

calculated by using the procedure of Ref. 30 and the ionization potential for iron of Ref. 32 is also given. The uncertainties in energy are estimated from the measured momentum width of the π -meson beam. Errors in range include straggling and the effects of the finite absorber thickness.

Figure 9 shows the experimental points on a plot of percentage straggling versus kinetic energy. For comparison, a curve based on a calculation of Refs. 32 and

33 for the percentage straggling in copper (which, in first approximation, should not be different from straggling in iron) is also given. The errors in energy are those noted above. Errors in percentage straggling arise from the finite absorber thickness and the limited statistical sample at each point.

³³ V. C. Burkig and K. R. MacKenzie, Phys. Rev. **106**, 848 (1957).

η Production in the Reaction $\pi p \rightarrow \pi p \eta$ at 1050 and 1170 MeV/c[†]

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We have investigated η production in the reaction $\pi^+ p \rightarrow \pi^+ p \eta$ at 1050 and 1170 MeV/c, and in the reaction $\pi^- p \rightarrow \pi^- p \eta$ at 1170 MeV/c. We obtain the following partial cross sections for the reaction $\pi^\pm p \rightarrow \pi^\pm p \eta$ followed by the charged decay $\eta \rightarrow \pi^+ \pi^- \pi^0$: $9.5 \pm 1.5 \mu\text{b}$ for 1050-MeV/c incident π^+ ; $53.3 \pm 5.0 \mu\text{b}$ for 1170-MeV/c π^+ ; $15.9 \pm 1.9 \mu\text{b}$ for 1170-MeV/c π^- . From our π^+ data we find that we need four $I = \frac{3}{2}$ amplitudes in order to obtain good agreement with our mass and angular distributions and to reproduce the threshold behavior of the cross section. We find the dominating amplitude to be that representing the reaction $\pi^+ p \rightarrow \eta \Delta(1238)$, $\Delta(1238) \rightarrow \pi^+ p$, where the η and the $\Delta(1238)$ are produced in s wave. The four amplitudes are extrapolated to higher energies where the predicted mass and angular distributions and cross sections are compared with the available data. From our π^- data we find that in addition to the four $I = \frac{3}{2}$ amplitudes (determined from the π^+ data), one $I = \frac{1}{2}$ amplitude is required: the amplitude in which all particles are in relative S states. We find that this $I = \frac{1}{2}$ amplitude in fact dominates the π^- reaction. We see no evidence for production of an $N^*(1550) \rightarrow \eta p$ resonance. We are able to parametrize the cross sections for $\pi p \rightarrow \pi p \eta$ from threshold to 4 BeV/c.

I. INTRODUCTION

SINCE the discovery of the η meson in 1961,¹ there have been several experiments²⁻⁷ at various energies involving the reaction $\pi^\pm p \rightarrow \pi^\pm p \eta$. In all of these ex-

periments the reaction appears to proceed mainly via

$$\begin{aligned} \pi p &\rightarrow \eta \Delta(1238), \\ \Delta(1238) &\rightarrow \pi p. \end{aligned} \quad (1)$$

There have also been a few experiments⁸⁻¹¹ involving the reaction $\pi^- p \rightarrow \pi^- p \eta$.

Reaction (1) is unusual in that there are no pseudo-scalar or vector mesons which can be exchanged in the t channel. Therefore, such models as the one-pion-exchange model or the vector-exchange model of

(1962); C. Alff-Steinberger, D. Berley, D. Colley, N. Gelfand, D. Miller, U. Nauenberg, J. Schultz, T. H. Tan, H. Brugger, P. Kramer, and R. Plano, Phys. Rev. **145**, 1072 (1966).

⁶ Maris Abolins, Richard L. Lander, Werner A. W. Melhop, Nguyen-huu Xuong, and Philip M. Yager, Phys. Rev. Letters **11**, 381 (1963).

⁷ Benjamin Ching-Chun Shen, Lawrence Radiation Laboratory Report No. UCRL-16170, 1965 (unpublished).

⁸ D. D. Carmony, F. Grard, R. T. Van de Walle, and Nguyen-huu Xuong, in *Proceedings of the International Conference on High-Energy Physics, 1962*, edited by J. Prentki (CERN Scientific Information Service, Geneva, 1962), p. 44.

⁹ P. H. Satterbloom, W. D. Walker, and A. R. Irwin, Phys. Rev. **134**, B207 (1964).

¹⁰ Suh Urk Chung, Orin I. Dahl, Janos Kirz, and Donald H. Miller, Phys. Rev. **165**, 1491 (1968).

¹¹ G. Ascoli, H. B. Crawley, D. W. Mortara, A. Shapiro, C. A. Bridges, B. I. Eisenstein, U. E. Kruse, E. D. Schafer, and B. Terreault, Phys. Rev. Letters **20**, 1321 (1968).

[†] Work done under the auspices of the U. S. Atomic Energy Commission.

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² Horst W. J. Foelsche and Henry L. Kraybill, Phys. Rev. **134**, B1138 (1964).

³ P. Daronian, A. Daudin, M. A. Jabiol, C. Lewin, C. Kochowski, B. Ghidini, S. Mongelli, and V. Picciarelli, Nuovo Cimento **41**, 502 (1966).

⁴ Frederick E. James and Henry L. Kraybill, Phys. Rev. **142**, 896 (1966).

⁵ C. Alff, D. Berley, D. Colley, N. Gelfand, U. Nauenberg, D. Miller, J. Schultz, J. Steinberger, T. H. Tan, H. Brugger, P. Kramer, and R. Plano, Phys. Rev. Letters **9**, 322 (1962); **9**, 325

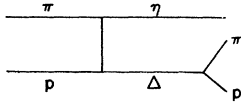


FIG. 1. Simple exchange diagram for $\pi p \rightarrow \Delta \eta$.

Stodolsky and Sakurai¹² do not apply to reaction (1). As can be seen from Fig. 1, the required quantum numbers for the exchange particle are $I^G=1^-$ and $J^P=0^+, 1^-, 2^+, \dots$. Possible mesons with these quantum numbers are the massive $A_2(1300)$, $J^P=2^+$, and $\pi_N(1016)$, $J^P=0^+$.

The A_2 meson is well established, and the quantum numbers are now fairly well agreed upon. (There is some evidence, however, that there might be more than one meson in this region.¹³) The compiled branching ratio for A_2 decay into $\eta\pi$ is listed¹⁴ as $(11 \pm 2)\%$, indicating that the coupling constant at the upper vertex is small.

The other meson candidate for the exchange particle, $\pi_N(1016)$,¹⁴ is seen as a $\bar{K}K$ enhancement near threshold and may be interpreted as due to a large scattering length. A reported $\eta\pi$ enhancement¹⁵ in this region has not been confirmed.

In addition to meson exchange in the t channel, it is also possible to have baryon exchange in the u channel. This would tend to peak the backward region of the production angular distribution. For baryon exchange, the nucleon and the $N^*(1550)$, S_{11} resonance have the required quantum numbers.

The fact that the masses of all the proposed exchange particles are large has the consequence that the interaction is of shorter range than interactions to which the peripheral model is usually applied. It is not surprising, therefore, that experiments involving reaction (1) are not highly peripheral, that is, the production angular distributions do not exhibit the great forward-peaking characteristic of peripheral interactions. In fact, in the low-energy data of Foelsche and Kraybill² and of this experiment, the production angular distributions are fairly flat.

We also note that in addition to $\Delta(1238)$ production the reaction $\pi p \rightarrow \pi p \eta$ might proceed via

$$\pi p \rightarrow \pi N^*(1550), \\ N^*(1550) \rightarrow \eta p.$$

¹² Leo Stodolsky and J. J. Sakurai, Phys. Rev. Letters **11**, 90 (1963); Leo Stodolsky, Phys. Rev. **134**, B1099 (1964).

¹³ R. Leprat, C. A. Tolstrup, P. Schübelin, C. Nef, M. Martin, B. C. Maglič, W. Kienzle, M. N. Focacci, L. Dubal, and G. Chikovani, Phys. Letters **22**, 714 (1966); G. Chikovani, M. N. Focacci, W. Kienzle, C. Lechanoine, B. Leprat, B. Maglič, M. Martin, P. Schübelin, L. Dubal, M. Fisher, P. Grieder, H. A. Neal, and C. Nef, *ibid.* **25**, B44 (1967); D. R. O. Morrison, *ibid.* **25**, B238 (1967).

¹⁴ Arthur H. Rosenfeld, Naomi Barash-Schmidt, Angela Barbaro-Galtieri, LeRoy R. Price, Paul Söding, Charles G. Wohl, Matts Roos, and William J. Willis, Rev. Mod. Phys. **40**, 77 (1968).

¹⁵ J. Alitti, J. P. Baton, B. Deler, M. Neveu-Rene, J. Crussard, J. Ginestet, A. H. Tran, R. Gessaroli, and A. Romano, Phys. Letters **15**, 69 (1965); also A. Astier, J. Cohen-Ganouna, M. Della Negra, B. Maréchal, L. Montanet, M. Tomas, M. Baubillier, and J. Duboc, *ibid.* **25**, B294 (1967).

In this paper we adopt the phenomenological approach of describing the production process of the reaction $\pi p \rightarrow \pi p \eta$ in terms of low partial waves. We find that our π^+ data at 1050 and 1170 MeV/c may be adequately represented by s -wave production of the η and $\Delta(1238)$, plus an admixture in which the η is produced in s or p wave and the π^+ and p are in a relative S wave. At 1170 MeV/c we find the cross-section ratio $\sigma(\pi^- p \rightarrow \pi^- p \eta)/\sigma(\pi^+ p \rightarrow \pi^+ p \eta) = 0.30 \pm 0.04$ instead of the value 0.11 expected (on the basis of Clebsch-Gordan factors) if the $\pi^- p$ reaction proceeded only through the $I=\frac{3}{2}$ amplitudes. This indicates that in addition to the $I=\frac{3}{2}$ amplitudes there is also an appreciable $I=\frac{1}{2}$ contribution. Our π^- data are adequately represented by the addition of only one $I=\frac{1}{2}$ amplitude, the amplitude in which all particles are in relative S states.

II. EXPERIMENTAL DETAILS

The Alvarez 72-in. liquid-hydrogen bubble chamber was exposed to beams of both π^+ and π^- mesons at a momentum of 1170 MeV/c and to π^+ mesons at a momentum of 1050 MeV/c. The beam setup has been described previously^{16,17}; we present here in Fig. 2 only a schematic diagram of the beam optics.

A total of 240 000 pictures were taken at the three momenta. The film was scanned for four-prong events. A total of 14 000 four-prong events were found and fitted by the LRL spatial reconstruction and kinematical fitting program SIOUX. Most of the events, 12 000, fit the four-constraint (4C) reaction

$$\pi^\pm p \rightarrow \pi^\pm p \pi^+ \pi^- \quad (2)$$

with a χ^2 less than 16. These events were discarded. The remaining 2000 events were then examined on the scanning table to determine the proton. Out of these 2000 events, 800 fit reaction (2) with a χ^2 between 16 and 35. For these events, if the proton identification

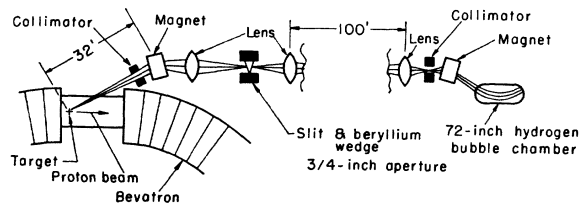


FIG. 2. Schematic diagram of the beam optics. A mass spectrometer, not shown in the diagram, is used in the π^+ beam to separate π^+ mesons from protons.

