# High-Energy Behavior of Scattering due to Massless (Neutrino) Exchange*

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We devise a method of calculating the high-energy behavior of some processes where massless particles are being exchanged and the normal theorems such as the Froissart bound cannot be directly applied. The processes are $\nu$-$\nu$ and $\nu$-$\bar{\nu}$ scattering, with a long-range potential due to the exchange of neutrinos. It is shown that as the energy $E$ tends to infinity, (i) the $\nu$-$\nu$ and $\nu$-$\bar{\nu}$ total cross sections behave as constants, (ii) the $\nu$-$\nu$ and $\nu$-$\bar{\nu}$ cross sections approach each other (a Pomeranchuk theorem), and (iii) the long-range two-neutrino exchange force is of the form $E/r^5$. The calculation is made with the assumption that the amplitudes obey the Mandelstam representation, but we do not restrict ourselves to any specific order in the weak coupling constant $G$, or to any specific form of the Hamiltonian. The high-energy behavior is deduced by constructing a consistency condition to it. Using analyticity and unitarity, the potential due to the massless neutrino exchange is shown to depend on the high-energy behavior of the $\nu$-$\nu$ and $\nu$-$\bar{\nu}$ processes. Since, in turn, the high-energy cross sections of these processes are themselves dependent on the potential, a consistency requirement is available on the former, leading to the results mentioned above.

## 1. INTRODUCTION

It is well known that in strong-interaction physics one can derive, from analyticity and unitarity, bounds on high-energy cross sections.[1–3] The Froissart bound, for instance, forbids the total cross sections from increasing faster than $\log^2 s$. However, these bounds require for their proof a finite ellipse with foci at $\pm 1$ in the $\cos\theta$ plane, in which the scattering amplitude is analytic, which in turn requires a finite region of analyticity around the origin in the $t$ plane. This condition is more than satisfied by the Mandelstam representation for strong interactions, since all hadrons are massive and the singularities in the $t$ plane begin at the square of the pion mass or beyond.

However, in weak interactions, with the possibility of massless neutrino exchanges the above proof cannot be directly applied, and the question of high-energy weak cross sections remains open. Of course, the same problem of long-range forces exists in electrodynamics. However, there *is* a renormalizable theory of quantum electrodynamics, whereas there is no equally acceptable theory for weak interactions. Thus, while the strong interactions have no satisfactory theory, but at least massive particles, and electrodynamics has massless photons but a satisfactory theory, weak interactions have both difficulties.

There is recently a renewed interest in the problems that beset the theory of weak interactions.[4–6] The well-known current-current Hamiltonian, while very useful in dealing with low-energy weak processes if used in first order, leads, nevertheless, to a nonrenormalizable theory. It further violates unitarity beyond a certain energy. It is therefore felt, with reason, that beyond a certain cutoff, the current-current form may have to be modified, and there has even been speculation that beyond a certain energy, weak processes may become "strong."

Therefore, the high-energy behavior of weak processes is of considerable interest theoretically, even though the experimental situation regarding weak scattering processes is still bleak.

We have attempted to devise a method in this work, whereby high-energy bounds can be obtained for some processes mediated by neutrino-exchange forces, where the Froissart bound cannot be applied. We will consider the scattering of neutrinos by neutrinos and by antineutrinos, where the "potential" [or the kernel for the Bethe-Salpeter (BS) equation] corresponds to the exchange of a neutrino-antineutrino pair (Fig. 1). The shaded "blobs" in Fig. 1 correspond to full $\nu$-$\nu$ (and $\nu$-$\bar{\nu}$) scattering amplitudes, not restricted to any particular order in any specific Hamiltonian. Therefore our calculation will be free from the limitations of the current-current Hamiltonian and perturbation expansions. The potential due to the diagram in Fig. 1 is essentially the discontinuity across the positive $t$ real axis of the amplitude for the diagram, i.e., $A_t^{\text{Born}}(s,t)$. Here $A^{\text{Born}}(s,t)$ is the Born amplitude represented by Fig. 1, which will have to be iterated to give the full scattering amplitude. We evaluate $A_t^{\text{Born}}(s,t)$ by using elastic unitarity in the $t$ channel and Mandelstam analyticity for the amplitudes. We show that $\lim_{t\to 0} A_t^{\text{Born}}(s,t)$, which is the "longest-range" potential, depends on the high-energy parameters of the $\nu$-$\nu$ and $\nu$-$\bar{\nu}$ scattering cross sections, which are fed in as inputs. This is done in Sec. 3. The potential obtained is then iterated in Sec. 4 by using a

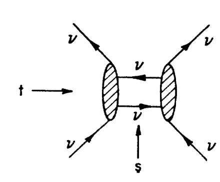


FIG. 1. Diagram representing the general two-neutrino exchange potential. The shaded loops are meant to indicate the full scattering amplitude to all orders in the Hamiltonian.

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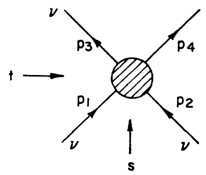



FIG. 2. Kinematics of the $\nu$-$\nu$ scattering process.

BS equation and various wave equations like the Klein-Gordon (KG) equation. A high-energy (eikonal) approximation is used to solve the BS and KG equations. It is shown that these various methods of getting the full amplitude from the potential or Born term give the same result in the eikonal approximation. This result, which once again gives high-energy cross sections for $\nu$-$\nu$ and $\nu$-$\bar{\nu}$ scattering in terms of the potential, therefore reproduces the high-energy parameters as output. By requiring the output and input values of the high-energy parameters to be the same, we evaluate them. Some very nice results come out, such as:

(i) The $\nu$-$\nu$ and $\nu$-$\bar{\nu}$ total cross sections approach constant values at high energies, despite the zero-mass-exchange or long-range forces.

(ii) The $\nu$-$\nu$ and $\nu$-$\bar{\nu}$ cross sections approach each other—a Pomeranchuk theorem for our case. Both these results are very reminiscent of the exchange of a Pomeranchukon.

(iii) The only consistent long-range potential possible behaves as $E/r^5$, where $E$ is the c.m. energy.

It should be mentioned here that the last result, viz., the $1/r^5$ dependence, was obtained by Feinberg and Sucher[7] for the neutrino-exchange potential. At the end of Sec. 5 we give a brief discussion of the relative independence of the shape $1/r^5$ on models. In any case, the main purpose of our work is to get the asymptotic behavior of the cross sections.

It should also be noted that in our potential in Fig. 1, although only a neutrino-antineutrino pair is present in the $t$ channel, as far as the $s$ channel is concerned, the two blobs could contain anything. Thus, all irreducible diagrams in the $s$ channel are included as long as only two neutrinos are exchanged. The question of what happens when potentials due to four or more neutrino exchanges are considered is also discussed briefly at the end of the paper.

