Asymptotic Form of the Wave Function for Three-Particle Scattering*

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The asymptotic form of the wave function for scattering from an initial state of one-particle incident on a bound pair at energies above the breakup threshold is given. Unknown corrections are rigorously square integrable. The discussion is valid only for short-range forces.

INTRODUCTION

In spite of considerable progress in recent years, the solution of the three-body problem at energies high enough to allow a final state of three free particles is still not easy. The propagator singularities in the Faddeev equations must be dealt with carefully. An alternative procedure is to work in coordinate space and use Schrödinger's partial differential equation along with suitable boundary conditions.

This paper gives the asymptotic form of the wave function for scattering from an initial state of one-particle incident on a bound pair with enough energy to break up the bound pair. All corrections to the form we give are rigorously square integrable. The discussion is limited to suitably well-behaved short-range potentials.

DERIVATION

The basic outline of the method we use to find the asymptotic form was set out by Nuttall.[1] We study the same problem, three different particles of equal mass ($m=\frac{1}{2}$) and use the same notation as I.[2] It is only when two of the particles may be relatively close together with the third one i far distant that any difficulty arises. To study this situation it is most convenient to use the exact representation of the wave function $\psi^+(\hat{\rho})$ given by Eq. I(4)

$$\psi^+(\hat{\rho}_i)=\chi(\hat{\rho}_i)+\psi_a(\hat{\rho}_i)+(2\pi)^{-3}(\tfrac{2}{3})^{3/2}\sum_{\alpha,i}\int d\mathbf{P}_i\chi_\alpha{}^i(\mathbf{Y}_i)\times\exp(i\mathbf{X}_i\cdot\mathbf{P}_i)(E+E_\alpha{}^i-\mathbf{P}_i{}^2+i\epsilon)^{-1}\times\langle\chi_\alpha{}^i,\mathbf{P}_i|T(E)|\chi\rangle,\quad(1)$$

where

$$\psi_a(\hat{\rho}_i)=(2\pi)^{-3}3^{-3/2}\int d\hat{K}_i\psi_{\mathbf{Q}_i}{}^-(\mathbf{Y}_i)\times\exp(i\mathbf{X}_i\cdot\mathbf{P}_i)(E-\hat{K}^2+i\epsilon)^{-1}\langle\hat{K}_i|\bar{T}_i{}^+|\chi\rangle.\quad(2)$$

This involves the two-body scattering wave functions $\psi_{\mathbf{Q}_i}{}^-(\mathbf{Y}_i)$ and the off-energy-shell T matrix $\langle\hat{K}_i|\bar{T}_i{}^+|\chi\rangle$ defined by I(5), i.e.,

$$\begin{aligned}\langle\hat{K}_i|\bar{T}_i{}^+|\chi\rangle&=\langle\psi_{\mathbf{Q}_i}{}^-,\mathbf{P}_i|[1+V^i(E-H+i\epsilon)^{-1}]V^1|\chi\rangle\\&=\langle\hat{K}_i|[1+V_i(\hat{K}^2-H_i+i\epsilon)^{-1}]\\&\quad\times[1+V^i(E-H+i\epsilon)^{-1}]V^1|\chi\rangle.\end{aligned}\quad(3)$$

The asymptotic form of $\psi^+(\hat{\rho})$ for large ρ is determined by the location and nature of the singularities of $\langle\hat{K}_i|\bar{T}_i{}^+|\chi\rangle$ near real $\hat{K}_i$. We note that this T matrix is not the same off the energy shell as the solution of the Faddeev equations, although it can be related to the Faddeev T. On the energy shell they are identical.

A diagrammatic analysis of $\langle\hat{K}_i|\bar{T}_i{}^+|\chi\rangle$ shows that two types of diagram contribute:

(i) those in which the last interaction before the final state of (3) involves particle i;

(ii) those in which particle i is not involved. It may be shown by standard Landau-diagram techniques[3] that the only propagators that can be involved in pinches giving rise to singularities of $\langle\hat{K}_i|\bar{T}_i{}^+|\chi\rangle$ are those to the left of the point marked by a wiggly line in Figs. 1(a) and (b), which show examples of the two types of diagram. Such propagators have the form

$$\text{(i)}\quad[E-\mathbf{P}_j{}^2-\mathbf{Q}_j{}^{(k)2}+i\epsilon]^{-1},\quad j\neq i\quad(4a)$$

$$\text{(ii)}\quad[\mathbf{Q}_i{}^2-\mathbf{Q}_i{}^{(k)2}+i\epsilon]^{-1},\quad(4b)$$

where $\mathbf{Q}_j{}^{(k)}$, $\mathbf{Q}_i{}^{(k)}$ are internal momenta to be integrated