¹⁶ Sanford E. Wolf, Norbert Schmitz, Lester J. Lloyd, William Laskar, Frank S. Crawford, Jr., Janice Button, Jared A. Anderson, and Gideon Alexander, Rev. Mod. Phys. **33**, 439 (1961).

¹⁷ The beam contaminations have been determined by Fernand Grard and Gerald A. Smith, Phys. Rev. **127**, 607 (1962), for the 1050-MeV/c π^+ data; Frank S. Crawford, Jr., Fernand Grard, and Gerald A. Smith, *ibid.* **128**, 368 (1962), for the 1170-MeV/c π^+ data; Jared A. Anderson, Lawrence Radiation Laboratory Report No. UCRL-10838, 1963 (unpublished), for the 1170-MeV/c π^- data.

made on the scanning table agreed with the track identified as the proton in the 4C fit of reaction (2), then the event was discarded. In addition, events identified on the scanning table as having Dalitz pairs and events identified as having four charged pions were also discarded.

The remaining events were retained if they fit the reaction

$$\pi^\pm p \rightarrow \pi^\pm p \pi^+ \pi^- \pi^0$$

or

$$\pi^\pm p \rightarrow \pi^\pm p \pi^+ \pi^- \gamma$$

with a one-constraint (1C) χ^2 of less than 8.6 and if the

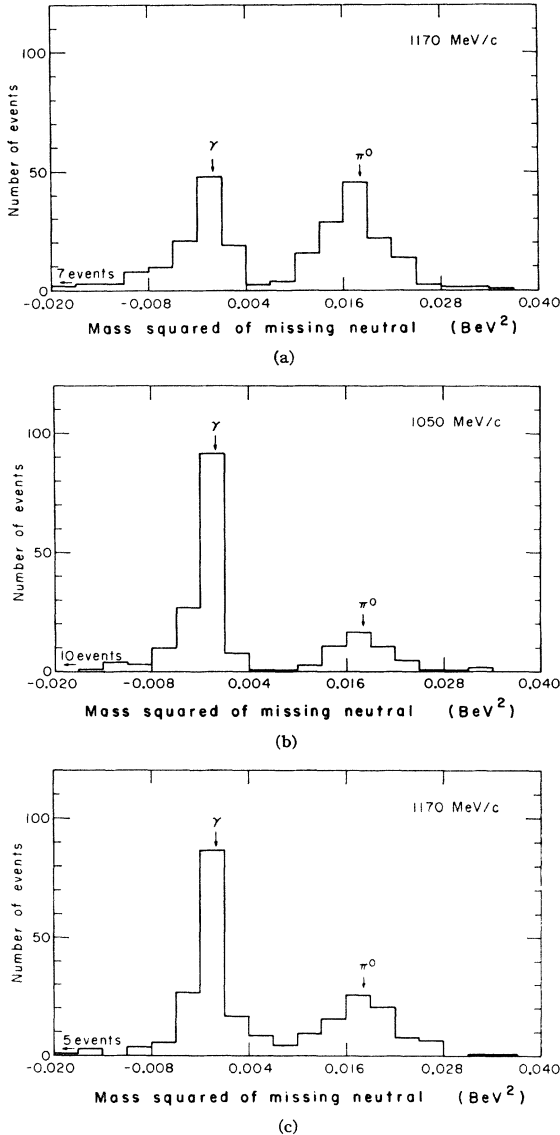


FIG. 3. Distributions in the square of the missing neutral mass (using unfitted quantities) for all four prongs which fit either of the reactions $\pi^\pm p \rightarrow \pi^\pm p \pi^+ \pi^-$ (π^0 or γ) with a $\chi^2 < 8.6$. (a) 1170-MeV/c incident π^+ ; (b) 1050-MeV/c incident π^+ ; and (c) 1170-MeV/c incident π^- .

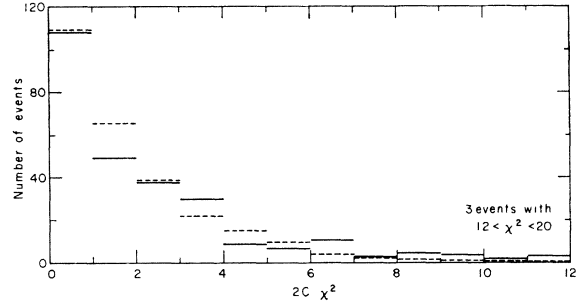


FIG. 4. 2C χ^2 distribution of the 273 events for reaction (3) (solid lines). The dashed lines represent the "theoretical" 2C χ^2 distribution normalized to 273 events.

events were inside the fiducial volume. The distributions in the missing neutral mass for these events are shown in Fig. 3. These events were then fitted to the two-vertex, two-constraint (2C) hypothesis

$$\begin{aligned} \pi^\pm p &\rightarrow \pi^\pm p \eta \\ \eta &\rightarrow \pi^+ \pi^- \pi^0 \end{aligned} \quad (3)$$

and

$$\begin{aligned} \pi^\pm p &\rightarrow \pi^\pm p \eta \\ \eta &\rightarrow \pi^+ \pi^- \gamma. \end{aligned} \quad (4)$$

An event was recorded as reaction (3) or (4) depending on which had the lower χ^2 . Events fitting reaction (4) are not used. All but four of the events that fit $\pi^\pm p \rightarrow \pi^\pm p \pi^+ \pi^- \pi^0$ also fit the 2C reaction (3) with $\chi^2_s < 20$ and are regarded as good η events. The four exceptions have χ^2_s over 200. Thus, essentially all events which fit the final-state hypothesis, $\pi p \pi^+ \pi^- \pi^0$, come from η 's. The number of events fitting reaction (3) are 135 for the 1170-MeV/c π^+ data, 51 for the 1050-MeV/c π^+ data, and 87 for the 1170-MeV/c π^- data.¹⁸ The 2C χ^2 distribution of these 273 events is given in Fig. 4.

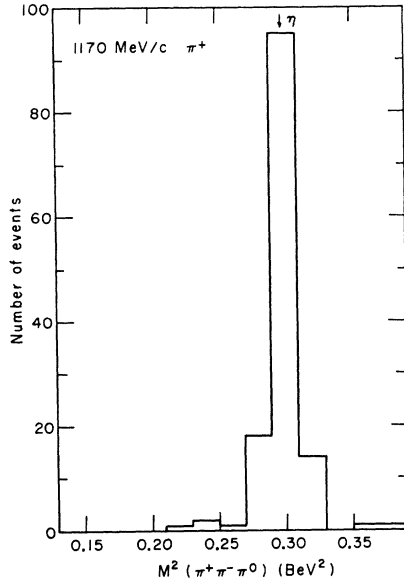
There is still the problem of "ambiguous" events. As there are two pions of the same charge in the final state, it is unknown which pion has come from the decay of the η ; thus reaction (3) can be fitted two ways. When both fits are good ($\chi^2 < 20$), the event is ambiguous. In this case, the interpretation with the lower χ^2 is taken to be the correct one. In the analysis which follows it is assumed that the right identification is always made. (By a Monte Carlo calculation we estimate that we choose the wrong pion about 3% of the time.)

Another possible way in which we could have obtained our final sample of events— $\pi^\pm p \rightarrow \pi^\pm p \eta$, $\eta \rightarrow \pi^+ \pi^- \pi^0$ —is to separate out the events with a missing γ

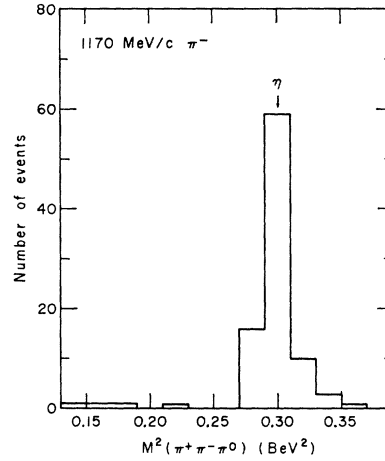
¹⁸ Most of these events have been analyzed previously in regard to η decay. See Earle C. Fowler, Frank S. Crawford, Jr., L. J. Lloyd, Ronald A. Grossman, and LeRoy R. Price, Phys. Rev. Letters **10**, 110 (1963); Frank S. Crawford, Jr., L. J. Lloyd, and Earle C. Fowler, *ibid.* **10**, 546 (1963); Frank S. Crawford, Jr., Ronald A. Grossman, L. J. Lloyd, and LeRoy R. Price, *ibid.* **11**, 564 (1963); Columbia-Berkeley-Purdue-Wisconsin-Yale Collaboration, Phys. Rev. **149**, 1044 (1966).

TABLE I. Scanning information and partial cross sections for $\pi p \rightarrow \pi p \eta$, $\eta \rightarrow \pi^+ \pi^- \pi^0$.

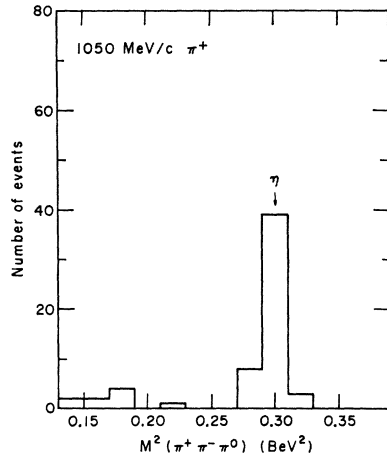
	$\pi^+p \rightarrow \pi^+p\eta$ $\eta \rightarrow \pi^+\pi^-\pi^0$ 1170 MeV/c	$\pi^-p \rightarrow \pi^-p\eta$ $\eta \rightarrow \pi^+\pi^-\pi^0$ 1170 MeV/c		$\pi^+p \rightarrow \pi^+p\eta$ $\eta \rightarrow \pi^+\pi^-\pi^0$ 1050 MeV/c
		Part I	Part II	
Total No. of pictures scanned	34 928	18 309	81 858	98 291
Beam tracks per frame	17.22±0.39	17.4±0.5	14.90±0.21	13.76±0.90
Length of beam track in fiducial volume	157.3±0.1 cm	145 cm	147.8 cm	157.3±0.1 cm
Total track length scanned (uncorrected)	(94.61±2.14)×10 ⁶ cm	(46.19±1.32)×10 ⁶ cm	(180.27±2.51)×10 ⁶ cm	(212.75±13.92)×10 ⁶ cm
Beam contamination ^a	5.1±2.0%	3.0±1.6%	3.0±1.6%	6.7±2.1%
Attenuation due to strong interactions ^b	7.6%	8.6%	8.6%	6.8%
Total corrected track length	(82.96±2.56)×10 ⁶ cm	(40.95±1.35)×10 ⁶ cm	(159.82±3.45)×10 ⁶ cm	(185.00±12.82)×10 ⁶ cm
No. of observed events	135	87		51
Scanning efficiency	95.0±1.9%		85.1±3.8%	90.4±4.3%
Selection criteria efficiency	91.5%		91.5%	91.5%
No. of corrected events	155.0±13.7		111.4±12.9	61.4±9.1
Partial cross section	53.3±5.0 μb		15.9±1.9 μb	9.5±1.5 μb

^a Reference 17.^b Reference 20.

