac{1}{2}(n-1) & \frac{1}{2}(n-1) & 0\\ \frac{1}{2}(-n_1+n_2) & \frac{1}{2}(n_1-n_2) & 0\end{pmatrix}$$
$$\times V_{\frac12,n_1+\frac12}{}^{\frac12}(\sinh(\tfrac12\theta_K-\tfrac12\theta_\pi))V_{\frac12,n_2+\frac12}{}^{\frac12}(-\sinh(\tfrac12\theta_K-\tfrac12\theta_\pi))$$
$$\times\{(\alpha_2p_\pi+\alpha_3p_\pi n\sinh\theta_\pi)V_{n_1+\frac12,\frac12}{}^{\frac12}(\sinh\tfrac12\theta_\pi)V_{n_2+\frac12,\frac12}{}^{\frac12}(-\sinh\tfrac12\theta_\pi)e^{-i\alpha(n_1-n_2)}$$
$$+\alpha_3p_\pi\cosh\theta_\pi[-\tfrac12(n_1+1)e^{-i\alpha(n_1-n_2+1)}V_{n_1+\frac32,\frac12}{}^{\frac12}(\sinh\tfrac12\beta)V_{n_2+\frac12,\frac12}{}^{\frac12}(-\sinh\tfrac12\beta)-\tfrac12n_1e^{-i\alpha(n_1-n_2-1)}V_{n_1-\frac12,\frac12}{}^{\frac12}(\sinh\tfrac12\beta)$$
$$\times V_{n_2+\frac12,\frac12}{}^{\frac12}(-\sinh\tfrac12\beta)+\tfrac12(n_2+1)e^{-i\alpha(n_1-n_2-1)}V_{n_1+\frac12,\frac12}{}^{\frac12}(\sinh\tfrac12\beta)V_{n_2+\frac32,\frac12}{}^{\frac12}(-\sinh\tfrac12\beta)$$
$$+\tfrac12n_2e^{-i\alpha(n_1-n_2+1)}V_{n_1+\frac12,\frac12}{}^{\frac12}(\sinh\tfrac12\beta)V_{n_2-\frac12,\frac12}{}^{\frac12}(-\sinh\tfrac12\beta)]$$
$$+\alpha_1(-\tfrac12i)[(n_2+1)e^{-i\alpha(n_1-n_2-1)}V_{n_1+\frac12,\frac12}{}^{\frac12}(\sinh\tfrac12\beta)V_{n_2+\frac32,\frac12}{}^{\frac12}(-\sinh\tfrac12\beta)-n_2e^{-i\alpha(n_1-n_2+1)}V_{n_1+\frac12,\frac12}{}^{\frac12}(\sinh\tfrac12\beta)$$
$$\times V_{n_2-\frac12,\frac12}{}^{\frac12}(-\sinh\tfrac12\beta)+(n_1+1)e^{-i\alpha(n_1-n_2+1)}V_{n_1+\frac32,\frac12}{}^{\frac12}(\sinh\tfrac12\beta)V_{n_2+\frac12,\frac12}{}^{\frac12}(-\sinh\tfrac12\beta)$$
$$-n_1e^{-i\alpha(n_1-n_2-1)}V_{n_1-\frac12,\frac12}{}^{\frac12}(\sinh\tfrac12\beta)V_{n_2+\frac12,\frac12}{}^{\frac12}(-\sinh\tfrac12\beta)]\,. \quad \text{(A10)}$$

The current conservation condition $g_3(\zeta)/g_0(\zeta)=\sinh\frac{1}{2}\zeta/\cosh\frac{1}{2}\zeta$ would be satisfied if the $K-\pi$ mass difference could be neglected. It follows from (A9) that the contribution of the intermediate state $|n00)$ goes like

$$\frac{[(m_K-m_\pi)^2-t]^{(n-1)/2}}{\{1-\cosh^2\theta_\pi[(m_K-m_\pi)^2-t]/4m_Km_\pi\}^{(n+1)/2}},$$

so that for small ζ, i.e., small $[(m_K-m_\pi)^2-t]$, higher n values can be neglected in the numerical evaluation. Thus, to a good approximation we find with $\Theta=\theta_K-\theta_\pi$