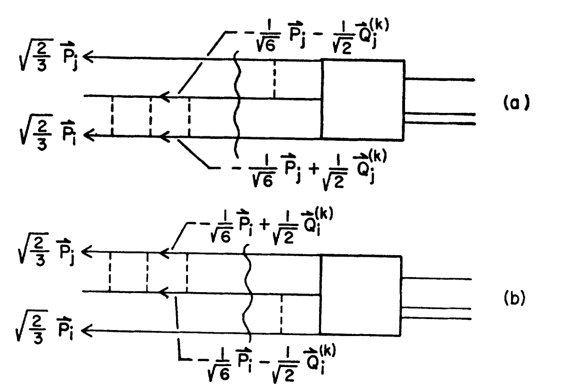

FIG. 1(a). A diagram contributing to $\langle\hat{K}_i\bar{T}_i{}^+|x\rangle$ in which particle i is involved in the last interaction. (b). A diagram contributing to $\langle\hat{K}_i|\bar{T}_i{}^+|x\rangle$ in which particle i is not involved in the last interaction.

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[1] J. Nuttall, Phys. Rev. Letters 19, 473 (1967), hereafter referred to as I.

[2] Note that a misprint appears in Eq. (3) of I. The factor $(\frac{2}{3})^{1/2}$ should read $(\frac{2}{3})^{-1/2}$.

[3] M. Rubin, R. Sugar, and G. Tiktopoulos, Phys. Rev. 146, 1130 (1966).

over. These give rise to singularities at

$$\text{(i)}\quad E-\mathbf{P}_j^2=0\,,\quad j\neq i \tag{5a}$$

$$\text{(ii)}\quad \mathbf{Q}_i^2=0\,. \tag{5b}$$

In addition to the above, there may also be two-body bound-state poles at

$$\mathbf{P}_j^2=E+E_\alpha{}^j\,,\quad j\neq i\,. \tag{6}$$

Bound-state poles in the i channel will not occur near real $\hat{K}$.

It is only the singularity at $\mathbf{Q}_i^2=0$ that concerns us and the type of singularity there is the same as that encountered at zero-energy two-particle scattering. If we expand $\psi_{\mathbf{Q}}^-(\mathbf{Y})$ as

$$\psi_{\mathbf{Q}}^-(\mathbf{Y})=\frac{(4\pi)^{1/2}}{(2\pi)^{3/2}}\sum_{l=0}^{\infty} g_l(Q,Y)Y_l^0(\hat{\mathbf{Q}},\hat{\mathbf{Y}})\,, \tag{7}$$

we know that $g_l(Q,Y)$ is analytic in Q and behaves like Q^l near $Q=0$.[4] Similarly, if

$$\langle\hat{K}_i|\bar{T}_i{}^+|\chi\rangle=(4\pi)^{1/2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} t_{lm}(Q,\mathbf{P}_i)Y_l{}^m(\hat{Q})\,, \tag{8}$$

we expect that $t_{lm}(Q,P_i)$ will be analytic in Q and near $Q=0$ behaves like Q^l.

Our main concern in this paper is with the second term of (1), $\psi_a(\hat{\rho}_i)$, whose asymptotic part describes an outgoing wave in which no pair of particles is bound, although they may still interact. We study the asymptotic part of $\psi_a(\hat{\rho}_i)$ for large X_i by determining first the leading part of the integral over $\mathbf{P}_i$ for each value of $\mathbf{Q}_i$. The result is familiar from the two-body case and is, as in I(9)

$$\begin{aligned}\psi_a(\hat{\rho})\underset{X_i\to\infty}{\sim}&-3^{-3/2}(4\pi X_i)^{-1}\int d\mathbf{Q}_i\psi_{\mathbf{Q}_i}{}^-(\mathbf{Y}_i)\\&\times\exp[iX_i(E-\mathbf{Q}_i^2)^{1/2}]\\&\times\langle(E-\mathbf{Q}_i^2)^{1/2}\mathbf{X}_{iu},\mathbf{Q}_i|\bar{T}^+|\chi\rangle\,.\end{aligned} \tag{9}$$

Errors introduced by the approximation (9) are of order $X_i^{-1}\psi_a$, which we shall see means $O(\rho^{-7/2})$. Thus that part of $\psi_a(\hat{\rho})$ which differs from (9) will be square integrable over the six-dimensional $\hat{\rho}$ space, even if the integral over ρ is carried out first. This is what we mean by rigorously square integrable.