(a)



(c)



(b)

FIG. 5. Distributions in the invariant mass of the $\pi^+ \pi^- \pi^0$ (using unfitted quantities) for events in Fig. 3 for which M^2 (missing neutral) > 0.010 BeV². There are two pions of the same charge in the final state; only that combination of M^2 ($\pi^+ \pi^- \pi^0$) which is closer to $M_\eta^2 = 0.30$ BeV² has been plotted. (a) 1170-MeV/c incident π^+ ; (b) 1050-MeV/c incident π^+ ; and (c) 1170-MeV/c incident π^- .

TABLE II. Partial cross sections for $\pi p \rightarrow \pi p \eta$, $\eta \rightarrow \pi^+ \pi^- \pi^0$ in mb.

Incident beam momentum (BeV/c)	$\pi^+ p \rightarrow \pi^+ p \eta$ $\eta \rightarrow \pi^+ \pi^- \pi^0$	$\pi^- p \rightarrow \pi^- p \eta$ $\eta \rightarrow \pi^+ \pi^- \pi^0$
1.05 ^a	0.0095 ± 0.0015	
1.17 ^a	0.0533 ± 0.0050	0.0159 ± 0.0019
1.22 ^b	0.069 ± 0.010	
1.39 ^b	0.130 ± 0.045	
1.59 ^c	0.12 ± 0.02	
2.03 ^d		0.031 ± 0.010
2.08 ^e	0.15 ± 0.03	
2.08 ^f		0.037 ± 0.015
2.34 ^g	0.23 ± 0.05	
2.62 ^g	0.18 ± 0.09	
2.9 ^g	0.15 ± 0.05	
3.2 ^h		0.030 ± 0.010
3.43 ⁱ	0.09 ± 0.04	
3.54 ⁱ	0.07 ± 0.02	
3.65 ^j	0.06 ± 0.02	
4.2 ^h		0.021 ± 0.007

^a This experiment.^b Reference 2.^c Reference 3.^d Reference 8.^e Reference 4.^f Reference 9.^g Reference 5.^h Reference 10.ⁱ Reference 6.^j Reference 7.

ray by taking all events in which the mass squared of the missing neutral is greater than 0.010 BeV² (see Fig. 3), and then plot the invariant mass of the $\pi^+ \pi^- \pi^0$. The resulting plots are shown in Fig. 5. Again, because of the pion ambiguity, there are two such combinations; only the one in which the invariant mass is closer to the η mass has been plotted. It is seen in Fig. 5 that this procedure likewise results in nearly all events being η 's. This procedure is more visually appealing than the χ^2 method. We use the χ^2 criterion, however, because it takes measurement errors into account systematically and because, using the Monte Carlo program FAKE,¹⁹ we can more easily calculate the effects of cutoffs based on χ^2 than those based on calculated errors in missing mass.

III. CROSS SECTIONS

The measured cross sections for the reaction $\pi p \rightarrow \pi p \eta$, $\eta \rightarrow \pi^+ \pi^- \pi^0$ for each of our three beams are given in Table I along with the pertinent scanning information.²⁰

As stated in Sec. II, an η event is obtained by insisting that the $4C \chi^2$ for "nothing missing" is greater than 35, that the $1C \chi^2$ for missing π^0 or missing γ is less

¹⁹ G. R. Lynch, Lawrence Radiation Laboratory Report No. UCRL-10335, 1962 (unpublished). This program generates "bubble-chamber events" by a Monte Carlo method. These events are then sent through the same programs used for the real events. In this manner one can easily calculate the effects of any set of correlated or uncorrelated cutoffs.

²⁰ The attenuation due to strong interactions is based on $\pi^+ p$ total cross sections of 27.04 ± 0.80 mb and 29.69 ± 0.44 mb at 1050 and 1178 MeV/c, respectively, determined by Thomas J. Devlin, Burton J. Moyer, and Victor Perez-Mendez, Phys. Rev. **125**, 690 (1962), and a $\pi^- p$ total cross section of 37.30 ± 0.55 mb at 1170 MeV/c determined by B. Amblard, P. Borgeaud, Y. Ducros, P. Falk-Vairant, O. Guisan, W. Lasker, P. Sonderugger, A. Stirling, M. Yvert, A. Tran Ha, and S. D. Warshaw, Phys. Letters **10**, 138 (1964).

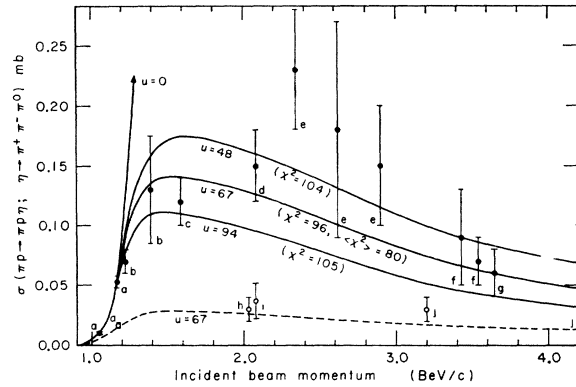


FIG. 6. Variation of $\sigma(\pi^\pm p \rightarrow \pi^\pm p \eta; \eta \rightarrow \pi^+ \pi^- \pi^0)$ as a function of incident beam momentum. The π^+ data points are indicated by closed circles, the π^- data points by open circles. See Table II for references. The $u=0$ curve is the result of the threshold parametrization of the π^+ data (see Sec. V B). The other solid curves are the results of the modified parametrization of Sec. V C. The dashed curve is the $u=67$ curve for the π^- data.

than 8.6, and that the $2C \chi^2$ for η decaying into $\pi^+ \pi^- \pi^0$ is less than for the decay into $\pi^+ \pi^- \gamma$. The last criterion introduces some misinterpretations if both χ^2 's are small. In order to obtain the efficiency of the above selection criteria, FAKE was used to simulate events of type (3). The efficiency was found to be 91.5%. In addition, events of type (4) were simulated. It was found that the number of $\pi^+ \pi^- \gamma \eta$ decays contaminating our final sample is negligible (< 0.8 event).

The partial cross sections obtained in this experiment for reaction (3) are listed in Table II along with the cross-section values given in Refs. 2-10. A plot of these values as a function of incident beam momentum is shown in Fig. 6. Note that the ratio

$$\sigma(\pi^- p \rightarrow \pi^- p \eta) / \sigma(\pi^+ p \rightarrow \pi^+ p \eta)$$

at 1170 MeV/c is

$$(15.9 \pm 1.9) / (53.3 \pm 5.0) = 0.30 \pm 0.04.$$

Were the $\pi^- p$ reaction to proceed via $I = \frac{3}{2}$ amplitudes only, such as through the reaction $\pi^- p \rightarrow \eta \Delta(1238)$, the ratio would be $\frac{1}{3} = 0.11$. Thus $I = \frac{1}{2}$ amplitudes are important at 1170 MeV/c.

IV. EXPERIMENTAL DISTRIBUTIONS

A. Coordinate System

In the analysis which follows, we regard the reactions $\pi^\pm p \rightarrow \pi^\pm p \eta$ as the two-step process

$$\pi^\pm p \rightarrow \eta \tilde{N}$$

followed by

$$\tilde{N} \rightarrow \pi^\pm p,$$

where we use the symbol \tilde{N} to indicate a π - p state of total angular momentum J and orbital angular mo-

TABLE III. Summary of the notation in Fig. 7.

	Particles	Linear momentum	Orbital angular momentum	Total angular momentum	Energy
Initial (over-all c.m. system)	π_B, p_T	\mathbf{k}	\mathfrak{L}	\mathfrak{J}	E
Final (over-all c.m. system)	η, \tilde{N}	\mathbf{p}	\mathbf{l}	\mathfrak{J}	E
\tilde{N} rest frame	π_D, p_D	\mathbf{q}	\mathbf{L}	\mathfrak{J}	M

mentum L . Thus when $J = \frac{3}{2}$ and $L = 1$, \tilde{N} corresponds to the $\Delta(1238)$.

Figure 7 is a mnemonic diagram in velocity space depicting the two-step process.²¹ The beam pion π_B and target proton p_T of orbital angular momentum \mathfrak{L} and linear momentum \mathbf{k} in the over-all c.m. system of energy E produce an η and an \tilde{N} of orbital angular momentum \mathbf{l} and linear momentum \mathbf{p} in the over-all c.m. system. The \tilde{N} of total angular momentum J then decays with orbital angular momentum \mathbf{L} and linear momentum \mathbf{q} in the \tilde{N} rest frame. The total angular momentum of the system is \mathfrak{J} . The final state $\pi^+ p \eta$ then consists of $I = \frac{3}{2}$ states, $(\mathbf{l} \mathbf{L}_J)_{\mathfrak{J}}$, and the final state $\pi^- p \eta$ consists of both $I = \frac{3}{2}$ and $I = \frac{1}{2}$ states, $(\mathbf{l} \mathbf{L}_J)_{\mathfrak{J}}$. The notation is summarized in Table III.

We choose as the z axis the line of flight of the beam particle in the c.m. system; that is,

$$\hat{z} = \mathbf{k} / |\mathbf{k}|.$$

We now define ξ as the angle in the c.m. system between the line of flight of the η and the line of flight of the incoming pion; that is,

$$\cos \xi = \mathbf{p} \cdot \mathbf{k} / |\mathbf{p}| |\mathbf{k}| = \hat{p} \cdot \hat{z}.$$

The vectors \mathbf{p} and \mathbf{k} define the production plane. The y axis is defined as the normal to the production plane,

$$\hat{y} = \hat{n} = \mathbf{k} \times \mathbf{p} / |\mathbf{k}| |\mathbf{p}| \sin \xi;$$

the x axis is defined so as to construct a right-handed coordinate system, $\hat{x} = \hat{y} \times \hat{z}$.

We now go to the \tilde{N} rest frame and define the angles θ and ϕ as the polar and azimuthal angles of the decay proton:

$$\begin{aligned} \cos \theta &= \hat{q} \cdot \hat{z}, \\ \phi &= \tan^{-1}(\hat{q} \cdot \hat{n} / \hat{q} \cdot \hat{x}). \end{aligned}$$

In this frame the z axis is defined to be in the production plane and at an angle ξ from the line of flight of the η .

A three-body final state has five external degrees of freedom: three particles, each with three components of momentum, minus the four energy-momentum equations of constraint. One of these degrees of freedom, the orientation of the production plane in space (say, in the bubble chamber), is of no interest. Thus the four

independent variables, $\cos \xi$, $\cos \theta$, ϕ , and the invariant mass-squared of the \tilde{N} , M^2 , are sufficient to completely describe an event.

In terms of these independent variables, the differential cross section can be written

$$d^4\sigma = \left(\frac{1}{kE} \right) |\mathfrak{M}|^2 \left(\frac{\rho}{E} dM^2 d \cos \xi d \cos \theta d \phi \right), \quad (5)$$

where the first factor is the Lorentz-invariant flux factor, the second is the absolute square of the matrix element, and the third is the Lorentz-invariant three-body phase-space factor. The factor ρ equals $p q / M$.

