Vector-Meson Decays and the Algebra of Fields

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The three-point function $\langle 0|T\{A_\mu(x)V_\nu(y)V_\lambda(0)\}|0\rangle$ is studied in the algebra-of-fields model. By considering the most general finite smoothness expansion in q^2, p^2, and k^2, we are able to give reasons for the vanishing of the decays $\phi\to\rho\pi$, $\phi\to\pi^0\gamma$, $\omega\to\pi^0\gamma$, $\omega\to\rho\pi$, $\rho\to\pi\gamma$, and $\pi^0\to 2\gamma$. We suggest plausible ways in which these decays may be allowed without the necessity of abandoning the algebra-of-fields commutation relations.

1. INTRODUCTION

THE study of the vacuum expectation values of time-ordered products has been shown recently by Schnitzer and Weinberg[1] to allow one to use current commutation relations in a rather "pure" way, the results then depending only on meson-dominance approximations. Although the results of such a treatment of n-point functions are thought to be valid for energies up to ~1 BeV, it is of interest to study the full consequences of these approximations to see when their applicability may cease to exist.

In this paper we shall study the three-point function $\langle 0|T\{A_\mu{}^a(x)V_\nu{}^b(y)V_\lambda{}^c(0)\}|0\rangle$, where $V_\mu{}^i(x)$ and $A_\mu{}^i(x)$ are vector and axial-vector currents obeying the commutation relations of the gauge field algebra, under the assumptions of meson dominance and smoothness of proper vertex functions. By considering the most general (but finite) expansion in the invariants q^2, p^2, and k^2 of the form factors, we are able to show explicitly how the result concerning the vanishing of decay constants for the decays $\phi\to\rho\pi$, $\phi\to\pi^0\gamma$, $\omega\to\pi^0\gamma$, $\omega\to\rho\pi$, $\rho\to\pi\gamma$, and $\pi^0\to 2\gamma$ found recently[2,3] depends on the pole-dominance approximation and suggest alternatives to the proposal that we should abandon the commutation relations to obtain nonvanishing decay constants. In Sec. 2 we describe the formalism in some detail and in Sec. 3 derive the results of applying the Bjorken limit[4] and a general smoothness expansion for the form factors. Section 4 summarizes our results and compares them with other recent papers.

2. THREE-POINT FUNCTION

In this section we repeat for clarity the relevant notation of Schnitzer and Weinberg.[1] We define $M_{\mu\nu\lambda}{}^\alpha$ and $N_{\nu\lambda}{}^\alpha$ by

$$M_{\mu\nu\lambda}{}^\alpha=\int d^4x\,d^4y\times e^{ipy-qx}\langle T\{A_\mu{}^3(x)V_\nu{}^3(y)V_\lambda{}^\alpha(0)\}\rangle_0 \quad (2.1)$$

and

$$N_{\nu\lambda}{}^\alpha=\int d^4x\,d^4y\times e^{ipy-qx}\langle T\{\partial^\mu A_\mu{}^3(x)V_\nu{}^3(y)V_\lambda{}^\alpha(0)\}\rangle_0 . \quad (2.2)$$

$M_{\mu\nu\lambda}{}^\alpha$ and $N_{\nu\lambda}{}^\alpha$ can in turn be related to the vertex functions for the decays $A_1\to\rho\gamma$, $\rho\to\pi\gamma$, $\omega\to\pi\gamma$, $\omega\to\rho\pi$, and $\pi^0\to 2\gamma$. Following Ref. 1, we write

$$M_{\mu\nu\lambda}{}^\alpha\equiv ig_A{}^{-1}g_\rho{}^{-1}\Delta_{A_1}{}^{\mu\tau}(q)\Delta_\rho{}^{\nu\sigma}(p)\Delta_{\alpha\beta}{}^{\lambda\eta}(k)\Gamma_{\tau\sigma\eta}{}^\beta(q,p)+i\frac{g_\rho{}^{-1}F_\pi}{q^2+m_\pi{}^2}q^\mu\Delta_\rho{}^{\nu\sigma}(p)\Delta_{\alpha\beta}{}^{\lambda\eta}(k)\Gamma_{\sigma\eta}{}^\beta(q,p) \quad (2.3)$$