$$\begin{aligned}g_0(\xi)=&[\alpha_1\cosh\theta_\pi+(m_K+E_\pi)(\alpha_2+\alpha_3\sinh\theta_\pi)]\cosh^{-2}(\tfrac{1}{2}\Theta)\cosh^{-2}(\tfrac{1}{2}\beta)\\&+[2\alpha_1\cosh\theta_\pi+(m_K+E_\pi)(\alpha_2+2\alpha_3\sinh\theta_\pi)]\sqrt{2}[\sinh(\tfrac{1}{2}\Theta)/\cosh^3(\tfrac{1}{2}\Theta)](-\sqrt{2}\cos\alpha[\sinh(\tfrac{1}{2}\beta)/\cosh^3(\tfrac{1}{2}\beta)])\\&+[3\alpha_1\cosh\theta_\pi+(m_K+E_\pi)(\alpha_2+3\alpha_3\sinh\theta_\pi)]\sqrt{3}[\sinh^2(\tfrac{1}{2}\Theta)/\cosh^4(\tfrac{1}{2}\Theta)]\tfrac{1}{3}\sqrt{3}(1+2\cos2\alpha)[\sinh^2(\tfrac{1}{2}\beta)/\cosh^4(\tfrac{1}{2}\beta)]\\&+[4\alpha_1\cosh\theta_\pi+(m_K+E_\pi)(\alpha_2+4\alpha_3\sinh\theta_\pi)]\\&\qquad\times2[\sinh^3(\tfrac{1}{2}\Theta)/\cosh^5(\tfrac{1}{2}\Theta)][-(\cos\alpha+\cos3\alpha)[\sinh^3(\tfrac{1}{2}\beta)/\cosh^5(\tfrac{1}{2}\beta)]]+\cdots\\&+[(\alpha_1\sinh\theta_\pi)+\alpha_3(m_K+E_\pi)\cosh\theta_\pi]\{-\cosh^{-2}(\tfrac{1}{2}\Theta)\cos\alpha[\sinh(\tfrac{1}{2}\beta)/\cosh^3(\tfrac{1}{2}\beta)]\\&\quad+\sqrt{2}[\sinh(\tfrac{1}{2}\Theta)/\cosh^3(\tfrac{1}{2}\Theta)]\tfrac{1}{2}\sqrt{2}\cosh^{-2}(\tfrac{1}{2}\beta)[1+\tanh^2(\tfrac{1}{2}\beta)(1+2\cos2\alpha)]\\&\quad-\sqrt{3}[\sinh^2(\tfrac{1}{2}\Theta)/\cosh^4(\tfrac{1}{2}\Theta)]\sqrt{3}[\sinh(\tfrac{1}{2}\beta)/\cosh^3(\tfrac{1}{2}\beta)][\cos\alpha+\tanh^2(\tfrac{1}{2}\beta)(\cos\alpha+\cos3\alpha)]\\&\quad+2[\sinh^3(\tfrac{1}{2}\Theta)/\cosh^5(\tfrac{1}{2}\Theta)][\sinh^2(\tfrac{1}{2}\beta)/\cosh^4(\tfrac{1}{2}\beta)]\\&\qquad\times[1+2\cos2\alpha+\tanh^2(\tfrac{1}{2}\beta)(1+2\cos2\alpha+2\cos4\alpha)]+\cdots\}.\end{aligned}\tag{A11}$$

$$\begin{aligned}g_3(\xi)=&-\alpha_1\{\cosh^{-2}(\tfrac{1}{2}\Theta)\sin\alpha[\sinh(\tfrac{1}{2}\beta)/\cosh^3(\tfrac{1}{2}\beta)]+\sqrt{2}[\sinh(\tfrac{1}{2}\Theta)/\cosh^3(\tfrac{1}{2}\Theta)](-\sqrt{2})\sin2\alpha[\sinh^2(\tfrac{1}{2}\beta)/\cosh^4(\tfrac{1}{2}\beta)]\\&\qquad+\sqrt{3}[\sinh^2(\tfrac{1}{2}\Theta)/\cosh^4(\tfrac{1}{2}\Theta)]\sqrt{3}\sin3\alpha[\sinh^3(\tfrac{1}{2}\beta)/\cosh^5(\tfrac{1}{2}\beta)]+\cdots\}\\&+(\alpha_2+\alpha_3\sinh\theta_\pi)p_\pi\cosh^{-2}(\tfrac{1}{2}\Theta)\cosh^{-2}(\tfrac{1}{2}\beta)\\&+(\alpha_2+2\alpha_3\sinh\theta_\pi)p_\pi\sqrt{2}[\sinh(\tfrac{1}{2}\Theta)/\cosh^3(\tfrac{1}{2}\Theta)](-\sqrt{2})\cos\alpha[\sinh(\tfrac{1}{2}\beta)/\cosh^3(\tfrac{1}{2}\beta)]\\&+(\alpha_2+3\alpha_3\sinh\theta_\pi)p_\pi\sqrt{3}[\sinh^2(\tfrac{1}{2}\Theta)/\cosh^4(\tfrac{1}{2}\Theta)]\tfrac{1}{3}\sqrt{3}(1+2\cos2\alpha)\sinh^2(\tfrac{1}{2}\beta)\cosh^{-4}(\tfrac{1}{2}\beta)\\&+\alpha_3\theta_\pi p_\pi\{-\cosh^{-2}(\tfrac{1}{2}\Theta)\cos\alpha[\sinh(\tfrac{1}{2}\beta)/\cosh^3(\tfrac{1}{2}\beta)]\\&\quad+\sqrt{2}[\sinh(\tfrac{1}{2}\Theta)/\cosh^3(\tfrac{1}{2}\Theta)]\tfrac{1}{2}\sqrt{2}\cosh^{-2}(\tfrac{1}{2}\beta)[1+\tanh^2(\tfrac{1}{2}\beta)(1+2\cos2\alpha)]\\&\quad+\sqrt{3}[\sinh^2(\tfrac{1}{2}\Theta)/\cosh^4(\tfrac{1}{2}\Theta)][-\sqrt{3}\sinh(\tfrac{1}{2}\beta)/\cosh^3(\tfrac{1}{2}\beta)][\cos\alpha+\tanh^2(\tfrac{1}{2}\beta)(\cos\alpha+\cos3\alpha)]\\&\quad+2[\sinh^3(\tfrac{1}{2}\Theta)/\cosh^5(\tfrac{1}{2}\Theta)][\sinh^2(\tfrac{1}{2}\beta)/\cosh^4(\tfrac{1}{2}\beta)]\\&\qquad\times[1+2\cos2\alpha+\tanh^2(\tfrac{1}{2}\beta)(1+2\cos2\alpha+2\cos4\alpha)]+\cdots\}.\end{aligned}\tag{A12}$$