B. Mass and Angular Distributions

The production Dalitz plots and their mass-squared projections for our three sets of data are shown in Fig. 8. The distributions in the angular variables, $d\sigma/d \cos \xi$, $d\sigma/d \phi$, and $d\sigma/d \cos \theta$, are shown in Figs. 9–11, respectively. All errors are statistical and are based only upon the number of events in each bin. The curves shown are the result of the best fit of the parametrization described in Sec. V.

In Fig. 8, we see that the events are clustered into the upper left-hand corners of the Dalitz plots as a result of the formation of the $\Delta(1238) \rightarrow \pi p$. It is seen that the enhancements in the low-mass region of the

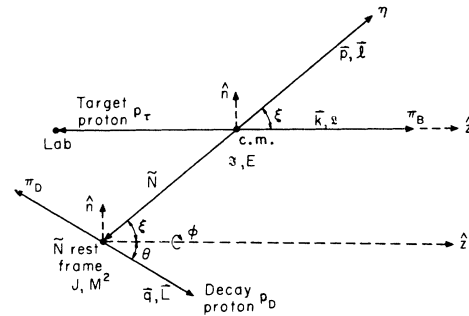


FIG. 7. Mnemonic (nonrelativistic) diagram in velocity space depicting the two-step reaction $\pi_B p_T \rightarrow \eta \tilde{N}$; $\tilde{N} \rightarrow \pi_D p_D$. In the over-all c.m. system, π_B has orbital angular momentum \mathfrak{L} and linear momentum \mathbf{k} along the z axis; η has orbital angular momentum \mathbf{l} and linear momentum \mathbf{p} . The two vectors \mathbf{k} and \mathbf{p} define the production plane. The normal to the production plane, $\hat{n} \propto \mathbf{k} \times \mathbf{p}$, is the y axis; $\hat{x} = \hat{y} \times \hat{z}$. In the \tilde{N} rest frame the z axis is in the production plane at an angle ξ from the line of flight of the η ; $\cos \xi = \mathbf{p} \cdot \hat{z}$. The \tilde{N} of mass M and total angular momentum \mathfrak{J} decays into π_D and p_D with orbital angular momentum \mathbf{L} and linear momentum \mathbf{q} ; $\cos \theta = \hat{q} \cdot \hat{z}$, $\phi = \tan^{-1}[\hat{q} \cdot \hat{n} / \hat{q} \cdot \hat{x}]$. The total angular momentum of the system is \mathfrak{J} ; the total energy is E .

²¹ The lab quantities were first transformed to the c.m. system and then to the \tilde{N} rest frame. Because of the nonlinearity of velocity addition in the Lorentz transformation, the z axis obtained by this procedure differs by a small rotation from that obtained by transforming from the laboratory to the \tilde{N} rest frame directly.

$M_{\eta p}^2$ distributions are a reflection of the Δ . The distributions in $M_{\pi\pi}^2$ (not shown) show no enhancements.

The distributions in $\cos\xi$, Fig. 9, are the production angular distributions. Production via only s wave would

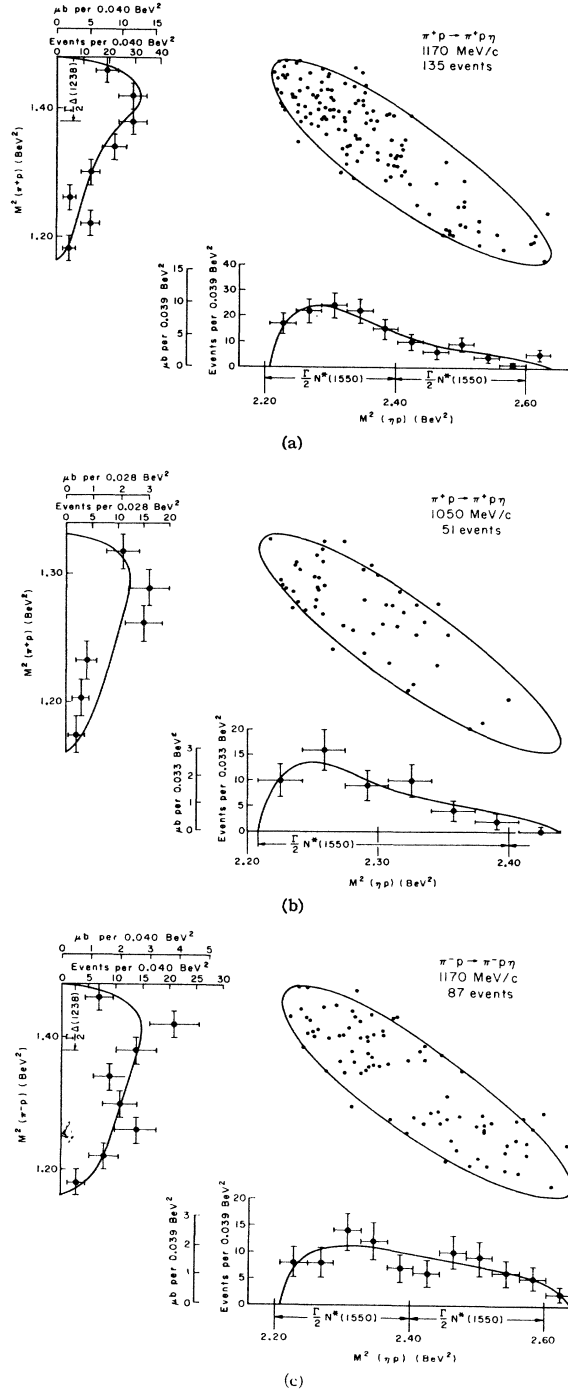


FIG. 8. Production Dalitz plots and mass-squared projections. The curves are given by Eqs. (15) and (16), using our best-fit parameters. (a) 1170-MeV/c incident π^+ ; (b) 1050-MeV/c incident π^+ ; and (c) 1170-MeV/c incident π^- .

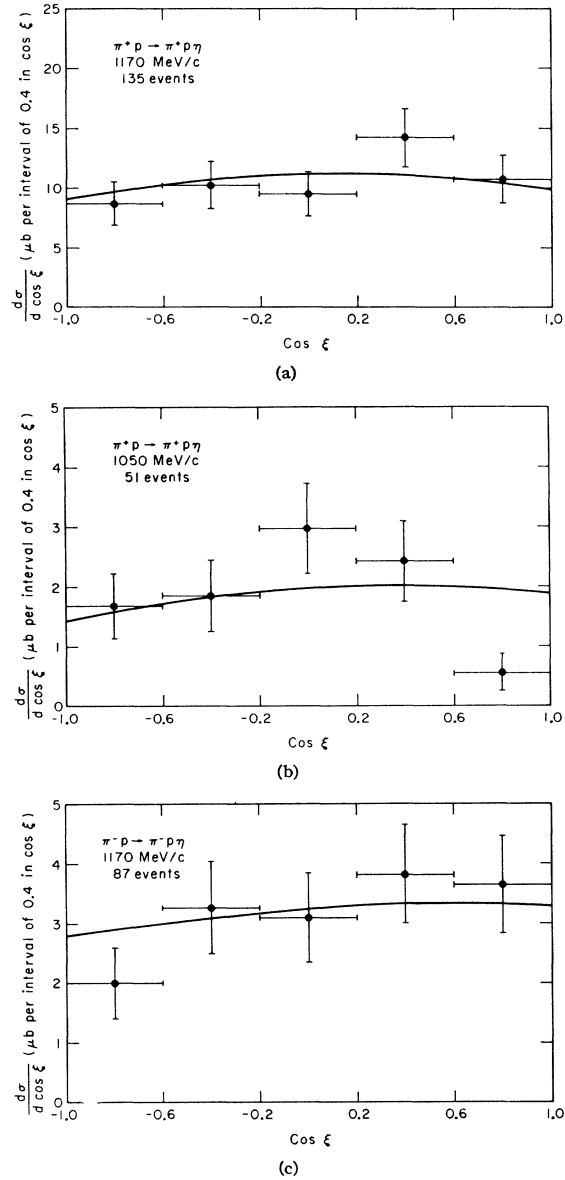


FIG. 9. Production angular distributions in the c.m. system. The curves are given by Eq. (12), using our best-fit parameters. (a) 1170-MeV/c incident π^+ ; (b) 1050-MeV/c incident π^+ ; and (c) 1170-MeV/c incident π^- .

require these distributions to be completely flat. (We will use lower-case letters to indicate the angular momentum state in which the η and \bar{N} are produced, and capital letters to indicate the state in which the \bar{N} decays.) We see that the "flat" hypothesis is a good one at 1170 MeV/c ($\chi^2=3.8$ for the π^+ data and 4.9 for the π^- data, with $\langle\chi^2\rangle=4.0$), indicating that higher l values may not be required.

However, the distributions in the azimuthal angle,²² ϕ , which are shown in Fig. 10, indicate that p -wave

²² Since parity is conserved in strong interactions, the ϕ distributions have been folded about 180° .

TABLE IV. Values of the D_i 's in μb . The values D_8 - D_{14} (the region within heavy brackets) would be zero if our final choice of amplitudes were correct.

	1050-MeV/ c π^+		1170-MeV/ c π^+		1170-MeV/ c π^-	
	1a Expt. values	1b Fitted values	2a Expt. values	2b Fitted values	3a Expt. values	3b Fitted values
D_1	9.50 ± 1.51	7.55	53.30 ± 5.02	54.92	15.85 ± 1.91	16.13
D_2	-1.14 ± 1.12	0.50	5.71 ± 4.49	1.12	2.90 ± 1.63	0.73
D_3	-3.31 ± 1.49	-1.23	-12.47 ± 4.89	-15.38	-0.79 ± 1.71	-1.03
D_4	-0.35 ± 1.27	0.20	5.16 ± 4.95	1.01	0.08 ± 1.70	0.11
D_5	-0.46 ± 1.46	-1.63	-22.01 ± 4.63	-20.44	-3.03 ± 1.85	-2.27
D_6	-3.05 ± 1.25	-0.46	-2.51 ± 4.11	-2.78	-1.34 ± 1.65	-0.31
D_7	2.76 ± 1.47	1.98	8.26 ± 4.75	17.99	0.25 ± 1.63	2.00
D_8	[3.46 ± 1.39		3.33 ± 4.36		-1.66 ± 1.71]	
D_9	[0.25 ± 1.25		3.84 ± 4.90		-0.68 ± 1.86]	
D_{10}	[0.63 ± 1.26		-2.88 ± 4.78		-0.57 ± 1.59]	
D_{11}	[-1.31 ± 1.45		4.02 ± 4.90		1.47 ± 1.98]	
D_{12}	[-0.68 ± 1.42		3.47 ± 4.16		1.31 ± 1.75]	
D_{13}	[-0.27 ± 1.37		1.38 ± 4.97		0.13 ± 1.87]	
D_{14}	[0.65 ± 1.35		-4.18 ± 4.83		1.68 ± 1.85]	

production amplitudes are also required. The 1170-MeV/ c π^+ data in particular show a very definite $\cos\phi$ dependence. Production via only s wave would require the ϕ distributions to be flat. As will become apparent in the next section, it is necessary to add p -wave production amplitudes in order to obtain a $\cos\phi$ dependence.