and

$$N_{\nu\lambda}{}^\alpha\equiv\frac{F_\pi m_\pi{}^2g_\rho{}^{-1}}{q^2+m_\pi{}^2}\Delta_\rho{}^{\nu\sigma}(p)\Delta_{\alpha\beta}{}^{\lambda\eta}(k)\Gamma_{\sigma\eta}{}^\beta(q,p)\,, \quad (2.4)$$

where $k=p-q$, and $\Delta_\rho{}^{\mu\nu}$, $\Delta_{A_1}{}^{\mu\nu}$, and $\Delta_{\alpha\beta}{}^{\mu\nu}$ are the covariant spin-1 parts of the unrenormalized vector and axial-vector propagators

$$\Delta_\rho{}^{\mu\nu}(p)\equiv\int d\mu^2\,\rho_V(\mu^2)\times\left(g^{\mu\nu}+\frac{p^\mu p^\nu}{\mu^2}\right)(\mu^2+p^2)^{-1}, \quad (2.5)$$

$$\langle V_\mu{}^a(x)V_\nu{}^b(0)\rangle_0=(2\pi)^{-3}\delta_{ab}\int d^4p\,\theta(p^0)\times e^{ipx}\rho_V(-p^2)\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right), \quad (2.6)$$

$$\Delta_{A_1}{}^{\mu\nu}(q)\equiv\int d\mu^2\,\rho_A{}^{(1)}(\mu^2)\times\left(g^{\mu\nu}+\frac{q^\mu q^\nu}{\mu^2}\right)(\mu^2+q^2)^{-1}, \quad (2.7)$$

$$\langle A_\mu{}^a(x)A_\nu{}^b(0)\rangle_0=(2\pi)^{-3}\delta_{ab}\int d^4p\,\theta(p^0)e^{ipx}\left[\rho_A{}^{(1)}(-p^2)\times\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)+\rho_A{}^{(0)}(-p^2)p_\mu p_\nu\right], \quad (2.8)$$

$$\Delta_{\alpha\beta}{}^{\mu\nu}(k)\equiv\int d\mu^2\,\rho_{\alpha\beta}(\mu^2)\times\left(g^{\mu\nu}+\frac{k^\mu k^\nu}{\mu^2}\right)(\mu^2+k^2)^{-1}, \quad (2.9)$$

[1] H. J. Schnitzer and S. Weinberg, Phys. Rev. **164**, 1828 (1967).
[2] R. Perrin, Phys. Rev. **170**, 1367 (1968).
[3] S. G. Brown and G. B. West, Phys. Rev. **174**, 1777 (1968).
[4] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966).

and

$$\langle V_\mu{}^\alpha(x)V_\nu{}^\beta(0)\rangle_0=(2\pi)^{-3}\int d^4p\,\theta(p^0)\times e^{ipx}\rho_{\alpha\beta}(-p^2)\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right),\quad(2.10)$$

where $\alpha,\beta=0,8$. The constants g_{A_1} and g_ρ are defined by

$$\langle 0|A_\mu(x)|A_1(q)\rangle=g_{A_1}\epsilon_\mu{}^{A_1}(q)\quad(2.11)$$

and

$$\langle 0|V_\mu(x)|\rho(p)\rangle=g_\rho\epsilon_\mu{}^\rho(p).\quad(2.12)$$

We now write the proper vertex function $\Gamma_{\mu\nu\lambda}{}^\beta$ in terms of the invariants $B_i{}^\beta$, where

$$\Gamma_{\mu\nu\lambda}{}^\beta=\epsilon_{\mu\nu\rho\lambda}(B_1{}^\beta q_\rho+B_2{}^\beta p_\rho)+\epsilon_{\mu\lambda\rho\sigma}q^\rho p^\sigma(B_4{}^\beta k_\nu+B_5{}^\beta q_\nu)+\epsilon_{\nu\lambda\rho\sigma}q^\rho p^\sigma(B_6{}^\beta k_\mu+B_7{}^\beta q_\mu)+\epsilon_{\mu\nu\rho\sigma}q^\rho p^\sigma(B_3{}^\beta q_\lambda+B_8{}^\beta k_\lambda).\quad(2.13)$$