Provided that we do not have a situation corresponding to Y_j not large, $j\neq i$, the singularities of $\langle\hat{K}_i|\bar{T}_i|\chi\rangle$ will not impede the contour distortions necessary to derive (9).

The third term of (1) can be treated in a similar manner to obtain the well-known asymptotic part of $\psi^+(\hat{\rho})$ corresponding to bound states in the i channel, correct to any required order in X_i^{-1}.

[4] L. S. Rodberg and R. M. Thaler, *The Quantum Theory of Scattering* (Academic Press Inc., New York, 1967).

The form (9) is a natural one to use if we are looking for a solution of Schrödinger's equation in a region where two of the particles may be close enough for their potential to be important, but in which the third one i is far away and moving freely. It has been given previously by Peterkop.[5] However, it is still rather complicated for use in numerical calculations, and the main purpose of this paper is to simplify it by keeping only the most important terms.

To do this we need to pick out from the on-shell scattering amplitude the S-wave part (with respect to $\mathbf{Q}$), $t_0(Q,\mathbf{X}_u)$, writing (omitting the subscript i),

$$\langle P(\mathbf{Q}^2)\mathbf{X}_u,\mathbf{Q}|\bar{T}^+|\chi\rangle=t_0(Q,\mathbf{X}_u)+t(\mathbf{Q},\mathbf{X}_u)\,. \tag{10}$$

We must further analyze $t_0(Q,\mathbf{X}_u)$ as

$$t_0(Q,\mathbf{X}_u)=\bar{t}_0(Q,\mathbf{X}_u)+Q\tau(\mathbf{X}_u)\,, \tag{11}$$

where

$$\left.\frac{d\bar{t}_0}{dQ}\right|_{Q=0}=0\,. \tag{12}$$

In addition, it is convenient to write

$$g_0(Q,Y)=\sin QY/QY+\phi(Q,Y)\,, \tag{13}$$

where

$$\phi(Q,Y)\underset{Y\to\infty}{\to}\frac{e^{-iQY}}{Y}f(Q) \tag{14}$$

and

$$f(Q)=-a-ia^2Q+\cdots\,, \tag{15}$$

where $\sqrt{2}a$ is the scattering length. Thus

$$\phi(0,Y)\underset{Y\to\infty}{\sim}-a/Y\,, \tag{16}$$

$$\phi'(0,Y)=\left.\frac{d\phi}{dQ}\right|_{Q=0}\underset{Y\to\infty}{\sim}ia(1-a/Y)\,. \tag{17}$$

Using these formulas, we may write $\psi_a(\hat{\rho})$ as

$$\psi_a(\hat{\rho})=-3^{-3/2}(4\pi)^{-1}X^{-1}\sum_{k=1}^{5}\psi_k(\hat{\rho})\,, \tag{18}$$

where

$$\begin{aligned}\psi_1(\hat{\rho})&=(2\pi)^{-3/2}\int d\mathbf{Q}\,(\sin QY/QY)\bar{t}_0(Q,\mathbf{X}_u)\\&\qquad\times\exp[iX(E-Q^2)^{1/2}]\,,\\ \psi_2(\hat{\rho})&=(2\pi)^{-3/2}\tau(\mathbf{X}_u)\int d\mathbf{Q}\,(\sin QY/QY)Q\\&\qquad\times\exp[iX(E-Q^2)^{1/2}]\,,\\ \psi_3(\hat{\rho})&=(2\pi)^{-3/2}\int d\mathbf{Q}\,\phi(Q,Y)\bar{t}_0(Q,\mathbf{X}_u)\\&\qquad\times\exp[iX(E-Q^2)^{1/2}]\,,\\ \psi_4(\hat{\rho})&=(2\pi)^{-3/2}\tau(\mathbf{X}_u)\int d\mathbf{Q}\,\phi(Q,Y)Q\exp[iX(E-Q^2)^{1/2}]\,,\\ \psi_5(\hat{\rho})&=\int d\mathbf{Q}\,\psi_{\mathbf{Q}}{}^-(\mathbf{Y})t(\mathbf{Q},\mathbf{X}_u)\exp[iX(E-Q^2)^{1/2}]\,.\end{aligned} \tag{19}$$

[5] R. K. Peterkop, Bull. Acad. Sci. USSR, Phys. Ser. **27**, 987 (1963).

