Implications of the Optical Model for the S-Matrix Approach to High-Energy pp Scattering

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The optical model is shown to indicate that the high-energy large-momentum-transfer pp elastic scattering amplitude obeys the Mandelstam representation and that the corresponding absorptive parts and double spectral functions are highly oscillatory.

At present, very accurate measurements of high-energy elastic pp differential cross sections as small as 10^{-33} cm^2/sr and at momentum transfers as large as 15 $(\text{GeV}/c)^2$ are being carried out.[1–3] One naturally wonders whether such an extensive experimental effort provides any insight into some of the basic postulates of S-matrix theory[4] such as Mandelstam analyticity and polynominal boundedness at high-energy and large-momentum transfer. Using analyticity in the $\cos\theta$ plane and boundedness conditions two important results have been obtained, which in fact are consistent with experimental observations on pp scattering. One is a lower bound on the fastest decreasing rate of the amplitude,[5] and the other is the asymptotic dependence on the scattering angle at high energies.[6] To go, however, beyond the consistency checks one has to devise a model which reproduces the main features of the actual data, and then examine what light the model throws on the basic postulates. An optical model has been found to provide a good description of present high-energy large-momentum-transfer pp elastic scattering data in terms of simple physical concepts.[7–9] The model explicitly satisfies the bound on the fastest decreasing rate of the amplitude and leads precisely to the theoretical asymptotic angular dependence.[9] It is natural, therefore, to ask ourselves what connection such a phenomenological model has with S-matrix theory and if there is any connection, what are its implications. This question assumes further importance in view of recent experimental evidence of diffractionlike behavior[2] at high-momentum transfers, indicating that an optical-model description may be more appropriate. The purpose of this paper is to discuss this particular question.

The Born amplitude which the optical model provides[9] is given by

$$f_B(s,t)=[g(s)/(t-\mu^2)]\beta(\mu^2-t-t^2/s)^{1/2}\times K_1[\beta(\mu^2-t-t^2/s)^{1/2}],\quad(1)$$

where s = square of c.m. energy, $-t$ = square of momentum transfer; $g(s)$ is the complex energy-dependent coupling strength, β determines the size of the hadronic distribution, and μ is the mass of the particle exchanged. $K_1(z)$ is the modified Bessel function of the second kind. Using the representation

$$(-z)^{1/2}K_1[\beta(-z)^{1/2}]=\tfrac{1}{2}\int_0^\infty \frac{x^{1/2}J_1(\beta x^{1/2})}{x-z}dx,\quad(2)$$

we can write for s in the physical region

$$f_B(s,t)=\tfrac{1}{2}g(s)\int_{t_0(s)}^\infty dt'\beta\left(t'+\frac{t'^2}{s}-\mu^2\right)^{1/2}J_1\left[\beta\left(t'+\frac{t'^2}{s}-\mu^2\right)^{1/2}\right]\left[\frac{1}{(t'-\mu^2)(t'-t)}-\frac{1}{(t'+s+\mu^2)(t'+s+t)}\right]+\frac{g(s)}{t-\mu^2}\beta\left(-\frac{\mu^4}{s}\right)^{1/2}K_1\left[\beta\left(-\frac{\mu^4}{s}\right)^{1/2}\right];\quad(3)$$

$t_0(s)=\frac{1}{2}[-s+(s^2+4\mu^2s)^{1/2}]$. This equation shows that the amplitude given by Eq. (1) is analytic in t plane with a right-hand cut from $t_0(s)$ to $+\infty$, and a left-hand cut from $-t_0(s)-s$ to $-\infty$, and a pole at $t=\mu^2$. For s large $t_0(s)\to\mu^2$, so that the analyticity in the t plane or the $\cos\theta$ plane is in agreement with the corresponding analyticity

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following from the Mandelstam representation. For s asymptotic,[10] Eq. (3) simplifies to

$$f_B(s,t)=\tfrac{1}{2}g(s)\int_{\mu^2}^{\infty}dt'\beta(t'-\mu^2)^{1/2}J_1[\beta(t'-\mu^2)^{1/2}]\frac{1}{(t'-\mu^2)(t'-t)}$$
$$-\tfrac{1}{2}g(s)\int_{\Sigma+\mu^2}^{\infty}du'\beta(u'-\Sigma-\mu^2)^{1/2}J_1[\beta(u'-\Sigma-\mu^2)^{1/2}]\frac{1}{(u'-\Sigma+s+\mu^2)(u'-u)}+\frac{g(s)}{t-\mu^2};\quad(4)$$

here $s+t+u=\Sigma$ and $\Sigma=4m^2$. Thus the Born amplitude satisfies a fixed-energy dispersion relation with t- and u-channel absorptive parts given by

$$A_t(s,t)=\tfrac{1}{2}\pi g(s)\beta J_1[\beta(t-\mu^2)^{1/2}]/(t-\mu^2)^{1/2}-\pi g(s)\delta(t-\mu^2),\quad(5a)$$

$$A_u(s,u)=-\tfrac{1}{2}\pi g(s)\beta(u-\Sigma-\mu^2)^{1/2}\times J_1[\beta(u-\Sigma-\mu^2)^{1/2}]/(u-\Sigma+s+\mu^2).\quad(5b)$$