## 2. KINEMATICAL PRELIMINARIES

Consider the scattering amplitude as shown in Fig. 2, which corresponds to $\nu$-$\nu$ scattering in the $s$ channel. As usual, $s=(p_1+p_2)^2$, $t=(p_1-p_3)^2$, and $(u=(p_1-p_4)^2$. Clearly, for the case of neutrinos, helicity and spinor amplitudes are the same. In order to avoid spurious singularities due to the zero mass of the external neutrino lines, we will normalize the spinor by

$$\bar{u}(\mathbf{p})u(\mathbf{p})=-\bar{v}(\mathbf{p})v(\mathbf{p})=m=0 \qquad (1)$$

instead of the usual $\bar{u}u=1$. In our normalization the $S$ matrix is given by

$$S=1-i\delta(p_3+p_4-p_1-p_2)$$
$$\times\frac{(2\pi)^4}{(E_1E_2E_3E_4)^{1/2}}A(p_3p_4;\,p_1p_2)\,. \qquad (2)$$

Since each neutrino has only one helicity, the problem has only one independent amplitude. The situation would be analogous to the case of scalar particles, except for questions of kinematical zeros and relative minus signs arising from crossing fermions. Thus, in the $u$-channel ($\nu\bar{\nu}\to\nu\bar{\nu}$) process, considered in its c.m. frame [Fig. 3(b)], the amplitude $A(p_3,\,-p_2;\,p_1,\,-p_4)$ can be expanded in terms of $d_{\lambda\mu}{}^J(\cos\theta_u)$, where $\lambda=\mu=1$. As is well known, this has a zero in the form of $\cos^2(\frac{1}{2}\theta_u)$ which is proportional to $s$ and has to be factored out. There are no kinematical zeros in $t$, since the $s$-channel process involves $d_{00}{}^J(cos\theta_s)$; the factorization of this zero in $s$ easily accomplished by writing

$$A^s(p_3p_4;\,p_1p_2)=2\bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma_\mu u(p_2)M(s,t,u)$$

and

$$A^u(p_3,\,-p_2;\,p_1,\,-p_4)=-2\bar{u}(p_3)\gamma^\mu n(p_1)$$
$$\times\bar{v}(-p_2)(\gamma_\mu v(-p_4)M(s,t,u)\,. \qquad (3)$$

The minus sign in the second equation is due to the crossing of two fermion lines. It can now be easily verified, by directly using the spinors in the c.m. frame, that

$$\bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_4)\gamma_\mu u(p_2)=\tfrac{1}{2}s$$

and

$$\bar{u}(p_3)\gamma^\mu u(p_1)\bar{v}(-p_2)\gamma_\mu v(-p_4)=-\tfrac{1}{2}s\,,$$

with the normalization of Eq. (1). Thus one can see that

$$\begin{aligned}A^s(p_3p_4p_1p_2)&=A^u(p_3,-p_2;\,p_1,-p_4)\\&=A^t(-p_2,p_4;\,-p_3,p_1)\\&=A(s,t,u)\\&=sM(s,t,u)\,.\end{aligned} \qquad (4)$$

Let us now suppose that

$$\mathrm{Im}A(s+i\epsilon,\,t,\,u)\xrightarrow[s\to\infty,\ t\to 0]{}\lambda s^\alpha. \qquad (5a)$$

This is a perfectly general assumption and corresponds to a hiqh-energy $\nu\nu$ total cross section which behaves

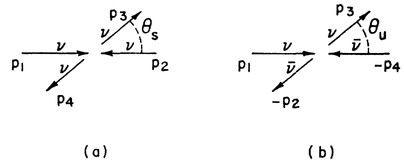



FIG. 3. The c.m. frames of the $s$- and the $u$-channel processes, respectively.

[7] G. Feinberg and J. Sucher, Phys. Rev. **166**, 1638 (1968).

as $s^{\alpha-1}$ asymptotically. Similarly, let

$$\mathrm{Im}A(s,\,t,\,u+i\epsilon)\xrightarrow[u\to\infty,\ t\to 0]{}\bar{\lambda}u^{\bar{\alpha}}, \tag{5b}$$

which gives a high-energy $\nu\bar{\nu}$ cross section behaving as $u^{\bar{\alpha}-1}$. Here $\lambda$ and $\bar{\lambda}$ are constants and *have the same sign* because both total cross sections are obviously positive. This relative sign, which corresponds in strong-interaction physics to the signature of the leading Regge trajectory, will be of importance in our subsequent discussion. Clearly, the corresponding asymptotic behavior of $\mathrm{Im}M(s,t,u)$ is

$$\mathrm{Im}M(s+i\epsilon,\,t,\,u)\xrightarrow[s\to\infty,\ t\to 0]{}\lambda s^{\bar{\alpha}-1}$$

and

$$\mathrm{Im}M(s,\,t,\,u+i\epsilon)\xrightarrow[u\to\infty,\ t\to 0]{}(1/s)\bar{\lambda}u^{\bar{\alpha}}=-\bar{\lambda}u^{-\bar{\alpha}1}. \tag{6}$$

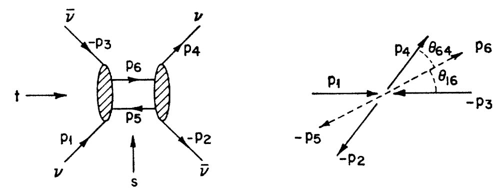
The main purpose of this work is to find a consistency condition on the parameters $\alpha$, $\bar{\alpha}$, $\lambda$, and $\bar{\lambda}$, assuming the scattering to be mediated by the two-neutrino exchange force. We now show in Sec. 3 that this force has a range and energy dependence which themselves are functions of $\lambda$, $\bar{\lambda}$, $\alpha$, and $\bar{\alpha}$.

## 3. LONG-RANGE TWO-NEUTRINO EXCHANGE POTENTIAL

Consider the diagram in Fig. 4, where the shaded blobs represent *full scattering amplitudes, not restricted to any given order of the perturbation.* Let $A^{\mathrm{Born}}(s,t)\equiv A^B(s,t)$ represent the amplitude for this diagram. We will be considering $\nu\nu$ and $\nu\bar{\nu}$ high-energy scattering process as being due to the iteration of this "Born term" or "potential." The precise procedure for iteration will be discussed in Sec. 4, but first we have to calculate this potential. In coordinate space, the potential $V(r)$ is related to the discontinuity $A_t{}^B(s,t)$ of the Born amplitude across the positive real axis in the $t$ plane by the familiar formula

$$V(r)\sim\frac{1}{(\sqrt{s})r}\int_0^\infty A_t{}^B(s,t)e^{-(\sqrt{t})r}dt. \tag{7}$$

In particular, the "longest-range" force is due to the threshold value of $A_t{}^B(s,t)$ as $t\to 0$. Needless to say, the use of a coordinate-space potential is only a convenient mnemonic which is helpful in visualizing the problem in coordinate space. As will be seen in Sec. 4, one can directly calculate the full scattering amplitude from this Born amplitude $A^B(s,t)$ in a high-energy approximation, which yields the same result as using the corresponding $V(\mathbf{r})$ and a suitable Schrödinger equation. With these introductory remarks, let us proceed to calculate the discontinuity $A_t{}^B(s,t)$ of the diagram in Fig. 4.




(a) (b)

FIG. 4. (a) The "Born term" corresponding to the two-neutrino exchange potential and (b) the momenta in the c.m. frame of the $t$ channel.

We start with elastic unitarity in the $t$ channel, which states in the c.m. frame of the $t$ channel that

$$A_t{}^B(s,t)=\mathrm{Im}A^B(s,t)$$
$$=K\int A^*(p_4,\,-p_2;\,p_6,\,-p_5)\times A(p_6,\,-p_5;\,p_1,\,-p_3)d\Omega_6, \tag{8}$$

where (i) the $A$'s in the integral are the full $\nu\bar{\nu}$ scattering amplitudes in the $t$ channels as represented by the shaded blobs, (ii) $\Omega_6=(\theta_6,\phi_6)$ is the direction of $\mathbf{p}_6$ as compared to $\mathbf{p}_1$ [see Fig. 4(b)], and (iii) $K$ is a numerical factor.

