Closed Forms for the Scattering Amplitudes and Bootstrap Based on Sum Rules*

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An infinite class of solutions of the superconvergence equations is constructed with no restrictions on masses and Regge residues. Implications for bootstraps based on sum rules are discussed.

IN the last year the use of superconvergence relations[1] together with Regge behavior and the duality principle gave a surprisingly successful method to study properties of hadron amplitudes. In particular, the application of this approach to the reactions $\pi\pi\to\pi\omega$, $\pi\pi\to\pi A_2$,[2] $\pi\pi\to\pi H$,[3] and other reactions[4] led to results in very good agreement with experiment (in whatever indirect consequences which can be tested).

The next step was advanced by Veneziano.[5] In his paper a closed form of the amplitude was proposed for $\pi\pi\to\pi\omega$, which satisfies all the requirements of analyticity, crossing symmetry, Lorentz invariance, and Regge behavior for linearly rising trajectories. In particular, a constraint was to be imposed, relating the slope of the Regge trajectories and the sum of the external masses, in order to eliminate unwanted poles at even values of angular momentum. Another form for the amplitude was proposed by Virasoro[6] and by one of us,[7] which, as discussed in Ref. 6, does not require any *ad hoc* constraint involving the particle masses. A different form for the Regge residue is implied in this case, which, in general, shows a Mandelstam-Wang mechanism at sense-nonsense points, and goes over into other mechanisms for particular values of the external masses.

Having these two solutions, one immediately wonders how restrictive the starting assumptions really are. In this paper, by constructing an infinite class of solutions for the amplitude that do not depend on the masses of the external particles, and that do not imply a particular form of the Regge residue, we show that the superconvergence equations as they stand do not contain any bootstrap information.

We define the invariant amplitude for $\pi\pi\to\pi\omega$ as usual:

$$T=\epsilon_{\mu\nu\rho\sigma}e_\mu{}^{(\lambda)}p_1{}^\nu p_2{}^\rho p_3{}^\sigma A(s,t,u)\,, \tag{1}$$

where the p_i are the pion momenta, and e is the polarization vector of the ω. Here $A(s,t,u)$ is completely symmetric in s, t, and u. We assume that only the ρ trajectory, together with its daughters, contributes to this process. We want an expression for A which is (a) Lorentz invariant, (b) crossing symmetric, (c) analytic for all finite values of s, t, and u apart from simple poles (in the narrow-resonance limit) at odd values of α with polynomial residua of degree $\alpha-1$, and (d) Regge behaved, i.e., $A(s,t,u)\underset{\substack{s\to\infty\\ t\text{ fixed}}}{\sim}\beta(t)s^{\alpha(t)-1}$. Our expression for A is

$$A(s,t,u)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}c_{nm}\frac{\Gamma(n+\frac12[1-\alpha(s)])\Gamma(n+\frac12[1-\alpha(t)])\Gamma(n+\frac12[1-\alpha(u)])}{\Gamma(n+m+\frac12[2-\alpha(s)-\alpha(t)])\Gamma(n+m+\frac12[2-\alpha(s)-\alpha(u)])\Gamma(n+m+\frac12[2-\alpha(t)-\alpha(u)])}. \tag{2}$$

The only limitations on c_{nm} are imposed by the convergence of the series. The only necessarily nonvanishing coefficient is c_{00} (this insures the existence of the ρ pole). The Regge trajectories are supposed to have a real part which is linear in s, with a positive imaginary part growing at infinity at any lower than linear rate.

Conditions (a)-(c) are easily seen to be fulfilled. We only remark that no double poles appear since we have demanded $m\leqslant n$. Furthermore, the residues of the simple poles are indeed polynomials in the narrow-resonance limit, i.e., when $\mathrm{Im}\alpha\to0$, because in this case the sum $\alpha(s)+\alpha(t)+\alpha(u)$ approaches a constant:

$$\alpha(s)+\alpha(t)+\alpha(u)=\gamma\,, \tag{3}$$

with

$$\gamma=3\alpha_0+\alpha'\sum_i m_i{}^2. \tag{4}$$

The asymptotic behavior is found to be

$$A(s,t,u)\simeq-\pi[\tfrac12\alpha(s)]^{\alpha(t)-1}[\tan\tfrac12\pi\alpha(s)+\tan\tfrac12\pi\alpha(t)]\times\sum_{n=0}^{\infty}c_{n0}(-1)^{n+1}\frac{1}{\Gamma(\frac12+\frac12\alpha(t)-n)\Gamma(n+1+\frac12\alpha(t)-\frac12\gamma)}. \tag{5}$$

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[2] See Ref. 1; also M. Ademollo, H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Rev. **176**, 1904 (1968).
[3] M. Bishari, H. R. Rubinstein, A. Schwimmer, and G. Veneziano, Phys. Rev. **176**, 1926 (1968).
[4] See Ref. 3; also C. Schmid, Phys. Rev. Letters **20**, 628 (1968).
[5] G. Veneziano, Nuovo Cimento **52A**, 190 (1968).
[6] M. A. Virasoro, Phys. Rev. **177**, 2309 (1968).
[7] H. R. Rubinstein (unpublished).

Note that, since $\alpha(s)$ has a positive imaginary part which diverges at infinity,[5] $\tan\frac{1}{2}\pi\alpha(s)\to i$ and the correct signature factor is reproduced. Terms with $m\neq 0$ lead to nonasymptotic contributions.