We now apply the conservation equations to obtain the generalized Ward identities for the proper vertices

$$q^\mu\Gamma_{\mu\nu\lambda}{}^\beta=-(F_\pi g_A/C_A)\Gamma_{\nu\lambda}{}^\beta,\quad(2.14)$$

$$p^\nu\Gamma_{\mu\nu\lambda}{}^\beta=0,\quad(2.15)$$

$$k^\lambda\Gamma_{\mu\nu\lambda}{}^\beta=0,\quad(2.16)$$

$$k_\lambda\Gamma_{\nu\lambda}{}^\beta=0,\quad(2.17)$$

$$p_\nu\Gamma_{\nu\lambda}{}^\beta=0.\quad(2.18)$$

From Eq. (2.15) we obtain

$$B_1{}^\beta-(p\cdot k)B_4{}^\beta-(p\cdot q)B_5{}^\beta=0,\quad(2.19)$$

and from Eq. (2.16)

$$B_1{}^\beta+B_2{}^\beta+(k\cdot q)B_3{}^\beta+k^2B_8{}^\beta=0.\quad(2.20)$$

In Eq. (2.14), C_A is defined by

$$C_A=\int\rho_A(\mu^2)\mu^{-2}d\mu^2,\quad(2.21)$$

and

$$\Gamma_{\nu\lambda}{}^\beta(q,p)=(g_A{}^{-1}C_A/F_\pi)\epsilon_{\nu\lambda\rho\sigma}q^\rho p^\sigma\times[B_2{}^\beta-(k\cdot q)B_6{}^\beta-q^2B_7{}^\beta],\quad(2.22)$$

so that Eqs. (2.17) and (2.18) are automatically satisfied.

3. SMOOTHNESS APPROXIMATION

We now consider the result of approximating the scalar functions $B_i{}^\alpha(q^2,p^2,k^2)$ by polynomials of arbitrary but finite order in q^2, p^2, and k^2

$$B_i{}^\alpha(q^2,p^2,k^2)=\sum_{l=0}^{L_i^\alpha}\sum_{m=0}^{M_i^\alpha}\sum_{n=0}^{N_i^\alpha}(A_i{}^\alpha)_{l,m,n}(q^2)^l(p^2)^m(k^2)^n.\quad(3.1)$$

This is a generalization of the smoothness approximation used by Schnitzer and Weinberg.[1] Inserting the expansion (3.1) into the conservation equations (2.19) and (2.20) leads to the result that the constant terms in $B_1{}^\alpha$ and $B_2{}^\alpha$ vanish.

Further restrictions on the coefficients occurring in (3.1) may be obtained by application of the Bjorken limit[4] to the functions

$$D_\lambda{}^{(\rho)\alpha}\equiv\lim_{p^2\to-m_\rho^2}[g_\rho{}^{-1}(p^2+m_\rho{}^2)\epsilon_{(\rho)}{}^\nu(p)N_{\nu\lambda}{}^\alpha],\quad(3.2)$$

$$D_\nu{}^{(\omega)}\equiv\lim_{k^2\to-m_\omega^2}[(g_\omega{}^\alpha)^{-1}(k^2+m_\omega{}^2)\epsilon_{(\omega)}{}^\lambda(k)N_{\nu\lambda}{}^\alpha],\quad(3.3)$$

$$D_\nu{}^{(\phi)}\equiv\lim_{k^2\to-m_\phi^2}[(g_\phi{}^\alpha)^{-1}(k^2+m_\phi{}^2)\epsilon_{(\phi)}{}^\lambda(k)N_{\nu\lambda}{}^\alpha],\quad(3.4)$$

$$D_{\nu\lambda}{}^{(\pi)\alpha}\equiv\lim_{q^2\to-m_\pi^2}[(F_\pi m_\pi{}^2)^{-1}(q^2+m_\pi{}^2)N_{\nu\lambda}{}^\alpha],\quad(3.5)$$

where, in (3.3) and (3.4), there is no summation over α.