Let us define, in the c.m. frame in the $t$ channel,

$$\begin{aligned}
&p_1=(k,0,0,k),\quad z_{16}=(\hat{p}_6\cdot\hat{p}_1),\quad z_{46}=(\hat{p}_4\cdot\hat{p}_6),\quad z=(\hat{p}_1\cdot\hat{p}_4),\\
&t=4k^2,\quad u=(p_1-p_4)^2=-2k^2(1-z)=-\tfrac{1}{2}t(1-z),\\
&s=[p_1-(-p_2)]^2=-\tfrac{1}{2}t(1+z),\\
&s_1=-\tfrac{1}{2}t(1+z_{16}),\quad u_1=-\tfrac{1}{2}t(1-z_{16}),\\
&s_2=-\tfrac{1}{2}t(1+z_{46}),\quad u_2=-\tfrac{1}{2}t(1-z_{46}).
\end{aligned} \tag{9}$$

The reader may find it convenient to note that all the $s$ variables are defined to be the (mass)$^2$ of $\nu\nu$ systems, while the $u$ variables are the (mass)$^2$ of $\nu\bar{\nu}$ systems. In terms of these variables,

$$\mathrm{Im}A^B(s,t)=K\int d\Omega_6\,A(s_1,t)A^*(s_2,t)$$
$$=K\int d\Omega_6\,s_1s_2M(s_1t)M^*(s_2,t). \tag{10}$$

We have ignored inelastic intermediate states in the $t$ channel, which would correspond to either massive-particle exchanges and four or more neutrino exchanges in the $s$-channel potential. As mentioned in the Introduction, we are dealing only with the longest-ranged (two-neutrino exchange) force.

We are really interested in the limit of Im $A^B(s,t)$ as $t\to 0$ and $s\to\infty$, but we cannot immediately substitute $t=0$ in the right-hand side of Eq. (10). It is clear from Eq. (9) that if $t=0$, then any finite $s_1$, $s_2$, or $s$ would correspond to infinite values of $z_{16}$, $z_{64}$, and $z$ and render the angular integration over $d\Omega_6$ in Eq. (10) undefined.

Therefore, we will keep $t$ nonzero in certain crucial places until after the angular integration has been completed. We obtain $M(s_1,t)$ and $M(s_2,t)$ by dispersing over the physical $s$-channel region, i.e.,

$$M(s_1,t)=\frac{1}{\pi}\int_0^\infty \frac{a(s',t)}{s'-s_1}ds'+\frac{1}{\pi}\int_0^\infty \frac{b(u',t)}{u'-u_1}du',$$
$$M(s_2,t)=\frac{1}{\pi}\int_0^\infty \frac{a(s'',t)}{s''-s_2}ds''+\frac{1}{\pi}\int_0^\infty \frac{b(u'',t)}{u''-u_2}du''. \quad (11)$$

Here $a(s',t)$ and $b(u',t)$ are the absorptive parts of $M$ in the $s$- and $u$-channel processes. We have made the important assumption that the dispersion integrals in Eq. (11) are unsubtracted. This will be justified later on, on the grounds that a subtracted dispersion relation will *not* yield a consistency on the high-energy behavior of the amplitudes.

Substituting Eq. (11) into Eq. (10), we have

$$A_t^B(s,t)=\frac{K}{\pi^2}\int\int ds'ds''\,a(s',t)a(s'',t)\int d\Omega_6\frac{s_1}{s'-s_1}\frac{s_2}{s''-s_2}$$
$$+\frac{K}{\pi^2}\int du'du''\,b(u',t)b(u'',t)\int d\Omega_6\frac{s_1}{u'-u_1}\frac{s_2}{u''-u_2}$$
$$+\frac{K}{\pi^2}\int\int ds'du''\,a(s',t)b(u'',t)\int d\Omega_6\frac{s_1}{s'-s_1}\frac{s_2}{u''-u_2}$$
$$+\frac{K}{\pi^2}\int\int ds''du'\,b(u',t)a(s'',t)\int d\Omega_6\frac{s_1}{u'-u_1}\frac{s_2}{s''-s_2}. \quad (12)$$

Consider for a moment just the angular integration in the first term of this equation in the limit of $t\to 0$. We have

$$\int\frac{s_1}{s'-s_1}\frac{s_2}{s''-s_2}d\Omega_2$$
$$=\int d\Omega_6\left(\frac{s'}{s'-s_1}-1\right)\left(\frac{s''}{s''-s_2}-1\right)$$
$$=s's''\int d\Omega_6\frac{1}{\frac{1}{4}t^2(z'-z_{16})(z''-z_{64})}$$
$$-\int d\Omega_6\frac{s'}{\frac{1}{2}t(z_{16}-z')}-\int d\Omega_6\frac{s''}{\frac{1}{2}t(z_{64}-z'')}+4\pi, \quad (13)$$

where $s'=-\frac{1}{2}t(1+z')$ and $s''=-\frac{1}{2}t(1+z'')$. Now,

$$\frac{2s'}{t}\int\frac{d\Omega_6}{z_{16}-z'}=\frac{4\pi s'}{t}\ln\frac{1-z'}{1+z'}=\frac{-4\pi s'}{t}\ln\frac{s'+t}{s'}$$
$$\to -4\pi \quad \text{as} \quad t\to 0.$$

Similarly,

$$\frac{2s''}{t}\int d\Omega_6\frac{1}{z_{64}-z''}\to -4\pi \quad \text{as} \quad t\to 0.$$

The integral $\int d\Omega_6\,[(z'-z_{16})(z''-z_{64})]^{-1}$ is well known[8] and is most conveniently expressed for one purpose in the form

$$\int d\Omega_6\frac{1}{(z'-z_{16})(z''-z_{64})}=4\pi\int_0^\infty\frac{d\zeta}{\zeta-z}\frac{\theta(\zeta-\lambda)}{[(\zeta-\lambda)(\zeta-\mu)]^{1/2}},$$

where

$$\lambda=z'z''+[(z'^2-1)(z''^2-1)]^{1/2},$$
$$u=z'z''-[(z'^2-1)(z''^2-1)]^{1/2},$$

$z=\hat{p}_1\cdot\hat{p}_4$ as before, and $\theta$ is the usual step function. Combining all this information, we have

$$\int\frac{s_1}{s'-s_1}\frac{s_2}{s''-s_2}d\Omega_6$$
$$\xrightarrow[t\to0]{} 4\pi\left(s's''\int_0^\infty\frac{d\zeta}{\zeta-z}\frac{\theta(\zeta-\lambda)}{\frac{1}{4}t^2[(\zeta-\lambda)(\zeta-\mu)]^{1/2}}-1\right).$$