Each integral will be approximated by the method of stationary phase. Thus for $\psi_1(\hat{\rho})$ we have

$$\psi_1(\hat{\rho})=\left(\frac{2}{\pi}\right)^{1/2}(2iY)^{-1}\int_0^K QdQ(e^{iQY}-e^{-iQY})\bar{t}_0(Q,X_u)\times\exp[iX(E-Q^2)^{1/2}]. \quad (20)$$

The phase of $\exp[\pm iQY+iX(E-Q^2)^{1/2}]$ is stationary at $Q=\pm Q_0$, where

$$Q_0=K(Y/\rho),\quad K=E^{1/2}. \quad (21)$$

This point lies outside the range of integration for the second term of (20), and so for large Y we take only the first term, in which we expand the phase about Q_0 to second order, and replace $\bar{t}_0(Q,\mathbf{X}_u)$ by $\bar{t}_0(Q_0,\mathbf{X}_u)$, obtaining

$$\psi_1(\hat{\rho})\sim\left(\frac{2}{\pi}\right)^{1/2}(2iY)^{-1}e^{iK\rho}\bar{t}_0(Q_0,\mathbf{X}_u)\int_{-\infty}^{\infty}QdQ\, e^{-i\lambda(Q-Q_0)^2}$$
$$=-XK^{3/2}e^{i\pi/4}\rho^{-5/2}e^{iK\rho}\bar{t}_0(Q_0,\mathbf{X}_u), \quad (22)$$

where $\lambda=\frac{1}{2}\rho^3X^{-2}K^{-1}$. A closer investigation shows that this result is valid to $O(\rho^{-7/2})$ even for small Y because a power series expansion of $\bar{t}_0(Q,X_u)$ contains no Q term.

A similar technique is applied to $\psi_2(\hat{\rho})$, but this time we must keep the factor Q inside the integral. While the phase of $\exp[-iQY+iX(E-Q^2)^{1/2}]$ is stationary outside the region of integration at $Q=-Q_0$, this point nears $Q=0$ as $Y/\rho\to 0$, so that, to obtain a form valid for all Y we shall include a contribution from both parts of the sine function. We find

$$\psi_2(\hat{\rho})\sim\left(\frac{2}{\pi}\right)^{1/2}(2iY)^{-1}\tau(\mathbf{X}_u)e^{iK\rho}\int_0^{\infty}Q^2dQ\times[e^{-i\lambda(Q-Q_0)^2}-e^{-i\lambda(Q+Q_0)^2}]$$
$$=-4\pi X(2\pi)^{-3/2}\tau(\mathbf{X}_u)e^{iK\rho}\rho^{-5/2}\{\pi^{1/2}K^{3/2}\rho^{-1}\times(KY-iX^2\rho^{-1}Y^{-1})[G+(\tfrac{1}{2}i)^{1/2}]+K^2\rho^{-3/2}Xe^{-i\mu}\}, \quad (23)$$

where

$$G=i\pi^{-1/2}\int_0^{(K\rho)^{1/2}YX^{-1}}e^{-\frac{1}{2}it^2}dt-(\tfrac{1}{2}i)^{1/2} \quad (24)$$

and

$$\mu=\lambda Q_0^2. \quad (25)$$

Corrections to this expression will be $O(\rho^{-7/2})$.