The distributions in $\cos\theta$, Fig. 11, are the \tilde{N} decay angular distributions. If the reaction proceeded solely via s -wave production of the $\Delta(1238)$, the distributions would be of the form $(1+3\cos^2\theta)$. If some p -wave Δ production were present, the distributions would no longer be of this form but would still be symmetrical. Asymmetry in the distributions is the result of both S - and P -wave decay of the \tilde{N} . The structure of these distributions is such as to be adequately described by only S - and P -wave decay of the \tilde{N} (i.e., partial waves with $L>1$ are not needed).

The qualitative remarks made in this section concerning the angular distributions are made more quantitative in Sec. V. On the basis of the distributions in $\cos\xi$, $\cos\theta$, and ϕ , we consider only partial waves with $l=0$ and 1, and $L=0$ and 1; i.e., s - and p -wave production of the η and \tilde{N} , and S - and P -wave decay of the \tilde{N} .

V. PARAMETRIZATION

If we list all possible final-state amplitudes which can be produced with $l=0$ and 1, and $L=0$ and 1, we find a total of 10 (see Appendix A). In addition, each complex amplitude $(lL_J)_g$ can exist in two over-all isospin states: $I=\frac{1}{2}$ and $I=\frac{3}{2}$. We want to determine which amplitudes are required by the data and what are the values of the magnitudes and phases of these amplitudes. The $I=\frac{3}{2}$ amplitudes can be determined from the π^+ data, because the π^+p state exists only in $I=\frac{3}{2}$. The π^-p state, however, consists of both $I=\frac{3}{2}$ and $I=\frac{1}{2}$. Once the $I=\frac{3}{2}$ parameters have been determined from

the π^+ data, this information can be used to determine the $I=\frac{1}{2}$ parameters from the π^- data at the same momentum.

At first glance it may seem presumptuous to consider 10 complex amplitudes with the order of only 100 events. This would be true if all we had to work with was one angular distribution, since we certainly could not fit the distribution beyond a few powers in the cosine of the angle; thus the number of amplitudes which could be determined would be severely limited. However, in our experiment we have four independent variables and therefore a total of four independent mass and angular distributions to work with, plus all the correlations which exist among the four variables. We do not fit beyond the second power in any one angular distribution. We will show that only four of the 10 complex $I=\frac{3}{2}$ amplitudes are required by the π^+ data and only one of the ten $I=\frac{1}{2}$ amplitudes is required by the π^- data. Thus we have seven $I=\frac{3}{2}$ parameters to be determined (four magnitudes and three phases) and two $I=\frac{1}{2}$ parameters (one magnitude and one phase). With four independent variables, and using the correlations which exist among the variables, we can determine seven parameters quite well with the order of only 100 events.

We will now describe our technique for examining these correlations. We then go on to describe the procedure used to determine which of the 10 complex amplitudes are present for each isospin state and how the values of the magnitudes and phases of these amplitudes were determined. The results of this parametrization are then examined.

A. Method of Moments

For each isospin state, the absolute square of the matrix element, $|\mathfrak{M}|^2$, composed of the 10 complex amplitudes, is derived as in Appendix A and has the

form

$$\begin{aligned}
 |\mathfrak{N}|^2 &= \sum_{i=1}^{14} C_i (M_{\pi p^2})^{\frac{1}{2}} [\mathfrak{Y}_i^m(\xi, 0) Y_L^{-M}(\theta, \phi) + \mathfrak{Y}_i^{-m}(\xi, 0) \\
 &\quad \times Y_L^M(\theta, \phi)] \\
 &= C_1 \mathfrak{Y}_0^0 Y_0^0 + C_2 \mathfrak{Y}_1^0 Y_0^0 + C_3 \mathfrak{Y}_0^0 Y_1^0 + C_4 \mathfrak{Y}_1^0 Y_1^0 \\
 &\quad + C_5 \frac{1}{2} (\mathfrak{Y}_1^1 Y_1^{-1} + \mathfrak{Y}_1^{-1} Y_1^1) \\
 &\quad + C_6 \mathfrak{Y}_2^0 Y_0^0 + C_7 \mathfrak{Y}_0^0 Y_2^0 \\
 &\quad + C_8 \mathfrak{Y}_2^0 Y_1^0 + C_9 \frac{1}{2} (\mathfrak{Y}_2^1 Y_1^{-1} + \mathfrak{Y}_2^{-1} Y_1^1) \\
 &\quad + C_{10} \mathfrak{Y}_1^0 Y_2^0 + C_{11} \frac{1}{2} (\mathfrak{Y}_1^1 Y_2^{-1} + \mathfrak{Y}_1^{-1} Y_2^1) \\
 &\quad + C_{12} \mathfrak{Y}_2^0 Y_2^0 + C_{13} \frac{1}{2} (\mathfrak{Y}_2^1 Y_2^{-1} + \mathfrak{Y}_2^{-1} Y_2^1) \\
 &\quad + C_{14} \frac{1}{2} (\mathfrak{Y}_2^2 Y_2^{-2} + \mathfrak{Y}_2^{-2} Y_2^2), \quad (6)
 \end{aligned}$$

where the C 's are real functions of $M_{\pi p^2}$ ($M_{\pi p^2} \equiv M^2$, one of our four independent variables) consisting of combinations of the 10 complex amplitudes (LL_J) $_g$ (see Appendix A), and where $\mathfrak{Y}_i^m(\xi, 0)$ and $Y_L^M(\theta, \phi)$ are the production and decay spherical harmonics, respectively. The quantities L , l , and the angular variables are defined in Sec. IV, and $-l \leq m \leq l$, $-L \leq M \leq L$.

One possible method for determining which of the 10 complex amplitudes are required by the data is to insert Eq. (6) into Eq. (5) and then integrate over three of the four variables to get expressions for the mass distribution and the three angular distributions in terms of the complex amplitudes. A fit to these distributions would then yield some information as to which amplitudes are present. This procedure, however, does not make use of all the information contained in the data, because the various correlations which exist among the four variables integrate out when one obtains a mass or angular distribution.

In order to take these correlations into account, we project out each of the 14 coefficients of the various combinations of the production and decay spherical harmonics present in $|\mathfrak{N}|^2$. We make use of the orthonormal property of the spherical harmonics,

$$\begin{aligned}
 \int (\mathfrak{Y}_i^m Y_L^M)^* (\mathfrak{Y}_{i'}^{m'} Y_{L'}^{M'}) d \cos \xi d \cos \theta d \phi \\
 = \delta_{ii'} \delta_{mm'} \delta_{LL'} \delta_{MM'}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 C_i(M_{\pi p^2}) &= \int \gamma_m [\mathfrak{Y}_i^m Y_L^{-M} + \mathfrak{Y}_i^{-m} Y_L^M]^* \\
 &\quad \times d \cos \xi d \cos \theta d \phi, \quad (7)
 \end{aligned}$$

where $\gamma_m = \frac{1}{2}$ when $m = M = 0$, and $\gamma_m = 1$ when $m = -M \neq 0$. We define the values

$$D_i \equiv \frac{1}{k E^2} \int dM^2 \rho C_i(M^2), \quad (8)$$

where $\rho = pq/M$ and, as defined before, k is the linear momentum of the beam pion and p is that of the η

TABLE V. Type of terms present in each $C_i(M^2)$ and D_i . The J and g values have been suppressed. For example, $\langle pS | pS \rangle$ includes the following terms: $|\langle pS_{1/2} \rangle_{1/2}|^2$, $|\langle pS_{1/2} \rangle_{3/2}|^2$, and $\langle \langle pS_{1/2} \rangle_{1/2} | \langle pS_{1/2} \rangle_{3/2} \rangle$. The terms in column b (within heavy brackets) become zero under the choice of amplitudes made in Sec. V.

	(a)	(b)
C_1	$ sS ^2, \sum sP ^2, \sum pS ^2,$	$[\sum pP ^2]$
C_2	$\langle sS pS \rangle,$	$[\langle sP pP \rangle]$
C_3	$\langle sS sP \rangle,$	$[\langle pS pP \rangle]$
C_4	$\langle pS sP \rangle,$	$[\langle sS pP \rangle]$
C_5	$\langle pS sP \rangle,$	$[\langle sS pP \rangle]$
C_6	$\langle pS pS \rangle,$	$[\langle pP pP \rangle]$
C_7	$\langle sP sP \rangle,$	$[\langle pP pP \rangle]$
C_8		$[\langle pS pP \rangle]$
C_9		$[\langle pS pP \rangle]$
C_{10}		$[\langle sP pP \rangle]$
C_{11}		$[\langle sP pP \rangle]$
C_{12}		$[\langle pP pP \rangle]$
C_{13}		$[\langle pP pP \rangle]$
C_{14}		$[\langle pP pP \rangle]$

in the over-all c.m. system, E is the total energy in that c.m. system, q is the linear momentum of the decay proton in the \tilde{N} rest frame, and M is the mass of the \tilde{N} .

We performed operations (7) and (8) on our three sets of data. That is, in each of the M^2 bins we added up the values of $\gamma_m (\mathfrak{Y}_i^m Y_L^{-M} + \mathfrak{Y}_i^{-m} Y_L^M)$ to get the distribution

$$\rho C_i(M_j^2) = \sum_{\substack{\text{No. of events} \\ \text{in bin } j}} \gamma_m (\mathfrak{Y}_i^m Y_L^{-M} + \mathfrak{Y}_i^{-m} Y_L^M)_{ij}.$$

The value of the error for each bin is given by

$$\begin{aligned}
 \delta[\rho C_i(M_j^2)] \\
 = \left\{ \sum_{\substack{\text{No. of events} \\ \text{in bin } j}} [\gamma_m (\mathfrak{Y}_i^m Y_L^{-M} + \mathfrak{Y}_i^{-m} Y_L^M)_{ij}]^2 \right\}^{1/2}.
 \end{aligned}$$

We then obtained the value D_i by summing over the M^2 bins and multiplying by a scale factor to convert from counts to cross section. Thus,

$$D_i = (\text{scale factor}) \sum_{j \text{ bins}} \rho C_i(M_j^2)$$

and

$$\begin{aligned}
 \delta D_i &= (\text{scale factor}) \\
 &\quad \times \left\{ \sum_{\substack{\text{total No.} \\ \text{of events}}} [\gamma_m (\mathfrak{Y}_i^m Y_L^{-M} + \mathfrak{Y}_i^{-m} Y_L^M)_i]^2 \right\}^{1/2}.
 \end{aligned}$$

The 14 D values for each of the three sets of data are presented in Table IV.

As stated at the beginning of this subsection, each mass-dependent coefficient $C_i(M^2)$ consists of combinations of the mass-dependent parts of all the complex amplitudes. The type of amplitudes present in each $C_i(M^2)$, and hence in each D_i , is given in Table V. (This table is derived in Appendix A.) In the following subsections, by comparing Tables IV and V we obtain the simplest set of amplitudes necessary to fit the data.