Let $\bar{u}=-\frac{1}{2}t(1-\zeta)$. Then

$$\zeta-\lambda=\frac{2\bar{u}}{t}+1-z'z''-[(z'^2-1)(z''^2-1)]^{1/2}$$
$$=\frac{2\bar{u}}{t}+1-\left(\frac{2s'}{t}+1\right)\left(\frac{2s''}{t}+1\right)$$
$$-\left[\left(\frac{4s''^2}{t^2}+\frac{4s''}{t}\right)\left(\frac{4s'^2}{t^2}+\frac{4s'}{t}\right)\right]^{1/2},$$

$$\tfrac{1}{2}t(\zeta-\lambda)=\bar{u}-\left\{s'+s''+2\frac{s's''}{t}\left[1+\left(1+\frac{t}{s''}\right)^{1/2}\left(1+\frac{t}{s'}\right)^{1/2}\right]\right\}$$
$$\to \bar{u}-4\frac{s's''}{t} \quad \text{as} \quad t\to 0.$$

Similarly, $\frac{1}{2}t(\zeta-\mu)\to\bar{u}$. Thus,

$$\int\frac{s_1s_2}{(s'-s_1)(s''-s_2)}d\Omega_6$$
$$\xrightarrow[t\to0]{} 4\pi\left(s's''\int_0^\infty\frac{d\bar{u}}{(\bar{u}-u)}\frac{\theta(\bar{u}-4s's''/t)}{\frac{1}{2}t[\bar{u}(\bar{u}-4s's''/t)]^{1/2}}-1\right).$$

[8] See, for instance, M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).

Thus, the first term of Eq. (12) becomes, as $t\rightarrow 0$,

$$\frac{4\pi K}{\pi^2}\int\int ds'ds''\, a(s',0)a(s'',0)$$
$$\times\left(s's''\int_0^\infty \frac{d\bar{u}}{(\bar{u}-u)}\frac{\theta(\bar{u}-4s's''/t)}{\frac{1}{2}t[\bar{u}(\bar{u}-4s's''/t)]^{1/2}}-1\right)\equiv R\text{ (say)}. \tag{14}$$

Two remarks are now in order. We know from Eq. (6) that

$$a(s',0)=\mathrm{Im}M(s',0)=\lambda(s')^{\alpha-1}$$

for large $s'$. We will show presently that the integrals over $s'$ and $s''$ in Eq. (14) for large $|s|$ or $|u|$ are dominated by the large $s'$ and $s''$ regions. Hence we will set $a(s',0)=\lambda(s')^{\alpha-1}$ and $a(s'',0)=\lambda(s'')^{\alpha-1}$ in Eq. (14). Secondly, since the power $\alpha$ was such as to allow the use of an unsubtracted dispersion integral in Eq. (11), the integral in Eq. (14) will not converge without a subtraction. Let us subtract at the point $u=t=s=0$. In other words, the first term in Eq. (12) for

$$A_t^B(s,t)\equiv R\xrightarrow[t\to 0,\ s\to\infty]{}\frac{4K}{\pi}\int\int ds'ds''\,\lambda^2(s's'')^\alpha$$
$$\times u\int\frac{d\bar{u}}{\bar{u}(\bar{u}-u)}\frac{\theta(\bar{u}-4s's''/t)}{\frac{1}{2}t[\bar{u}(\bar{u}-4s's''/t)]^{1/2}}$$
$$+\text{first term of }A_t^B(0,0)$$
$$=u\frac{8K\lambda^2}{\pi}\int_0^\infty\frac{d\bar{u}}{\bar{u}(\bar{u}-u)}\bar{u}^\alpha t^\alpha\int\int dxdy$$
$$\times\frac{\theta(1-4xy)}{(1-4xy)^{1/2}}(xy)^\alpha+\text{first term of }A_t^B(0,0),$$

where $s'=(\bar{u}t)^{1/2}x$ and $s''=(\bar{u}t)^{1/2}y$. Let

$$N(\alpha)=\int_0^\infty\int_0^\infty dxdy\,\frac{\theta(1-4xy)}{(1-4xy)^{1/2}}(xy)^\alpha.$$

Then,

$$R=ut^\alpha\frac{8K\lambda^2}{\pi}N(\alpha)\int_0^\infty\frac{d\bar{u}}{\bar{u}(\bar{u}-u)}\bar{u}^\alpha+\text{first term of }A_t^B(0,0)$$
$$=8K\lambda^2N(\alpha)t^\alpha\frac{us^{\alpha-1}}{\sin\pi(\alpha-1)}+\text{first term of }A_t^B(0,0), \tag{15}$$

where

$$\frac{1}{\pi}\int_0^\infty\frac{d\bar{u}(\bar{u})^{\alpha-1}}{\bar{u}-u}=\frac{(-u)^{\alpha-1}}{\sin\pi(\alpha-1)}=\frac{s^{\alpha-1}}{\sin\pi(\alpha-1)}\quad\text{as } t\rightarrow 0.$$

This is the contribution of the first term on the right-hand side of Eq. (12). An analogous treatment can be given to the other three terms in Eq. (12), using $b(u,0)=-\bar{\lambda}u^{\bar{\alpha}-1}$ and keeping careful track of minus signs. We then get

$$A_t^B(s,t)-A_t^B(0,0)=A_t^B(s,t)\xrightarrow[t\to 0,\ s\to\infty]{}$$
$$K\left(\frac{\lambda^2N(\alpha)t^\alpha us^{\alpha-1}}{\sin\pi(\alpha-1)}+\frac{\bar{\lambda}^2N(\bar{\alpha})t^{\bar{\alpha}}us^{\bar{\alpha}-1}}{\sin\pi(\bar{\alpha}-1)}\right.$$
$$\left.+2\lambda\bar{\lambda}\frac{N(\alpha,\bar{\alpha})t^{(\alpha+\bar{\alpha})/2}}{\sin\frac{1}{2}\pi(\alpha+\bar{\alpha}-2)}su^{(\alpha+\bar{\alpha})/2-1}\right), \tag{16}$$

where

$$N(\alpha,\bar{\alpha})=\int\int dxdy\,x^\alpha y^{\bar{\alpha}}\frac{\theta(1-4xy)}{(1-4xy)^{1/2}}.$$

We further note that, as observed in Sec. 2, $A^B(s,t)$ has a zero in $s$, i.e., $A^B(0,t)=0$, and hence the first equality in Eq. (16). Therefore, the long range high-energy potential which is essentially $A_t^B(s,t)$ as $t\rightarrow 0$, $s\rightarrow\infty$ is given by the right-hand side of Eq. (16). It is clear that, as advertised, both the range and the energy dependence of the potential are functions of the high-energy scattering parameters $\alpha$ and $\bar{\alpha}$. For instance, if one considers the first term of the potential in Eq. (16), it has the form const$\times s^\alpha t^\alpha$ as $t$ tends to zero and $s$ tends to infinity. This corresponds in coordinate space to

$$V(r)=\frac{\text{const}}{(\sqrt{s})r}s^\alpha\int_0^\infty e^{-(\sqrt{t})r}t^\alpha dt$$
$$=\frac{s^{\alpha-1/2}}{r^{2\alpha+3}}\times\text{const}. \tag{17}$$

The high-energy scattering cross section obtained by iterating such a potential will clearly also be a function of $\alpha$, $\bar{\alpha}$, $\lambda$, and $\bar{\lambda}$, and will thus provide the necessary consistency on these parameters. We note that Eq. (16) gives the potential for $\nu$-$\nu$ scattering and for $\nu$-$\bar{\nu}$ scattering as well. The $u$-channel process in Fig. 4(a) is $\nu$-$\bar{\nu}$ scattering. The momentum transfer for this process is once again $t=(p_3-p_1)^2$. Thus, $A_t^B(s,t,u)$ calculated in Eq. (16) gives the $\nu$-$\bar{\nu}$ potential as well, with the stipulation that now $u$ refers to the total mass squared instead of $s$. In general, even though the right-hand side of Eq. (16) gives both potentials, the $\nu$-$\nu$ and $\nu$-$\bar{\nu}$ potentials are not equal, since the roles of $s$ and $u$ are interchanged.