In potential theory it has been proved[11] that $g(s)$ has a right-hand cut starting from an inelastic threshold up to ∞. If we assume this to be true in our present case, then the absorptive parts $A_t(s,t)$ and $A_u(s,u)$ can be written as dispersion integrals in s, with double spectral functions given by

$$\rho(s,t)=\tfrac{1}{2}\pi\,\mathrm{Im}g(s)\beta J_1[\beta(t-\mu^2)^{1/2}]/(t-\mu^2)^{1/2}-\pi\,\mathrm{Im}g(s)\delta(t-\mu^2),\quad(6a)$$

$$\rho(s,u)=\tfrac{1}{2}\pi\,\mathrm{Im}g(s)\beta(u-\Sigma-\mu^2)^{1/2}\times J_1[\beta(u-\Sigma-\mu^2)^{1/2}]/(\Sigma-u-\mu^2-s),\quad(6b)$$

and

$$\rho(t,u)=-\tfrac{1}{2}\pi^2\beta(u-\Sigma-\mu^2)^{1/2}J_1[\beta(u-\Sigma-\mu^2)^{1/2}]\times g(\Sigma-u-t)\delta(t-\mu^2).\quad(6c)$$

Therefore, the Born amplitude $f_B(s,t)$ obeys a Mandelstam representation with highly oscillatory double-spectral functions[12] given by Eqs. (6).

In actual analysis of pp elastic scattering, a distorted-wave Born approximation was used. We point out that for large-momentum transfer this approximation yields an amplitude[9] which is actually the Born amplitude multiplied by the factor $(1-\sigma_T{}^d/\pi R^2)$. Asymptotically, this factor is a constant. Hence, the high-energy large-momentum-transfer scattering amplitude in this model obeys the Mandelstam representation.

The optical model implies that the electromagnetic form factor of the nucleon falls off exponentially in momentum transfer for $t\to-\infty$, vanishes for $|t|\to\infty$, and has only right-hand cut.[13] The spectral function of such a form factor is highly oscillatory. Oscillatory spectral functions are, indeed, needed to remove the difficulty that current-algebra models predict infinite electromagnetic mass shifts.[14] Exponentially falling form factors are also obtained in crossing-symmetric bootstrap models[15] and in models of an infinitely composite nucleon.[16]

For t fixed and negative, and $s\to+\infty$, Eq. (1) shows that the s and t dependence separates out. Such factorization in Regge pole theory occurs only when one Regge trajectory with all the infinite cuts associated with it are taken into account.[17] Therefore, the optical model indicates that in applying Regge pole theory to high-energy large-momentum-transfer pp scattering, all the branch cuts associated with a Regge trajectory should be considered.

Summarizing, the important conclusions are as follows:

(1) If the optical-model description is valid, then the high-energy large-momentum-transfer pp elastic scattering amplitude obeys the Mandelstam representation.

(2) The t- and u-channel absorptive parts as well as the double spectral functions are highly oscillatory.

(3) The difficulty with current-algebra models in predicting finite electromagnetic mass shifts is probably due to the existence of oscillatory spectral functions.

(4) In applying Regge pole theory to large-momentum-transfer pp scattering, the contributions of all the infinite cuts associated with a Regge pole should be taken into account.

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[10] An estimate of how large s has to be in order to be considered asymptotic is obtained from the multiplication theorem of Bessel functions. It is $(1-0.25/\beta^2\mu^2)\gg(\beta^2 s)^{-1}$.

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[12] Similar highly oscillatory double spectral functions have been obtained by other authors starting from an ansatz for the s-channel absorptive part; see L. K. Chavda and S. D. Narayan, Nuovo Cimento **43A**, 382 (1966).

[13] All these properties follow by first noticing that $\beta(\mu^2-t)^{1/2}\times K_1[\beta(\mu^2-t)^{1/2}]$ is an analytic function with a right-hand cut and vanishes for $|t|\to\infty$. In this cut plane this function has no zero, since the argument of $(\mu^2-t)^{1/2}$ lies between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$. Hence, $F(t)=\{\beta(\mu^2-t)^{1/2}K_1[\beta(\mu^2-t)^{1/2}]\}^{1/2}$ is also analytic in the cut plane. Present experimental data on ep scattering are consistent with such form factors [M. M. Islam and Kashyap V. Vasavada, preceding paper, Phys. Rev. **178**, 2140 (1969).

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