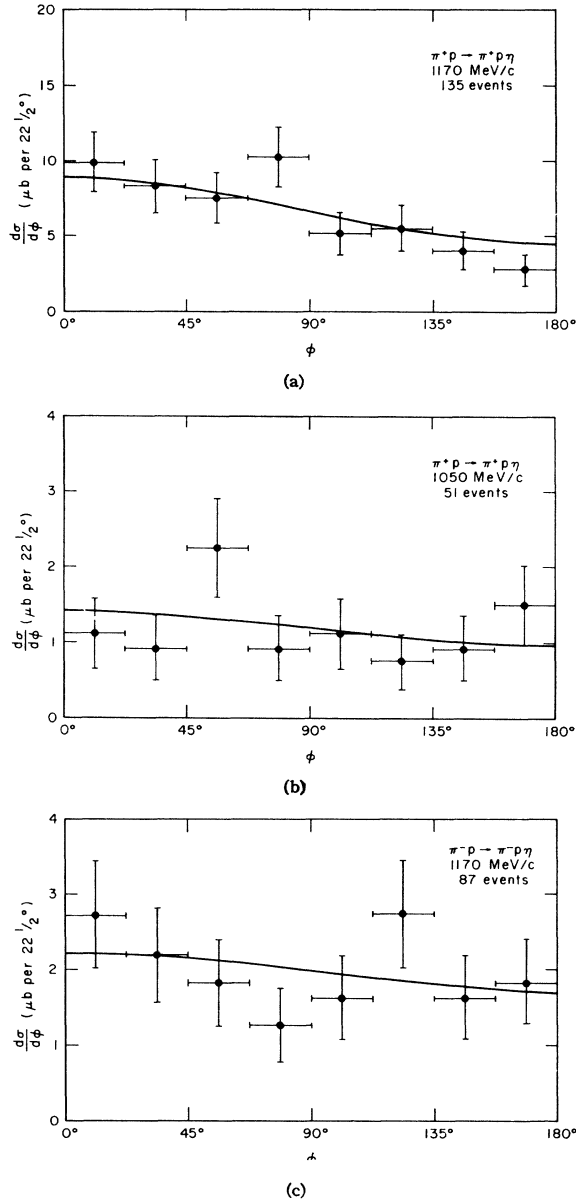


FIG. 10. \bar{N} azimuthal-decay angular distributions. The curves are given by Eq. (13), using our best-fit parameters. (a) 1170-MeV/c incident π^- ; (b) 1050-MeV/c incident π^+ ; and (c) 1170-MeV/c incident π^- .

B. π^+ Data

1. Determination of the $I=\frac{3}{2}$ Amplitudes

From the Dalitz plots and angular distributions we believe that the $\Delta(1238)$ is produced. We therefore need at least one amplitude representing Δ production. Of the ten $I=\frac{3}{2}$ amplitudes, $(LL_J)_g$, under consideration, four of them, $(sP_{3/2})_{3/2}$, $(pP_{3/2})_{1/2,3/2,5/2}$, represent the production of the Δ . [When states differ only in the total angular momentum g we use the notation $(LL_J)_{g_1, g_2, \dots}$] We begin by choosing the lowest orbital-

angular-momentum state amplitude $(sP_{3/2})_{3/2}$. Next we see from the π^+ part of Table IV (columns 1a and 2a) that D_3 , D_5 , D_7 (and of course D_1 , since $D_1=\sigma$) are nonzero. We see from Table V that in order to make D_3 nonzero we need sS - sP and/or pS - pP interference. (The values of J and g have been suppressed.) To obtain sS - sP interference we need only to assume one additional amplitude, $(sS_{1/2})_{1/2}$. However, in order to obtain pS - pP interference there are seven additional amplitudes which need to be considered, all of which

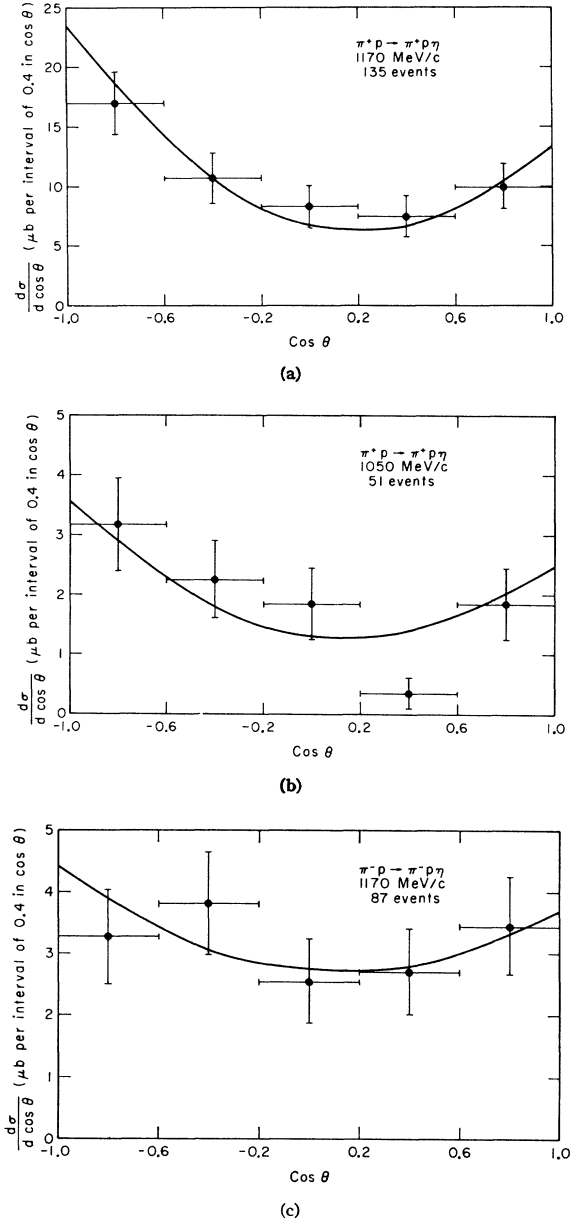


FIG. 11. \bar{N} polar-decay angular distributions. The curves are given by Eq. (14), using our best-fit parameters. (a) 1170-MeV/c incident π^+ ; (b) 1050-MeV/c incident π^+ ; and (c) 1170-MeV/c incident π^- .

TABLE VI. The $I=\frac{3}{2}$ amplitudes which we assume to be present.

$\mathcal{L}_J \rightarrow (lL_J)g$	M^2 and E dependence
$\mathcal{P}_{1/2} \rightarrow (sS_{1/2})_{1/2}$	$A_{00}e^{i\lambda_{00}}k$
$\mathcal{D}_{3/2} \rightarrow (sP_{3/2})_{3/2}$	$A_{03}e^{i\lambda_{03}}k^2(M/q\Gamma)^{1/2}[1/(\epsilon-i)]$
$\mathcal{S}_{1/2} \rightarrow (pS_{1/2})_{1/2}$	$A_{10}e^{i\lambda_{10}}p$
$\mathcal{D}_{3/2} \rightarrow (pS_{1/2})_{3/2}$	$A_{30}e^{i\lambda_{30}}pk^2$

possess higher angular momentum barriers than the sS amplitude. On the basis of always choosing the lowest angular-momentum state amplitudes, we choose the former alternative.

In order to make D_5 nonzero, we see from Table V that we have to add either pS or pP amplitudes. Again choosing the amplitudes with the lower angular momentum barriers, we add the pS amplitudes to our set. The $(pS_{1/2})_{1/2}$ final state originates from the $\mathcal{S}_{1/2}$ initial state, the $(pS_{1/2})_{3/2}$ final state from the $\mathcal{D}_{3/2}$ initial state. Because the momentum of the beam particle in the over-all c.m. system is of the order of 600 MeV/c, there is sufficient energy for the initial state to exist in \mathcal{D} wave. We therefore include both the $(pS_{1/2})_{1/2}$ and $(pS_{1/2})_{3/2}$ amplitudes.

Next we see from Table V that in order for D_7 to be nonzero, we need sP - sP and/or pP - pP interference. Because we have already chosen the $(sP_{3/2})_{3/2}$ amplitude, no additional amplitudes are required.

Thus in order to obtain nonzero values of D_1 , D_3 , D_5 , and D_7 we need only the amplitudes $(sS_{1/2})_{1/2}$, $(sP_{3/2})_{3/2}$, and $(pS_{1/2})_{1/2,3/2}$. We assume that all other amplitudes are absent. This means that when the \tilde{N} is in S wave, we have both s - and p -wave production; when the \tilde{N} is in P wave, it has only $J=\frac{3}{2}$, and is produced only in s wave.

With this choice of amplitudes, all portions of Tables IV, V, and IX within heavy brackets should be zero. We see that our assumption that the five (pP) amplitudes are zero requires D_8 - D_{14} to be zero. Except for the 1050-MeV/c value of D_8 , this requirement is consistent with the D values of Table IV. The nonzero value at 1050 MeV/c could be a statistical fluctuation.

2. Determination of Values of Magnitudes and Phases of Complex Amplitudes

Now that we have described the procedure used to determine which complex amplitudes are present in the π^+ data, we describe how the magnitudes and phases of these amplitudes were determined. Near threshold the M^2 and E dependence of the amplitude for $\mathcal{L}_J \rightarrow (lL_J)g$ can be parametrized by $A_{\alpha\beta}e^{i\lambda_{\alpha\beta}}k^\alpha p^\beta q^L$, where the quantities $A_{\alpha\beta}$ are real positive parameters (assumed to be energy-independent), $\lambda_{\alpha\beta}$ are phase angles (also assumed to be energy-independent), and α and β are indices used to differentiate the various states ($\alpha=0$ for s wave and $2J$ for p wave; similarly $\beta=0$ for S wave and $2J$ for p wave). Specifically, see Table VI

for our set of four amplitudes, where the factor

$$(M/q\Gamma)^{1/2}[1/(\epsilon-i)]$$

is the Breit-Wigner amplitude for the $\Delta(1238)$ resonance.²³ The factor ϵ is the number of half-widths removed from the resonant energy. That is,

$$\epsilon = \frac{M_0^2 - M^2}{M_0\Gamma} \approx \frac{M_0 - M}{\frac{1}{2}\Gamma},$$

where²⁴

$$\Gamma = \Gamma_0 \left(\frac{q}{q_0} \right)^3 \left(\frac{[(M+m_p)^2 - m_\pi^2]/M^2}{[(M_0+m_p)^2 - m_\pi^2]/M_0^2} \right),$$

with (Ref. 14) $M_0=1236$ MeV, $\Gamma_0=120$ MeV, and $q_0=231$ MeV. Because there is an over-all arbitrary phase, we define $\lambda_{03} \equiv 0$. We further define $F \equiv (M/q\Gamma) \times [1/(\epsilon^2+1)]$ and $G \equiv (q\Gamma/M)^{1/2}$. Inserting these definitions into the expressions for the C 's as derived in Appendix A, and using the above table, we obtain for the $I=\frac{3}{2}$ amplitudes

$$\begin{aligned} C_1(M^2) &= (A_{00}^2 k^2) + (2A_{03}^2 k^4)F + (A_{10}^2 + 2A_{30}^2)p^2, \\ C_2(M^2) &= (\frac{4}{3}\sqrt{3}A_{00}A_{10}k \cos(\lambda_{00}-\lambda_{10}) \\ &\quad + \frac{4}{3}\sqrt{3}A_{00}A_{30}k^3 \cos(\lambda_{00}-\lambda_{30}))p, \\ C_3(M^2) &= (\frac{2}{3}\sqrt{3}A_{00}A_{03}k^3 \cos\lambda_{00})GF\epsilon \\ &\quad + (\frac{4}{3}\sqrt{3}A_{00}A_{03}k^3 \sin\lambda_{00})GF, \\ C_4(M^2) &= [\frac{4}{3}A_{10}A_{03}k^2 \cos\lambda_{10} + (8/3)A_{30}A_{03}k^4 \cos\lambda_{30}] \\ &\quad \times pGF\epsilon + [\frac{4}{3}A_{10}A_{03}k^2 \sin\lambda_{10} \\ &\quad + (8/3)A_{30}A_{03}k^4 \sin\lambda_{30}]pGF, \quad (9) \\ C_5(M^2) &= (\frac{4}{3}A_{10}A_{03}k^2 \cos\lambda_{10} - \frac{4}{3}A_{30}A_{03}k^4 \cos\lambda_{30})pGF\epsilon \\ &\quad + (\frac{4}{3}A_{10}A_{03}k^2 \sin\lambda_{10} - \frac{4}{3}A_{30}A_{03}k^4 \sin\lambda_{30})pGF, \\ C_6(M^2) &= [(4/5^{1/2})A_{10}A_{30}k^2 \cos(\lambda_{10}-\lambda_{30}) \\ &\quad + (2/5^{1/2})A_{30}^2 k^4]p^2, \\ C_7(M^2) &= [(2/5^{1/2})A_{03}^2 k^4]F, \\ D_i &= \frac{1}{kE^2} \int C_i(M^2) \rho dM^2. \quad (8) \end{aligned}$$