We conclude this section by justifying the statement made in connection with Eq. (14), viz., that the integrals over $s'$ and $s''$ are dominated by their high-energy domains, and that we can use the asymptotic forms $a(s,0)=\lambda s^{\alpha-1}$ in the integrals of Eq. (12). Suppose $a(s,0)$ was of the form $\lambda_1s^{\alpha-1}+\lambda_2s^{\beta-1}$. Then the integral in Eq. (14) will lead, analogous to Eq. (15), to a result

$$R\sim[\lambda_1^2u^\alpha t^\alpha+\lambda_2^2u^\beta t^\beta+2\lambda_1\lambda_2u^{(\alpha+\beta)/2}t^{(\alpha+\beta)/2}]$$

in abbreviated notation. Now, if $\alpha>\beta$, then at high energies (as $|u|\rightarrow\infty$) the $u^\alpha t^\alpha$ term will dominate. But this corresponds to using just the high-energy behavior of $a(s,0)$, since

$$a(0,s)=\lambda_1 s^{\alpha-1}+\lambda_2 s^{\beta-1} \xrightarrow[s\to\infty]{} \lambda_1 s^{\alpha-1}.$$

It is therefore clear that in evaluating the integrals in Eqs. (12) and (14), for the high-energy potential ($|s|,|u|\to\infty$), one can use the asymptotic forms for $a(s',0)$, $b(u',0)$, etc.

We will now proceed to iterate the long-range potential in Eq. (16) to obtain the high-energy cross sections.

## 4. EIKONAL METHOD

In order to evaluate the high-energy scattering amplitude from the Born term or "potential," we will use the eikonal approximation. The pros and cons of using this approximation have been discussed in literature,[8,9] and we will not go into them here. Qualitatively, the eikonal method is likely to work when the energy is large compared to the potential and the wavelength is small compared to the range of the force. With our long-range weak-interaction force, these conditions are satisfied for high-energy scattering.

There are several ways available for using the eikonal principle to iterate the potential.[9] One could use the Born term as a kernel in a BS equation and solve the latter in the high-energy approximation. Alternatively, one could use the potential translated into coordinate space, in some relativistic Schrödinger equation, and once again solve the latter in the eikonal approximation. In this alternative, there are once again several candidates for the wave equation. Altogether, we considered (i) the BS equation with $A^B(s,t)$ as the kernel, (ii) $(E^2-p^2)\Psi=V\psi$ (KG equation with a world-scalar potential), (iii) $[(E-V)^2-p^2]\psi=0$ (KG equation with a time-component potential), and (iv) $(|p|+V)\psi=E\psi$.

It is encouraging that the BS equation, as well as these several wave equations, yields the same high-energy eikonal result for the scattering amplitude in terms of the two-neutrino exchange potential. We will now demonstrate this briefly for cases (i) and (ii) before using the result to evaluate $\alpha$, $\bar{\alpha}$, $\lambda$, and $\bar{\lambda}$. Cases (iii) and (iv) are trivially related to case (ii) and give the same result.

Let us start with the KG equation with the world-scalar potential, which, in time-independent form, is

$$[-\nabla^2+V(\mathbf{r})]\psi(\mathbf{r})=E^2\psi(\mathbf{r}). \tag{18}$$

This is identical in form to the nonrelativistic Schrödinger equation

$$[\nabla^2+k^2-V(r)]\psi(\mathbf{r})=0,$$

which has the well-known expression[8,9] for the scattering amplitude in the eikonal approximation

$$f(k,\mathbf{\Delta})=-ik\int_0^\infty db\, bJ_0(b\Delta)\times\left[\exp\left(\frac{1}{2ik}\int_{-\infty}^{\infty}V((b^2+z^2)^{1/2})dz\right)-1\right], \tag{19}$$

where $\mathbf{\Delta}$ is the momentum transfer. We note that the $k$ in Eq. (19) corresponds in the KG equation (18) to $k=E=\sqrt{s}$ and that the invariant amplitude $A(s,t)=(\sqrt{s})f(E,\mathbf{\Delta})$. Therefore,

$$A(s,t)=-is\int_0^\infty bdb\, J_0(b\Delta)\times\left[\exp\left(\frac{1}{2i\sqrt{s}}\int_{-\infty}^{\infty}V((b^2+z^2)^{1/2})dz\right)-1\right] \tag{20}$$

is the eikonal approximation for the scattering amplitude for the KG equation with a scalar potential.

A completely analogous result comes from using the BS equation. To see this, it is best to first note what the eikonal approximation in Eq. (19) for the Schrödinger equation corresponds to in momentum space. We have, in momentum space,

$$f(k,\mathbf{\Delta})=f^B(\mathbf{\Delta})+\frac{1}{2\pi^2}\int f^B(\mathbf{\Delta}-\mathbf{q})\times\frac{1}{(\mathbf{k}+\mathbf{q})^2-k^2-i\epsilon}f(k,\mathbf{q})d^3q, \tag{21}$$

which is just the Lippmann-Schwinger equation, with

$$f^B(\mathbf{\Delta})=f^{\text{Born}}(\mathbf{\Delta})=\frac{1}{4\pi}\int e^{-i\Delta\cdot\mathbf{r}}V(\mathbf{r})d^3r.$$

Now, in high-energy scattering under "eikonal conditions," we expect the momentum transfer $\mathbf{\Delta}$ to be small compared to $\mathbf{k}$. Further, by energy conservation, $(\mathbf{k}+\mathbf{\Delta})^2\simeq k^2+2\mathbf{k}\cdot\mathbf{\Delta}=k^2$, so that we also expect $\mathbf{k}\cdot\mathbf{\Delta}\simeq 0$. In other words, if $\mathbf{k}$ is along the $z$ axis, then $\Delta_z\simeq 0$ and $\mathbf{\Delta}_\perp=(\Delta_x,\Delta_y)$ is small. We may also expect the same approximation to be valid for the intermediate-state momentum $\mathbf{k}+\mathbf{q}$ in the integral in Eq. (21). Thus, let us set $q_z\simeq 0$, with $\mathbf{q}_\perp$ being small, so that Eq. (21) becomes

$$f(k,\mathbf{\Delta}_\perp)=f^B(\mathbf{\Delta}_\perp)+\frac{1}{2\pi^2}\int d^2q_\perp\int dq_z\, f^B(\mathbf{\Delta}_\perp-\mathbf{q}_\perp)\times\frac{1}{2kq_z-i\epsilon}f(k,\mathbf{q}_\perp) \tag{22}$$

$$=f^B(\Delta_\perp)+\frac{i\pi}{4\pi^2k}\int d^2q\, f^B(\mathbf{\Delta}_\perp-\mathbf{q}_\perp)f(k,\mathbf{q}_\perp).$$

This equation is easily solved for $f(k,\mathbf{\Delta}_\perp)$ by going to the

[9] A detailed study of the eikonal approximation for the Schrödinger and BS equations is available in L. H. Domash, Ph.D. thesis, Princeton University, 1967 (unpublished). We acknowledge borrowing some wisdom from this work.