To expand $\psi_3(\hat{\rho})$, we first assume that Y has some value smaller than the range of the potential, so that the phase of $\phi(Q,Y)$ does not oscillate rapidly with varying Q. The phase of the exponent is stationary at $Q=0$, so that we expand $\phi(Q,Y)$ and $\bar{t}_0(Q,X_u)$ about this point to obtain

$$\psi_3(\hat{\rho})\sim\left(\frac{2}{\pi}\right)^{1/2}\bar{t}_0(0,X_u)e^{iKX}\int_0^{\infty}Q^2dQ\, e^{-i\frac{1}{2}XK^{-1}Q^2}\times[\phi(0,Y)+Q\phi'(0,Y)]$$
$$=-4\pi X(2\pi)^{-3/2}\bar{t}_0(0,X_u)X^{-5/2}e^{iKX}\times\{\pi^{1/2}(\tfrac{1}{2}i)^{1/2}K^{3/2}\phi(0,Y)+2K^2X^{-1/2}\phi'(0,Y)\}, \quad (26)$$
$$X\to\infty\quad Y\text{ fixed.}$$

Corrections from higher terms in the expansion of $\bar{t}_0(Q,X_u)$, $\phi(Q,Y)$, and $(E-Q^2)^{1/2}$ will be $O(X^{-7/2})$.

On the other hand, if Y is outside the range of the potential, we may use the asymptotic form of $\phi(Q,Y)$ in (14) to obtain again to order $\rho^{-7/2}$

$$\psi_3(\hat{\rho})\sim-4\pi X(2\pi)^{-3/2}\bar{t}_0(Q_0,\mathbf{X}_u)e^{iK\rho}\rho^{-5/2}\times\{a[\pi^{1/2}K^{3/2}\rho^{-2}(X^2Y^{-1}+iK\rho Y)G+iK^2\rho^{-3/2}Xe^{-i\mu}]+a^2[\pi^{1/2}K^{5/2}\rho^{-2}(KY^2-3i\rho^{-1}X^2)G+K^2\rho^{-5/2}(KXY-2i\rho^{-1}X^3Y^{-1})e^{-i\mu}]\}. \quad (27)$$

It is possible to construct from (26) and (27) a function which represents $\psi_3(\hat{\rho})$ to $O(\rho^{-7/2})$ for all values of Y. This will be given in the final formula for $\psi_a(\hat{\rho})$.

The same method must be applied to $\psi_4(\hat{\rho})$, except that again the factor Q is to be kept inside the integral.

We obtain an adequate approximation to $\psi_5(\hat{\rho})$ by replacing $\psi_Q^-(Y)$ by $(2\pi)^{-3/2}e^{i\mathbf{Q}\cdot\mathbf{Y}}$ and using for $t(\mathbf{Q},\mathbf{X}_u)$ its value at the point $\mathbf{Q}=Q_0\mathbf{Y}_u$ where the phase of $\exp[i\mathbf{Q}\cdot\mathbf{Y}+iX(E-Q^2)^{1/2}]$ is stationary. Thus

$$\psi_5(\hat{\rho})\sim-4\pi X(4\pi)^{-1}K^{3/2}e^{i\pi/4}t(Q_0\mathbf{Y}_u,\mathbf{X}_u)\rho^{-5/2}e^{iK\rho}. \quad (28)$$

Again it may be shown that corrections to this formula are of order $\rho^{-7/2}$, even near $Y=0$. It remains to collect together the several terms into the final result, which is

$$\psi_a(\hat{\rho})\underset{\rho\to\infty}{\sim}(2\pi)^{-3/2}3^{-3/2}\rho^{-5/2}e^{iK\rho}\times\{\bar{t}_0(Q_0,\mathbf{X}_u)[(\tfrac{1}{2}\pi)^{1/2}K^{3/2}e^{i\pi/4}\times(1+\rho^{-2}X^2\phi(0,Y)+a\rho^{-2}X^2Y^{-1})+2K^2\rho^{-7/2}X^3e^{-i\mu}(\phi'(0,Y)-ia)+a(\pi^{1/2}K^{3/2}\rho^{-2}X^2Y^{-1}G+i\pi^{1/2}K^{5/2}\rho^{-1}Yh(Y)G+iK^2\rho^{-3/2}Xe^{-i\mu})]+\tau(\mathbf{X}_u)[2K^2\rho^{-1/2}\phi(0,Y)+\pi^{1/2}K^{3/2}\rho^{-1}(KY-i\rho^{-1}X^2Y^{-1})(G+(\tfrac{1}{2}i)^{1/2})+K^2\rho^{-3/2}Xe^{-i\mu}]+(\tfrac{1}{2}\pi)^{1/2}e^{i(\pi/4)}K^{3/2}\times t(Q_0\mathbf{Y}_u,\mathbf{X}_u)\}, \quad (29)$$

where $h(Y)$ is a smooth function which approaches zero for small Y and unity for large Y. Any corrections to this result fall off faster than ρ^{-3} for large ρ. We stress that this result is valid even if the pair of particles distance $\sqrt{2}Y$ apart should be close together, but does not hold if the other two pairs interact or if X is not large. Forms to cover these cases can be obtained by using the above result for another pair of particles.