Note that only the terms with $n=0, 1, 2, \cdots, N$ have the pole at $\alpha(s)=2N+1$. The residue is, in the narrow-resonance limit, a polynomial in t of degree $2N$, with coefficients depending on c_{nm} ($n=0, 1, 2, \cdots, N$), symmetric under the exchange $t\leftrightarrow u=\sum_i m_i^2 -2N-1-t$. Hence one can adjust the $2N+1$ numbers c_{Nm} in order to construct the most general polynomial residue, without altering the residua of poles at lower values of $\alpha(s)$. Thus the imaginary part of our amplitude is the most general one, in the narrow-resonance limit. However, the real part is still not said to be as general.[8]

The triple-product formula (TPF) of Refs. 6 and 7 is obtained when c_{00} is the only nonzero coefficient in (2). The Veneziano formula coincides with the TPF for $\gamma=2$, i.e., it is the only case when it is a solution of the problem.

It is also clear that no limitations so far as mechanisms at sense-nonsense points are concerned are implied by our solution. For instance, we can make the asymptotic amplitude to vanish at $\alpha(t)=0$ by fixing the ratio of c_{00} and c_{10}, and putting all other c_{nm} to zero, or similarly, by adding more terms, to make it vanish at any finite number of negative even values of $\alpha(t)$. Also, mechanisms implying multiple zeros, like Chew's mechanism, are contained in the expression (2). Of course, this implies restrictions on possible forms of the residues of the lowest-lying resonances, thus explaining the results obtained by the sum-rule technique.

Since the Γ functions converge very rapidly to their asymptotic limit, our solutions have the property of satisfying the superconvergence sum rules to a good approximation even when the cutoff energy $\bar{k}$ is chosen very low. Following the method of Refs. 2 and 5, we find the condition for obeying all superconvergence conditions in our case to be (when only $c_{00}\neq 0$)

$$\phi_k\equiv\frac{\Gamma(\bar{k}+\frac{3}{2}-\frac{1}{2}\gamma)\Gamma(\bar{k}+1)}{\Gamma(\bar{k}+2+\frac{1}{2}\alpha(t)-\frac{1}{2}\gamma)\Gamma(\bar{k}+\frac{3}{2}+\frac{1}{2}\alpha(t))}\times[1+\bar{k}+\tfrac{1}{4}\alpha(t)-\tfrac{1}{4}\gamma]^{\alpha(t)+1}=1. \quad (6)$$

This function is indeed constant and equal to 1 in a region around $\alpha(t)=0$ increasing with $\bar{k}$ and the equality holds for a large range of values of γ.

Explicit solutions of the form (2) can be easily derived for other processes where more spin and/or isospin amplitudes appear and the crossing properties are more complicated. For instance, in the case of $\pi\pi\to\pi H$ (H is a meson with $J^{PG}=1^{+-}$, $T=0$, whose experimental confirmation is still questionable), we have two invariant amplitudes

$$T=ie_\mu^{(\lambda)}[A(s,t,u)(p_2^\mu+p_3^\mu)+B(s,t,u)(p_2^\mu-p_3^\mu)]. \quad (7)$$

The crossing relations are

$$\begin{aligned} A(s,t,u)&=A(u,t,s)\,,\\ B(s,t,u)&=-B(u,t,s)\,,\\ A(s,t,u)+B(s,t,u)&=A(s,u,t)+B(s,u,t)\,,\\ 3A(s,t,u)-B(s,t,u)&=-3A(s,u,t)+B(s,u,t)\,, \end{aligned} \quad (8)$$

and the asymptotic behavior is

$$A\simeq\beta_A(t)s^{\alpha(t)-1},\quad B\simeq\beta_B(t)s^{\alpha(t)}. \quad (9)$$

If $F(s,t,u)$ is a function of exactly the same form as in (2), then a solution for A and B is[9]

$$\begin{aligned} A(s,t,u)&=F(s,t,u)[\alpha(s)+\alpha(u)-2\alpha(t)]\,,\\ B(s,t,u)&=3F(s,t,u)[\alpha(s)-\alpha(u)]\,. \end{aligned} \quad (10)$$

In conclusion, we have produced a variety of forms for the amplitudes of the simplest meson processes, which comply with all the requirements of crossing, analyticity, and Regge behavior and lead to no restrictions on masses or residues, thus stressing once more that a program of bootstrap based on these principles above cannot give any information on hadron dynamics. The question of what has been accomplished in the works with the sum rules is then relevant. First, contrary to what was once believed, one has imposed the Gell-Mann mechanism and not obtained it. Second, if we accept the universal validity of this form of the Regge residue, and if we demand that the superconvergence sum rules hold for values of the cutoff as low as just above the first resonance, then all the results derived there remain valid. This prescription, through arbitrary *a priori*, presumably reflects a true property of the real scattering amplitudes, which may be very different in form from any proposed until now, and which for their characterization probably need the use of the unitarity condition.

Note added in proof. Since there is interest in using double-product forms with no restrictions on trajectories spaced by one unit of J, it should be noticed that a formula analogous to (2) can be derived as well for double-product forms. These expressions have also the most general imaginary part by the same arguments.

[8] On different grounds, the existence of a large number of solutions to the superconvergence equations has been discussed by J. Kupsch, Bonn Report, 1968 (unpublished); see also N. N. Khuri, Rockefeller University Report (unpublished).

[9] The Veneziano form in this case predicts the wrong mass for the H ($m_H=m_\omega$).