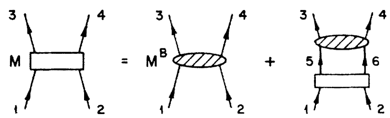



FIG. 5. The BS equation.

impact-parameter representation.[10] Let

$$H(b)=\frac{1}{2\pi}\int f(k,\mathbf{\Delta}_\perp)e^{i\mathbf{b}\cdot\mathbf{\Delta}_\perp}d^2\Delta_\perp$$

and

$$K(b)=\frac{1}{(2\pi)}\int f^B(\mathbf{\Delta}_\perp)e^{i\mathbf{b}\cdot\mathbf{\Delta}_\perp}d^2\Delta_\perp.$$

Then, by Fourier-inverting Eq. (22) with respect to $\mathbf{\Delta}_\perp$, we have

$$H(b)=K(b)+(i/2k)K(b)H(b),$$

so that

$$H(b)=\frac{K(b)}{1-(i/2k)K(b)}\simeq -ik(e^{iK(b)/k}-1).$$

Going back to the $\Delta_\perp$ representation,

$$f(k,\mathbf{\Delta}_\perp)=\frac{1}{2\pi}\int d^2b\ e^{i\mathbf{b}\cdot\mathbf{\Delta}_\perp}(-ik)[e^{iK(b)/k}-1]$$

$$=-ik\int_0^\infty bdb\ J_0(b\Delta_\perp)\left[\exp\left(\frac{iK(b)}{k}\right)-1\right],$$

but

$$K(b)=\frac{1}{2\pi}\int e^{i\mathbf{b}\cdot\mathbf{\Delta}_\perp}d^2\Delta_\perp\int\frac{1}{4\pi}e^{i\Delta_\perp\cdot\mathbf{b}'}v(\mathbf{b}',z)d^3r'$$

$$=-\frac{1}{2}\int d^3r'\ v(\mathbf{b}',z)\delta(\mathbf{b}'-\mathbf{b})$$

$$=-\frac{1}{2}\int_{-\infty}^{\infty}v(\mathbf{b},z)dz,$$

where $\mathbf{r}'=(\mathbf{b}',z)$. Thus,

$$f(k,\Delta_\perp)=-ik\int bdb\ J_0(b\Delta_\perp)\left[\exp\left(\frac{-i}{2k}\int_{-\infty}^{\infty}v(\mathbf{b},z)dz\right)-1\right],$$

which is the result in Eq. (19).

The purpose of this digression into momentum space, by no means original, is only intended to convince ourselves that the eikonal approximation corresponds in momentum space to (i) ignoring the dependence on $q_z$ of the amplitude $f(k,\mathbf{q})$ and the Born term $f^B(\mathbf{\Delta}-\mathbf{q})$ and (ii) ignoring terms of order $q^2$ in the propagator, in the integral equation for $f(k,\mathbf{\Delta})$.

We now proceed to use the BS integral equation in full generality to iterate the Born term instead of using a KG equation, but with analogous eikonal approximations.

The BS equation for our problem may be written (see Fig. 5)

$$A(p_4p_3,p_2p_1)\equiv 2\bar{u}(4)\gamma^\mu u(2)M(p_4p_3;\ p_2p_1)\bar{u}(3)\gamma_\mu u(1)$$

$$=2\bar{u}(4)\gamma^\mu u(2)M^{\text{Born}}(p_4p_3;\ p_2p_1)\bar{u}(3)\gamma_\mu u(1)+\frac{4i}{(2\pi)4}\int\int dp_5dp_6\ \bar{u}(4)\gamma^\mu\frac{1}{p_6+i\epsilon}\gamma^\nu u(2)\bar{u}(3)\gamma_\mu\frac{1}{p_5+i\epsilon}\gamma_\nu u(1)$$

$$\times M^B(p_4p_3;\ p_6p_5)M(p_6p_5;\ p_2p_1)\delta^4(p_1+p_2-p_5-p_6).\quad(23)$$

The $M$ amplitudes are related to the invariant amplitudes $A$ as in Eq. (3). Let us go to the c.m. frame and define

$$p_1=(k,00k),\quad p_2=(k,\ 00-k).\quad (0,\mathbf{\Delta})=p_3-p_1,$$

$$p_5=p_1+q=(k+q_0,\ q_x,\ q_y,\ k+q_z),$$

$$p_6=p_2-q=(k-q_0,\ -q_x,\ -q_y,\ -k-q_z),\quad s=4k^2.$$

We also recollect from Sec. 2 that $\bar{u}(4)\gamma^\mu u(2)\bar{u}(3)\gamma_\mu u(1)=\frac{1}{2}s$. Thus, Eq. (23) becomes

$$sM(s,\mathbf{\Delta})=sM^B(s,\mathbf{\Delta})+\frac{2i}{(2\pi)4}\int\bar{u}(4)\gamma^\mu(p_2-q)\gamma^\nu u(2)\bar{u}(3)\gamma_\mu(p_1+q)\gamma_\nu u(1)\frac{1}{(p_1+q)^2+i\epsilon}\frac{1}{(p_2-q)^2+i\epsilon}$$

$$\times M^B(s,\ \Delta-q)M(s,q)d^4q.\quad(24)$$

Thus far, everything is general. Now we proceed to use an eikonal-type approximation. Equation (24) is an integral equation in momentum space, similar to Eq. (21) for the KG, or Schrödinger, equation, with the difference that now there is also a time component $q_0$ to be integrated over. From our experience in the momentum-space eikonal approximation for Eq. (21), we drop terms of order $q^2$ in the propagators. Further, we ignore the dependence of $M$ and $M^B$ on $q_z$ and $q_0$, retaining only the $q_\perp^2=q_x^2+q_y^2$ dependence. In this limit, it can

[10] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962).

be checked that

$$\bar{u}(4)\gamma^\mu(p_2-q)\gamma^\nu u(2)\bar{u}(3)\gamma_\mu(p_1+q)\gamma_\nu u(1)\to 16k^4(=s^2)+\text{terms of order }(q,\Delta).$$

Once again we expect $\Delta_z\simeq 0$ and $\Delta_\perp$ to be small. The eikonalized amplitude is then

$$sM(s,\Delta_\perp)=sM^B(s,\Delta_\perp)+\frac{2is^2}{(2\pi)^4}\int d^2q_\perp\, M^B(s,\,\Delta_\perp-q_\perp)\times M(q_\perp)\int dq_0dq_z\frac{1}{2(kq_0-kq_z)+i\epsilon}\,\frac{1}{2(-kq_0-kq_z)+i\epsilon},$$

but

$$\int\int dq_0dq_z\,\frac{1}{-4k^2(q_0-q_z+i\epsilon)(q_0+q_z-i\epsilon)}=\frac{1}{4k^2}\int dq_z\,\frac{2i\pi}{2(q_z-i\epsilon)}=-4\pi^2/8k^2=-2\pi^2/s.$$

These last steps are, of course, only formal, in the sense that the $dq_0$ and $dq_z$ integrals are assumed to converge, which they would if the exact dependence on $q_z$ and $q_0$ of the amplitudes were inserted. Thus, we have

$$sM(s,\Delta_\perp)=sM^B(s,\Delta_\perp)+\frac{4s\pi^2 i}{(2\pi)^4}\int d^2q_\perp\, M^B(s,\,\Delta_\perp-q_\perp)M(s,q_\perp).$$

Remembering that the scattering amplitude $f=(1/\sqrt{s})\times A=(\sqrt{s})M$, we get

$$f(s,\Delta_\perp)=f^B(s,\Delta)+\frac{i}{8\pi^2k}\int d^2q_\perp\, f^B(s,\,\Delta_\perp-q_\perp)f(s,q_\perp).$$

This is the same as Eq. (22) for the Schrödinger (KG) equation case, except for factors of $2\pi$, which can be absorbed into the relative definitions of $f$ and $A$.