The Bjorken limit applied to $D_\lambda{}^{(\rho)\alpha}$, for example, results in

$$D_\lambda{}^{(\rho)\alpha}\sim\frac{1}{q_0}\int d^3x\,e^{i\mathbf{q}\cdot\mathbf{x}}\langle\rho(p)|[\partial^\mu A_\mu{}^3(\mathbf{x},0),V_\lambda{}^\alpha(0)]|0\rangle+O(1/q_0{}^2),\quad(3.6)$$

as $q_0\to\infty$, with $\mathbf{q}$ and p being held fixed. Similar expressions may be derived for $D_\nu{}^{(\omega,\phi)}$ and $D_{\nu\lambda}{}^{(\pi)\alpha}$. Assuming the algebra-of-fields equal-time commutation relations[5]

$$[\partial^\mu A_\mu{}^3(\mathbf{x},0),V_i{}^\alpha(0)]=0\quad(3.7a)$$

and

$$[V_i{}^3(\mathbf{x},0),V_j{}^\alpha(0)]=0\quad(3.7b)$$

then restricts the combination of $B_2{}^\alpha$, $B_6{}^\alpha$, and $B_7{}^\alpha$ occurring in (2.22) to be of the form

$$B_2{}^\alpha-(k\cdot q)B_6{}^\alpha-q^2B_7{}^\alpha=a^\alpha q^2+b^\alpha p^2+c^\alpha k^2,\quad(3.8)$$

subject to the conditions

$$G_{\alpha\beta}(a^\beta+c^\beta)=0,\quad(3.9a)$$

$$g_\omega{}^\alpha(a^\alpha+b^\alpha)=0,\quad(3.9b)$$

$$g_\phi{}^\alpha(a^\alpha+b^\alpha)=0,\quad(3.9c)$$

$$G_{\alpha\beta}(b^\beta+c^\beta)=0,\quad(3.9d)$$

where

$$G_{\alpha\beta}=\int d\mu^2\,\rho_{\alpha\beta}(\mu^2)\quad(3.10)$$

is assumed to exist, and where $g_\omega{}^\alpha$ is defined by

$$\sqrt{\tfrac{1}{3}}\langle 0|V_\mu{}^\alpha(0)|\omega(k)\rangle=g_\omega{}^\alpha\epsilon_\mu{}^{(\omega)}(k),\quad(3.11)$$

with a similar definition for $g_\phi{}^\alpha$.

Although the smoothness approximation (3.1) is motivated in part by the concept of vector dominance, we have not made explicit use of vector dominance in arriving at Eqs. (3.8) and (3.9). If we now assume ω and

[5] T. D. Lee, S. Weinberg, and B. Zumino, Phys. Rev. Letters 18, 1029 (1967).

ϕ dominance of $G_{\alpha\beta}$, that is,

$$G_{\alpha\beta}=g_\omega{}^\alpha g_\omega{}^\beta+g_\phi{}^\alpha g_\phi{}^\beta\,, \tag{3.12}$$

it is straightforward to show that either

$$a^\alpha=b^\alpha=c^\alpha=0 \tag{3.13}$$

or

$$g_\omega{}^0/g_\phi{}^0=g_\omega{}^8/g_\phi{}^8. \tag{3.14}$$

However, (3.14) implies that the mixing angles[6] θ_Y and θ_N are related by

$$\theta_Y-\theta_N=\tfrac{1}{2}\pi\pm n\pi\,, \tag{3.15}$$

which is not allowed, since $m_\omega{}^2\neq m_\phi{}^2$.[7] Thus, we have shown that the smoothness assumption (3.1) and the algebra-of-fields commutation relations lead to (3.8) and (3.9), which in the ω-ϕ dominance approximation result in the vertex function $\Gamma_{\nu\lambda}{}^\alpha$, defined in (2.4), vanishing identically.