DISCUSSION

Our result should be useful in coordinate-space calculations of three-particle scattering above the two-body breakup threshold, using methods such as the Kohn variational method or its modifications. It is interesting to note that such a calculation can proceed without the prior need to solve the complete two-body problem. Only bound-state and zero-energy scattering properties are required.

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Consistent Dynamical Formulation of Current-Current Theories and the Validity of Current Algebra

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It is shown that the usual formulations of current-current theories are not dynamically consistent, because of the occurrence of Schwinger terms. On the basis of the Poincaré-group algebra, an alternative and consistent formulation is constructed and generalized to allow for parity and isospin violation. It is found that, with the appropriate normalization of currents and the corresponding definitions of coupling constants, the chiral algebra of currents for time components is always satisfied in four-dimensional theories of the current-current type. In the two-dimensional case, this statement needs modification, but it is essentially valid there as well. The algebra involving spatial components is highly sensitive to the dynamics, and consequently is not identical to the free-quark-field case. In certain cases of parity violation, the usual currents which couple in the field equations do not transform as four-vectors even though the theory itself is covariant. This is somewhat analogous to radiation-gauge electrodynamics. It is found, however, that it is possible to construct new covariant currents as linear combinations of the components of the old currents which have the property that they couple covariantly into the field equation. Because of these complications, the spatial components of the covariant currents do not have a definite parity and do not transform as vectors in isospin.

I. INTRODUCTION

It is well known that in theories involving fermion fields ψ a strictly local current definition of the form $j^\mu(x)=\frac{1}{2}\psi(x)\beta\gamma^\mu q\psi(x)$ cannot be valid because it leads to the result that $[j^0(x),j^k(y)]_{x^0=y^0}=0$, which is not consistent with Lorentz invariance and positive-definiteness. (Note that in this paper we use Hermitian fields ψ and the matrices $\beta\gamma^\mu=\alpha^\mu$ are real symmetric while β is imaginary antisymmetric.) In the free-field case it is easy to establish that the definition

$$j^\mu(x)=\lim_{\boldsymbol{\epsilon}\to 0}\tfrac{1}{2}\psi(x-\boldsymbol{\epsilon})\beta\gamma^\mu q\psi(x+\boldsymbol{\epsilon})\,, \tag{1.1}$$

with $\boldsymbol{\epsilon}$ pure spatial and the limit $\boldsymbol{\epsilon}\to 0$ measuring the limit of the spatially symmetric average of the direction of $\boldsymbol{\epsilon}$, is consistent with a nonvanishing Schwinger term[1] while also having the proper translation and Lorentz rotation properties. In the case of interactions with the electromagnetic field and many other elementary boson fields it is known that merely separating the points as above is not adequate and that an extrapolating exponential involving the boson field must be included as well.[2] In particular, for an electromagnetic interaction of the form $ej^\mu A_\mu$ a correct gauge-invariant current definition is

$$j^\mu(x)=\lim_{\boldsymbol{\epsilon}\to 0}\tfrac{1}{2}\psi(x+\boldsymbol{\epsilon})\alpha^\mu q \times\exp\left[-ieq\int_{x-\boldsymbol{\epsilon}}^{x+\boldsymbol{\epsilon}} d\xi^\mu A_\mu(\xi)\right]\psi(x-\boldsymbol{\epsilon})\,. \tag{1.2}$$

It thus should not be surprising that in Sec. II we demonstrate in the case of the interaction $\frac{1}{2}\lambda j^\mu(x)j_\mu(x)$ that the mere separation of Fermi field points does not yield a consistent theory. Indeed, in the soluble case of the two-dimensional Thirring model[3,4] it has been shown

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[1] J. Schwinger, Phys. Rev. Letters 3, 296 (1959).

[2] See, for example, D. G. Boulware and S. Deser, Phys. Rev. 151, 1278 (1966).

[3] W. Thirring. Ann. Phys. (N. Y.) 3, 91 (1958).

[4] K. Johnson, Nuovo Cimento 20, 773 (1961); C. Sommerfield, Ann. Phys. (N. Y.) 26, 1 (1964).