Thus, the BS equation yields the same result in the eikonal approximation for the full amplitude in terms of the Born amplitude or "potential" as the KG equation does. We can therefore use with some confidence this result for the scattering amplitude, which is given in Eq. (20) by

$$A(s,\Delta_\perp)=-is\int_0^\infty bdb\,J_0(b\Delta_\perp)\times\left[\exp\left(\frac{1}{2i\sqrt{s}}\int_0^\infty V((b^2+z^2)^{1/2})dz\right)-1\right].\quad(25)$$

The potential to be used is

$$V(r)=\frac{1}{r\sqrt{s}}\int A_t{}^B(s,t)e^{-(\sqrt{t})r}dt,$$

where $A_t{}^B(s,t)$ has been derived in Eq. (16). Consider the expression in Eq. (16). Suppose $\alpha>\bar{\alpha}$. (The argument would work just as well if $\bar{\alpha}>\alpha$.) Then the potential at very high energies is dominated by the $us^{\alpha-1}$ term, and is of the form const$\times(\lambda^2t^\alpha|s|^\alpha)$, which is equivalent, according to Eq. (17), to a $\nu$-$\nu$ potential,

$$V_{\nu\nu}(r)=Q\lambda^2s^{\alpha-1/2}/r^{2\alpha+3},$$

where $Q$ is a constant. Let us find the high-energy $\nu$-$\nu$ scattering amplitude by using this potential in Eq. (25). We are interested only in the forward scattering amplitudes, since we started out by assuming asymptotic forms for these in Eq. (5). We have

$$A(s,0)\underset{s\to\infty}{\longrightarrow}-is\int_0^\infty bdb\left[\exp\left(\frac{1}{2i\sqrt{s}}\int_{-\infty}^{\infty}\frac{Q\lambda^2s^{\alpha-1/2}}{(b^2+z^2)^{(2\alpha+3)/2}}dz\right)-1\right]$$

$$=-is\int_0^\infty bdb\left[\exp\left(\frac{Q\lambda^2s^{\alpha-1}}{2i}\int_{-\infty}^{\infty}\frac{bdf}{b^{2\alpha+3}(1+f^2)^{(2\alpha+3)/2}}\right)-1\right]$$

$$=-is\int_0^\infty \tfrac{1}{2}dg\left[\exp\left(\frac{Q'\lambda^2s^{\alpha-1}}{2ig^{\alpha+1}}\right)-1\right],$$

where

$$Q'=Q\int_{-\infty}^{\infty}\frac{df}{(1+f^2)^{(2\alpha+3)/2}},\quad bf=z,\quad\text{and}\quad b^2=g.$$

Now, let

$$g^{\alpha+1}/s^{\alpha-1}\lambda^2=l,$$

$$(\alpha+1)g^\alpha/s^{\alpha-1}\lambda^2=dl/dg,$$

$$dg=\frac{dl}{\alpha+1}\,\frac{(\lambda^2s^{\alpha-1})^{1-\alpha/(\alpha+1)}}{l^{\alpha/(\alpha+1)}}.$$

Hence

$$A(s,0)\underset{s\to\infty}{\longrightarrow}-is(\lambda^2s^{\alpha-1})^{1/(\alpha+1)}\int_0^\infty dl\frac{e^{-iQ''/l}-1}{l^{\alpha/(\alpha+1)}(\alpha+1)},\quad(26)$$

where $Q''=$const. But we assumed at the very beginning, in Eq. (5a), that $\text{Im}A_{\nu\nu}(s,0)=\lambda s^\alpha$ as $s\to\infty$. Comparing the $s$ dependence of Eqs. (26) and (5a), we have

$$s^{(\alpha-1)/(\alpha+1)+1}=s^{\alpha},$$

$$(\alpha-1)/(\alpha+1)=\alpha-1, \tag{27a}$$

and hence

$$\alpha=1 \quad \text{or} \quad 0. \tag{27b}$$

## 5. HIGH-ENERGY BOUNDS, A POMERANCHUK THEOREM, AND THE TWO-NEUTRINO POTENTIAL

Now we are ready to reap the fruits of all the preceding algebra. We derived at the end of Sec. 4 that if $\alpha>\bar{\alpha}$, then by looking at the consistency of the high-energy behavior of $\nu$-$\nu$ scattering, one can get $\alpha=0$ or 1. However, we noted in Sec. 3 that the potential in Eq. (16) is also valid for the $\nu$-$\bar{\nu}$ system, where $u$ is now the total mass squared. Thus, the $\nu$-$\bar{\nu}$ potential, with $\alpha>\bar{\alpha}$, is also of the form const$\times(\lambda^2t^\alpha u^\alpha)$; in other words,

$$V_{\nu-\bar{\nu}}(r)=Q\lambda^2u^{\alpha-1/2}/r^{2\alpha+3}.$$

Therefore, using this in Eq. (25) and following the same steps as in the $\nu$-$\nu$ case will lead to the very similar result

$$A_{\nu\bar{\nu}}(u,0)\underset{u\to\infty}{\sim}\lambda^{2/(\alpha+1)}u^{(\alpha-1)/(\alpha+1)+1}=\lambda u^{\bar{\alpha}}$$

from Eq. (5b), or

$$\bar{\alpha}-1=(\alpha-1)/(\alpha+1).$$

Comparing with Eq. (27a), we get

$$\alpha=\bar{\alpha}. \tag{28}$$

We have further derived that

$$\alpha=\bar{\alpha}=0 \quad \text{or} \quad 1.$$

The choice between zero and one is made as follows: We note that for both $\bar{\alpha}=\alpha=0$ and $\alpha=1=\bar{\alpha}$, the potential in Eq. (16) diverges due to the sine functions in the denominator and that the dispersion integral $\int_0^\infty d\bar{u}\,\bar{u}^\alpha/\bar{u}(\bar{u}-u)$ in Eq. (15) is not convergent. The source of this divergence is the original unsubtracted dispersion relation for $M(s,t)$ in Eq. (11), where the integrals are not individually convergent. This is clear from the fact that $a(s,0)=\text{Im}M(s,0)=\lambda s^{\alpha-1}\sim$const as $s\to\infty$ and $\alpha=1$, which leads to an ultraviolet logarithmic divergence, or if $\alpha=0$, $a(s,0)=\lambda/s$, which gives an infrared divergence. However, in each dispersion relation in Eq. (11), there are two terms, the $s$ and the $u$ integrals, which are both divergent, since $\alpha=\bar{\alpha}$. Consequently, it is possible that if $|\lambda|=|\bar{\lambda}|$ and their relative sign is the right one, then the two divergences in the dispersion relation cancel. This is once again clear by looking at the potential in Eq. (16), which may be written, for $\alpha=\bar{\alpha}$, as

$$A_t^B(s,t)\xrightarrow[s\to\infty,\ t\to 0]{}4KN(\alpha)t^\alpha[\sin\pi(\alpha-1)]^{-1}\times[-\lambda^2s^\alpha-\bar{\lambda}^2s^\alpha+2\lambda\bar{\lambda}s^\alpha e^{-i\pi(\alpha-1)}].$$

This appears to diverge as $\alpha\to 0$ or 1, unless there is a corresponding zero in the numerator as $\alpha\to 0$ or 1.