4. DISCUSSION AND CONCLUSIONS

In the above we have shown that for *any* arbitrary, though finite, expansion in q^2, p^2, and k^2 of the scalar functions $B_i{}^\alpha$ defined in (2.13), the algebra-of-fields commutation relations together with the Bjorken limit applied to the three-point functions defined in (3.2)–(3.5) lead to the proper vertex function $\Gamma_{\mu\lambda}{}^\alpha$ being expressible as

$$\Gamma_{\mu\lambda}{}^\alpha(q,p)=(g_A{}^{-1}C_A/F_\pi)\times(a^\alpha q^2+b^\alpha p^2+c^\alpha k^2)\epsilon_{\mu\lambda\rho\sigma}q^\rho p^\sigma\,, \tag{4.1}$$

subject to the conditions (3.9). This is not sufficient to allow us to conclude that the vertex function $\Gamma_{\mu\lambda}{}^\alpha$ vanishes, although we can see clearly from our method that assuming, further, the complete dominance of the spectral function $\rho_{\alpha\beta}(\mu^2)$ by the ω and ϕ mesons is sufficient to give such a result. It is interesting to note that we make no use of ρ dominance in this approach.

In two recent papers[2,3] the authors have also noticed that the $\omega\rho\pi$ vertex vanishes when the algebra-of-fields commutation relations are assumed to hold. The paper by Perrin[2] treats the three-point function by representing the current commutators by means of the Dyson representation and, assuming ρ, π, ω, and ϕ dominance, arrives at $\omega\to\rho\pi$, $\phi\to\rho\pi$, etc., being forbidden. He shows that $U(12)$ commutation relations allow a nonzero result for the decays to be obtained within the pole-dominance and smoothness approximations. Brown and West,[3] on the other hand, assume unsubtracted dispersion relations when an arbitrary linear combination of q^2 and p^2 is held fixed, impose the smoothness and pole-dominance assumptions, and then arrive at the result $g_{\omega\rho\pi}=0$. They obtain a nonzero result by using a model in which $[\partial^\mu A_\mu{}^i(\mathbf{x},0),V_i{}^\alpha(0)]\neq0$. From our work it is clear that the concept of smoothness is not in general equivalent to the assumption of unsubtracted dispersion relations in the sense of Brown and West.[3] When the algebra-of-fields commutation relations are assumed, the two concepts are equivalent, but if, for example, the quark field commutation relations are taken to hold, it is easy to see the nonequivalence, for in our method one could still have smoothness and the Bjorken limit being satisfied, yet we should require, in general, one subtraction in the fixed-μ dispersion relations of Brown and West.

Although the results proved in Sec. 3 are quite general for a smoothness expansion in q^2, p^2, and k^2 and would seem to support the conclusions of Perrin[2] and of Brown and West,[3] it seems to us that they also suggest plausible arguments for not abandoning the algebra-of-fields commutation relations because of the observed (nonzero) $\omega\to\pi\gamma$, etc., decays. From the relative simplicity of our method it is easy to see that nonzero $\omega\to\rho\pi$, $\omega\to\pi\gamma$, etc., decays are possible, when the algebra-of-fields commutation relations hold, if (i) strict ω and ϕ dominance is not a complete representation for $G_{\alpha\beta}$ defined in (3.10) and/or (ii) one can find functions which are smooth enough to be included in the expansion (3.1) and which tend to zero at least as fast as $q_0{}^{-1}$ when $q_0\to\infty$. Indeed, a function like

$$\int dq'^2\,dp'^2\,\frac{\rho(q'^2,p'^2)}{(q'^2+q^2)(p'^2+p^2)}+(\text{cyclic permutations of }q^2,\ p^2,\text{ and }k^2),$$

where

$$\rho(q'^2,p'^2)=[(q'^2+4m_\pi{}^2)^{-1/2}(q'^2+16m_\pi{}^2)^{-1/2}(q'^2+32m_\pi{}^2)^{-1/2}]\times[(p'^2+4m_\pi{}^2)^{-1/2}(p'^2+16m_\pi{}^2)^{-1/2}(p'^2+32m_\pi{}^2)^{-1/2}]\,,$$

would probably be "smooth" enough to satisfy condition (ii), as well as having an acceptable singularity structure in the complex q^2-p^2-k^2 space.

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[6] N. Kroll, T. D. Lee, and B. Zumino, Phys. Rev. **157**, 1376 (1967).

[7] If (3.14) holds, one cannot simultaneously satisfy all of the conditions arising from considering matrix elements of the type $\langle 0|j_\mu{}^{(\omega,\phi)}|\omega\rangle$, etc., unless $m_\omega{}^2=m_\phi{}^2$.