The A 's and λ 's were determined by fitting the 1050- and 1170-MeV/c π^+ data. Equating the numerical value of each D_i as given in Table IV with the algebraic expression for the D_i as given by Eqs. (8) and (9), we obtain a set of seven quadratic equations involving the

²³ The factor $(M/q)^{1/2}$ arises from dividing out the two-body phase space factor; the factor $1/\Gamma^{1/2}$ appears because we choose the model in which the Δ is made "directly" in the same strong reaction in which the η is made. If we choose the model in which the three final-state particles are made directly and then the Δ is formed by "rescattering," the $\Gamma^{1/2}$ factor would not appear.

²⁴ J. D. Jackson, Nuovo Cimento 34, 1644 (1964).

TABLE VII. Partial cross sections and percentages of the four partial waves present in the reaction $\pi p \rightarrow \pi p \eta$.

	1050-MeV/c π^+	1170-MeV/c π^+	1170-MeV/c π^-
$(sS_{1/2})_{1/2}$	2.2 μb (24 \pm 7)%	6.7 μb (13 \pm 4)%	10.8 μb (67 \pm 3)%
$(sP_{3/2})_{3/2}$	5.6 μb (59 \pm 6)%	39.0 μb (73 \pm 7)%	4.5 μb (27 \pm 4)%
$(pS_{1/2})_{1/2}$	1.2 μb (13 \pm 3)%	5.0 μb (9 \pm 2)%	0.6 μb (4 \pm 1)%
$(pS_{1/2})_{3/2}$	0.5 μb (4 \pm 1)%	2.6 μb (5 \pm 1)%	0.3 μb (2 \pm 1)%

four A 's and three λ 's. For example,

$$D_1 = A_{00}^2 \left(\frac{k}{E^2} \int \rho dM^2 \right) + 2A_{03}^2 \left(\frac{k^3}{E^2} \int \rho F dM^2 \right) + (A_{10}^2 + 2A_{30}^2 k^4) \left(\frac{1}{kE} \int \rho p^2 dM^2 \right), \quad (10)$$

which at 1170 MeV/c becomes

$$53.3 \pm 5.0 \mu\text{b} = A_{00}^2 (1.58 \times 10^{-3} \text{ BeV}^2) + 2A_{03}^2 (7.49 \times 10^{-3} \text{ BeV}^3) + A_{10}^2 (1.88 \times 10^{-4} \text{ BeV}^2) + 2A_{30}^2 (2.83 \times 10^{-5} \text{ BeV}^6).$$

The integrations have been performed numerically. At each momentum we have seven such equations. In addition, $C_1(M^2)$, $C_3(M^2)$, $C_4(M^2)$, and $C_5(M^2)$ each have two or more final-state momentum dependences; therefore, the relative amounts of each dependence can be fitted, thereby yielding more information on the values of the parameters. [The distributions $C_2(M^2)$, $C_6(M^2)$, and $C_7(M^2)$ each have only one momentum dependence and therefore yield no additional information on the values of the A 's and λ 's]. We fitted $C_1(M^2)$, $C_3(M^2)$, $C_4(M^2)$, and $C_5(M^2)$ and the seven D_i 's at both π^+ momenta simultaneously and obtained the fitted values²⁵

$$\begin{aligned} A_{00} &= (66_{-13}^{+9}) (\mu\text{b}/\text{BeV}^2)^{1/2}, \\ \lambda_{00} &= (4.83_{-0.14}^{+0.08}) \text{ rad}, \\ A_{10} &= (166_{-21}^{+19}) (\mu\text{b}/\text{BeV}^2)^{1/2}, \\ \lambda_{10} &= (4.39_{-0.37}^{+0.44}) \text{ rad}, \\ A_{30} &= (221_{-17}^{+21}) (\mu\text{b}/\text{BeV}^6)^{1/2}, \\ \lambda_{30} &= (0.67_{-0.41}^{+0.20}) \text{ rad}, \\ A_{03} &= (51.7_{-3.7}^{+3.0}) (\mu\text{b}/\text{BeV}^3)^{1/2}, \end{aligned} \quad (11)$$

with $\chi^2 = 88$, $\langle \chi^2 \rangle = 71$.

3. Comparison of π^+ Data with Results of Fit

With knowledge of the A 's and λ 's, by Eqs. (8) and (9), we should be able to predict the distributions of

²⁵ In order to convert from $\eta \rightarrow \pi^+ \pi^- \pi^0$ decay to $\eta \rightarrow \text{all}$ decays, multiply the magnitudes by $1/R^{1/2}$, where

$$R = \Gamma(\eta \rightarrow \pi^+ \pi^- \pi^0) / \Gamma(\eta \rightarrow \text{all}).$$

In order to convert the magnitude values to fermis, divide out the radius of interaction.

all seven $C_i(M^2)$ and the values of the seven D_i 's, and hence the mass and angular distributions and the total cross section for the reaction $\pi^+ p \rightarrow \pi^+ p \eta$ at any momentum near threshold.

Figure 6 shows the variation in cross section with beam momentum. The curve labeled $u=0$ is the cross section as given by the values of the parameters in Eq. (11). We see that the curve follows the data well from threshold to just above 1170 MeV/c, whereupon it continues to rise steadily, departing from the data. (The other curves will be discussed in the following subsections.)

For the angular distributions the differential cross sections are obtained by integrating Eq. (5) with respect to the appropriate variables. Thus,

$$d\sigma/d \cos \xi = [(D_1 - \frac{1}{2} \times 5^{1/2} D_6) + (\sqrt{3} D_2) \cos \xi + (\frac{3}{2} \times 5^{1/2} D_6) \cos^2 \xi] / (\text{No. of bins}), \quad (12)$$

$$d\sigma/d\phi = [D_1 + (-\frac{3}{2} \pi^2 D_5) \cos \phi] / (\text{No. of bins}), \quad (13)$$

$$d\sigma/d \cos \theta = [(D_1 - \frac{1}{2} \times 5^{1/2} D_7) + (\sqrt{3} D_3) \cos \theta + (\frac{3}{2} \times 5^{1/2} D_7) \cos^2 \theta] / (\text{No. of bins}). \quad (14)$$

The values of the seven D_i 's at 1050 and 1170 MeV/c, as given by the values of the parameters, Eq. (11), are listed in Table IV beside the experimental values. The curves shown in the angular distributions, Figs. 9–11, are obtained by using the fitted D_i 's in Eqs. (12)–(14), respectively. It is seen that the curves follow the data quite well.

The mass distributions $d\sigma/dM_{\pi p}^2$ and $d\sigma/dM_{\eta p}^2$ can be obtained by integrating the expression

$$d^2\sigma = \frac{1}{kE^2} \frac{dM_{\pi p}^2 dM_{\eta p}^2}{4E} |\Im \Pi(M_{\pi p}^2, M_{\eta p}^2)|^2,$$

which is a differential element of the Dalitz plot, with respect to the appropriate variable. Thus,

$$\frac{d\sigma}{dM_{\pi p}^2} = \frac{1}{kE^2} \rho C_1(M_{\pi p}^2), \quad (15)$$

$$\frac{d\sigma}{dM_{\eta p}^2} = \frac{1}{4kE^3} \int_{\min(M_{\eta p}^2)}^{\max(M_{\eta p}^2)} C_1(M_{\pi p}^2) dM_{\pi p}^2. \quad (16)$$

These parametrized curves are shown in Fig. 8.

From Eq. (10) we see that D_1 consists of the sum of the intensities of the four $I = \frac{3}{2}$ amplitudes and is there-

fore the sum of four partial cross sections.²⁶ These partial cross sections are listed in Table VII. We see that the $\Delta(1238)$ accounts for more than half of the reaction at 1050 MeV/c and about three-fourths of the reaction at 1170/c, with the rest accounted for by non-resonant three-body states in which the η is produced in both s wave and p wave with the $(\pi-p)$ system in a relative S state.

C. Other Experiments Involving the Reaction $\pi^+p \rightarrow \pi^+p\eta$

We found in Sec. V B that the cross section, as given by the parameters of Eq. (11), rises steeply beyond 1170 MeV/c, departing from the data. In an attempt to get a parametrized cross section which better follows the data above 1170 MeV/c, we introduce an energy dependence into the quantities $A_{\alpha\beta}$. We replace $A_{\alpha\beta}$ with $A_{\alpha\beta}/[1+(u/\pi)^{1/2}f_{\alpha\beta}(E)]$, where $f_{\alpha\beta}(E)$ is a function of energy such that at threshold $f_{\alpha\beta}(E)=0$. The function $f_{\alpha\beta}(E)$ is chosen such that at high energy it cancels the threshold kinematical factors; hence,

$$\sigma_{\alpha\beta} \xrightarrow{k \rightarrow \infty} (1/u)(\mathcal{J}_{\alpha\beta} + \frac{1}{2})\pi/k^2,$$

which is $1/u$ times the unitarity limit for that partial wave. Thus, we choose

$$f_{\alpha\beta} = n_{\alpha\beta}(A_{\alpha\beta}/E)(k^2 \mathcal{L}^{+1} p_M^{2l+1} q^{2L+1} Q)^{1/2}$$

for the nonresonant amplitudes, and

$$f_{\alpha\beta} = n_{\alpha\beta}(A_{\alpha\beta}/E)(k^2 \mathcal{L}^{+1} p_M^{2l+1} Q)^{1/2}$$

for the resonant amplitude, where p_M and q_M are the maximum values of p and q , respectively, at energy E and $Q = E - m_\eta - m_\pi - m_p$. The constant $n_{\alpha\beta}$ is chosen such that, at high energy,

$$\left[(A_{\alpha\beta}^2/E^2) k^{2\mathcal{L}+1} \int \rho p^{2l} g dM^2 \right] / [f_{\alpha\beta}(E)]^2 = 1,$$

where $g = F$, the Breit-Wigner intensity, for the resonant amplitude, and $g = 1$ for the nonresonant amplitude.