Now, it was shown from the positivity of cross sections that $\lambda$ and $\bar{\lambda}$ must have the same sign. Thus, if $\lambda=\bar{\lambda}$, the above potential reduces to

$$-8KN(\alpha)t^\alpha\lambda^2s^\alpha\left[\frac{1-e^{-i\pi(\alpha-1)}}{\sin\pi(\alpha-1)}\right].$$

As $\alpha\to 0$, this still diverges, but as $\alpha\to 1$, it does not, and leads to

$$\lim_{t\to 0,\ s\to\infty}A_t^B(s,t)=-i[8KN(1)]\lambda^2st$$

or

$$V(r)=\text{const}\times[-i\lambda^2(\sqrt{s})/r^5]. \tag{29}$$

We therefore conclude that $\alpha=\bar{\alpha}=1$ and $\lambda=\bar{\lambda}$. This corresponds to *constant* and *equal* cross sections for $\nu$-$\nu$ and $\nu$-$\bar{\nu}$ at high energies. The above argument is identical to the one invoked in strong-interaction physics about the signature of the leading Regge pole. Our pole has $\alpha(t=0)=1$ and positive signature and is exactly like the Pomeranchuk pole, although we are not by any means suggesting that a Regge theory is valid for weak processes.

There is another nice consequence of $\alpha=1=\bar{\alpha}$. We note that when we compare the total cross section $\lambda s^{\alpha-1}$ assumed at the beginning with the cross section implied by Eq. (26), we get

$$\text{const}\times\lambda^{2/(\alpha+1)}s^{(\alpha-1)/\alpha+1)+1}=\lambda s^{\alpha}.$$

This was, of course, used to get $\alpha=1$ by comparing the power of $s$ on both sides. But we also note that for $\alpha=1$, the powers of $\lambda$ automatically balance on both sides, i.e.,

$$\lambda^{2/(\alpha+1)}=\lambda \quad \text{for} \quad \alpha=1.$$

This is a good result, since otherwise one might expect to evaluate $\lambda$ by the above method, which would be unreasonable. Since the total cross section is $\lambda s^{\alpha-1}$, when $\alpha=1$, $\lambda$ has the dimensions of $1/s$. Since there are no masses in this problem, a good candidate for $\lambda$ is the weak coupling constant $G$ times a number, where $G$ is well known to have the dimensions of $s^{-1}$. Thus, evaluation of $\lambda$ would have implied bootstrapping $G$, which would have been an improbable result in a theory such as this. We are therefore relieved rather than concerned that the $\lambda$ dependence cancels out of our consistency requirements.

Our results are therefore the following:

(i) The $\nu$-$\nu$ and $\nu$-$\bar{\nu}$ total cross sections due to neutrino exchange approach constant values at high energies.

(ii) The $\nu$-$\nu$ and $\nu$-$\bar{\nu}$ cross sections approach each other at high energies.

(iii) The long-range two-neutrino exchange force can be interpreted in coordinate space as being of the form $E/r^5$, where $E$ is the total energy in the c.m. system.

We conclude by substantiating a couple of assumptions made in our model. It has been assumed that the dispersion relations in Eq. (11) are unsubtracted. This is not really an assumption, since a subtracted dispersion relation would not give any constant high-energy behavior at all in our model. Suppose we wrote once-subtracted dispersion relations for $M(s,t)$ instead of Eq. (11). For $\bar{\alpha}$, $\alpha$ not integers, this is merely an analytic continuation in $\bar{\alpha}$, $\alpha$, and hence the rest of the calculation would follow, once again forcing a value of $\alpha=\bar{\alpha}=1$. Now, if $\lambda\neq\bar{\lambda}$, then the potential corresponding to Eq. (15) would involve integrals of the form

$$\int_0^\infty \frac{d\bar{u}\,\bar{u}}{(\bar{u}-u_1)(\bar{u}-u)\bar{u}},$$

which will clearly involve $u\log u$ and $s\log s$. Such a potential, when substituted into Eq. (25), will not reproduce an $s^\alpha$-dependent amplitude and will give no consistent high-energy dependence. Therefore, only $\lambda=\bar{\lambda}$, or essentially unsubtracted dispersion relations where the most divergent term is cancelled between the $s$ and $u$ integrals, will give consistency.

We have also not used four or more neutrino exchanges. These are still zero-mass exchanges. However, in the $t$-channel unitarity relation, where these form four or more neutron intermediate states, the phase space to be integrated over would be much greater and likely to involve higher powers of $t$. Thus, in the limit of $t\rightarrow 0$ (longest-range forces), these would vanish in comparison to the two-neutrino contribution. In coordinate space, although four-neutrino exchanges, etc., would give $1/r^n$-type potentials, it is reasonable to conjecture that $n$ will be greater than 5. We are interested in the longest-range forces, and have therefore taken only the two-neutrino-exchange case, since these are the conditions furthest removed from massive exchange forces, where the well-established theorems could be used. As mentioned in the Introduction, the $1/r^5$ dependence of the long-range neutrino-pair potential is not unique to our theory. Not only has it been proved by Feinberg and Sucher with the usual current-current interaction, but it also follows from Eq. (2.4) of *their* work that the result is more general, and essentially kinematical in origin.[11] A pair of neutrino propagators in the $t$ channel, as in Fig. 1, considered as a Feynman diagram would give a projection operator of the form $(\mathbf{k}/k^2)(\mathbf{k}'/k'^2)$. Here $k$ and $k'$ are the momenta of the exchanged neutrinos. In evaluating the imaginary part in the $t$ channel, this would give essentially a $\mathbf{k}\mathbf{k}'$ dependence which is proportional to $t$. Thus, unless the "blobs" in Fig. 1 have zeros or singularities as $t\rightarrow 0$, the potential will behave as $t$, when $t\rightarrow 0$. These kinematical factors are, of course, also present in our work, in the form of the kinematical zeros $s_1$ and $s_2$ shown in Eq. (10). Obviously, for a given $\theta_t$ both $s_1$ and $s_2$ go to zero linearly as $t\rightarrow 0$. This may appear as a double zero in $t$, but it must be remembered that the limit $s\rightarrow\infty$, $t\rightarrow 0$ is a tricky one when all the masses are zero. When the limiting process is applied carefully as in Sec. 3, only a linear zero in $t$ survives.

Further, given a $1/r^5$ dependence of the potential, one can even find the energy dependence trivially, *provided* one is working to a given order of the dimensional coupling constant $G$. Since we do not restrict ourselves to any given order in perturbation in this work, the energy dependence arises because of our dynamical model.

Finally, it should be stressed that the principal motivation and result of this work deal with high-energy behavior of certain weak amplitudes and not the derivation of the potential.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[11] The author is indebted to Dr. J. Sucher for first pointing this out to him.