²⁶ It is interesting to note that the fact that there are no interference terms between one partial wave and another in the cross-section expression D_1 is a general result and not a consequence of the particular partial waves chosen. (For example, see Table IX.) In order for D_1 , and hence the mass distributions $d\sigma/dM_{\pi p^2}$ and $d\sigma/dM_{\pi p^2}$, to exhibit interference terms, one of two conditions must exist. The first is the presence of two or more amplitudes which belong to the same partial wave, such as resonant and nonresonant amplitudes or two or more amplitudes of different l spin. There will then be interference terms arising from these amplitudes of the same partial wave. The second condition which gives rise to interference terms is that the matrix element consists of both an amplitude which is a function of M^2 (one projection of the production Dalitz plot) and M'^2 (another projection of the Dalitz plot). (See Appendix B.) There will then be interference terms between amplitudes of M^2 and amplitudes of M'^2 which come from the same initial state $\mathcal{L}g$.

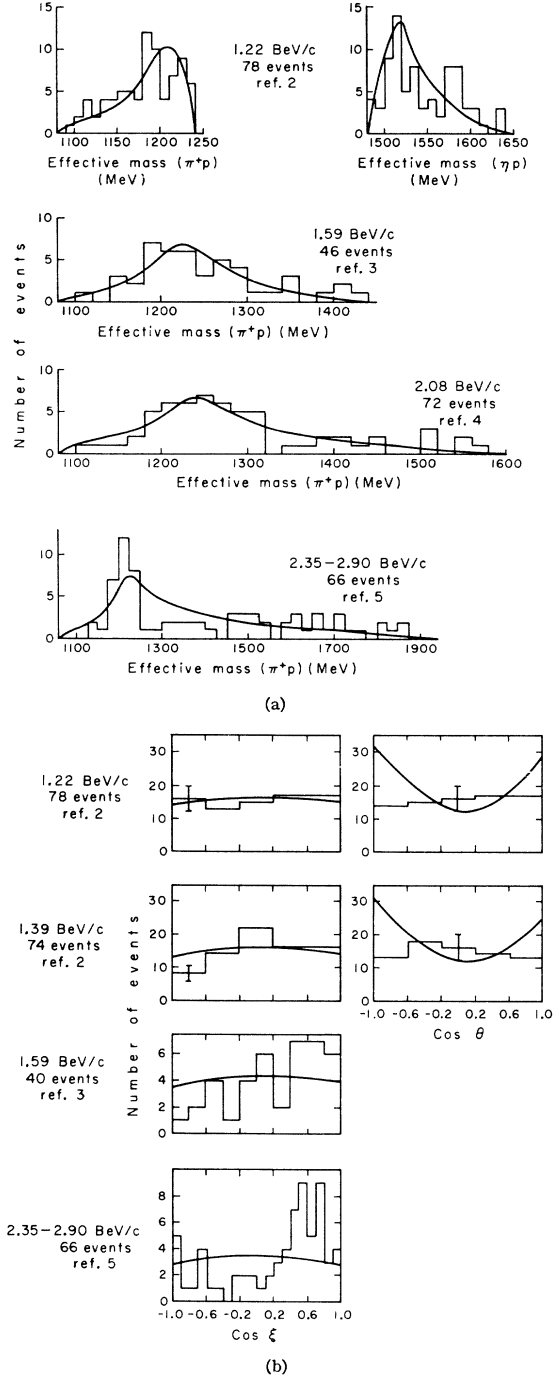


FIG. 12. Mass and angular distributions for the reaction $\pi^+p \rightarrow \pi^+p\eta$ published in Refs. 2-7. The curves are those given by our parametrization of Sec. V C, and are normalized to the total number of events. (a) Mass distributions; (b) angular distributions.

For simplicity we take the dimensionless parameter u to be the same for all four amplitudes. Since $u=1$ for the unitarity limit, we must have $u \geq 1$ for nonviolation of unitarity.

Fitting only our 1050- and 1170-MeV/ c data to this hypothesis, we obtain

$$\begin{aligned}
 A_{00} &= (71_{-25}^{+27})(\mu\text{b}/\text{BeV}^2)^{1/2}, \\
 \lambda_{00} &= (4.89_{-0.37}^{+0.28}) \text{ rad}, \\
 A_{10} &= (212_{-62}^{+74})(\mu\text{b}/\text{BeV}^2)^{1/2}, \\
 \lambda_{10} &= (4.36_{-0.28}^{+0.21}) \text{ rad}, \\
 A_{30} &= (329_{-90}^{+87})(\mu\text{b}/\text{BeV}^2)^{1/2}, \\
 \lambda_{30} &= (0.61_{-0.38}^{+0.23}) \text{ rad}, \\
 A_{03} &= (296_{-35}^{+80})(\mu\text{b}/\text{BeV}^3)^{1/2}, \\
 u &= (94_{-32}^{+17}),
 \end{aligned} \tag{17}$$

with $\chi^2 = 80$, $\langle\chi^2\rangle = 70$.

Figure 6 displays the cross-section curve resulting from the parameters of Eq. (17). We see that our best-fit value, $u = 94$, gives a predicted cross section that lies below the ten measured cross sections located between 1.2 and 3.7 BeV/ c . (The over-all χ^2 is $80 + 25$, with $\langle\chi^2\rangle = 70 + 10$.) By decreasing u to 67, we decrease the over-all χ^2 to $81 + 15 = 96$, where the contribution from our data alone is 81. This is in good agreement with $\langle\chi^2\rangle = 80 \pm (2\langle\chi^2\rangle)^{1/2} = 80 \pm 13$. Thus our simple model for the energy dependence of the parameters $A_{\alpha\beta}$, involving just one additional parameter u , gives good agreement with all of our data and with the 10 additional cross sections lying between 1.2 and 3.7 BeV/ c . However, in order for the parameter values to be meaningful, we must be able to predict not only the value of the total cross section at a particular beam momentum, but also all the mass and angular distributions at that momentum.

The published distributions of Refs. 2-7 are reproduced in Fig. 12; the curves are those predicted by the parameters which produce the $u = 67$ curve of Fig. 6. It is seen that all mass distributions are in good agreement. The production angular distributions are in good agreement at low energy, but our fit is unable to produce the forward peaking that arises with increased beam momentum. The only decay angular distributions which are available are those of Ref. 2, and they are not well represented by the curves. Thus, the assumption of phases which are constant with beam momentum probably becomes inadequate, and/or more partial waves become necessary at these higher energies.

D. π^- Data

While the π^+p state consists of only $I = \frac{3}{2}$ amplitudes, the π^-p state has both $I = \frac{3}{2}$ and $I = \frac{1}{2}$ amplitudes. Thus, for the π^-p reaction *each* complex amplitude (LL_J) becomes

$$(LL_J) \rightarrow \frac{1}{3}(LL_J) g^{I=1/2} + \frac{2}{3}(LL_J) g^{I=3/2}, \tag{18}$$

and therefore

$$\begin{aligned}
 [(LL_J) g][(\ell'L'J') g']^* &\rightarrow \frac{1}{9}[(LL_J) g^{I=3/2}] \\
 &\quad \times [(\ell'L'J') g'^{I=3/2}]^* + (2/9)[(LL_J) g^{I=3/2}] \\
 &\quad \times [(\ell'L'J') g'^{I=1/2}]^* + (2/9)[(LL_J) g^{I=1/2}] \\
 &\quad \times [(\ell'L'J') g'^{I=3/2}]^* + (4/9)[(LL_J) g^{I=1/2}] \\
 &\quad \times [(\ell'L'J') g'^{I=1/2}]^*.
 \end{aligned}$$

We see that if all the $I = \frac{1}{2}$ amplitudes were zero, then each $C_i(M^2)$ for the π^- data would be $\frac{1}{9}$ of the corresponding $C_i(M^2)$ for the π^+ data (at the same momentum). Of course, the same relationship would hold for each D_i .

From Table IV we see that at 1170 MeV/ c , $D_1(\pi^-)$ is not equal to $\frac{1}{9}$ of $D_1(\pi^+)$; i.e., the ratio of the total cross sections is not $\frac{1}{9}$. Therefore, at least one $I = \frac{1}{2}$ amplitude is required. We also see that, like the π^+ data, each $D_i(\pi^-)$, $i = 8$ to 14, is consistent with zero so that there is no need to consider $I = \frac{1}{2}$ pP amplitudes. Next, we see from Table V that if the $I = \frac{1}{2}$ ($sS_{1/2}$) $_{1/2}$ amplitude is present, we expect it to contribute to $D_2(\pi^-)$ through sS - pS interference (where the pS amplitude exists in $I = \frac{3}{2}$) and to $D_3(\pi^-)$ through sS - sP interference (where the sP amplitude exists in $I = \frac{3}{2}$). Therefore, if the $I = \frac{1}{2}$ ($sS_{1/2}$) $_{1/2}$ amplitude contributes, $D_1(\pi^-)$, $D_2(\pi^-)$, and $D_3(\pi^-)$ are not constrained to be $\frac{1}{9}$ of their corresponding π^+ values. Similarly, from Table V we see that if an $I = \frac{1}{2}$ (pS) amplitude is present, then $D_1(\pi^-)$, $D_2(\pi^-)$, $D_4(\pi^-)$, $D_5(\pi^-)$, and $D_6(\pi^-)$ are not constrained to be $\frac{1}{9}$ of their corresponding π^+ values. Finally, if an $I = \frac{1}{2}$ (sP) amplitude is present, then $D_1(\pi^-)$, $D_3(\pi^-)$, $D_4(\pi^-)$, $D_5(\pi^-)$, and $D_7(\pi^-)$ are not constrained to the $\frac{1}{9}$ value. Examination of Table IV shows that, except for D_1 , each experimental value $D_i(\pi^-)$ (column 3a) is *consistent* with $\frac{1}{9}$ of the corresponding experimental value $D_i(\pi^+)$ (column 2a); the errors involved, however, are so large as to make the values insensitive to any inconsistencies with the $\frac{1}{9}$ value. Thus, other than the fact that at least one $I = \frac{1}{2}$ amplitude is required, comparison of the experimental π^- values with the experimental π^+ values (comparison of column 3a with column 2a of Table IV) leads to no new information as to which $I = \frac{1}{2}$ amplitudes are needed.

However, if we now make use of the values of the $I = \frac{3}{2}$ amplitudes as determined by our fit to the π^+ data, we can compare the $D_i(\pi^-)$ values (column 3a) with the $D_i(\pi^+)$ values of column 2b. If this is done, we see that $D_2(\pi^-) = 2.90 \pm 1.63$ is almost two standard deviations away from $\frac{1}{9}$ of $D_2(\pi^+) = 0.13$, indicating (from Table V) that sS and/or pS $I = \frac{1}{2}$ amplitudes may be present.

Considering the three $I = \frac{1}{2}$ complex amplitudes, ($sS_{1/2}$) $_{1/2}$, ($pS_{1/2}$) $_{1/2, 3/2}$, we proceeded, as we did with the π^+ data, to determine the magnitudes and phases (or the real and imaginary parts) of these amplitudes. Each $C_i(M^2)$ of Eq. (12) was modified to include the $I = \frac{1}{2}$ parameters. As prescribed by Eq. (18),

$$A_{\alpha\beta}e^{i\lambda_{\alpha\beta}} \rightarrow \frac{1}{3}A_{\alpha\beta}e^{i\lambda_{\alpha\beta}} + \frac{2}{3}B_{\alpha\beta}e^{i\nu_{\alpha\beta}} \quad (\alpha\beta=00, 10, 30)$$

and

$$A_{\alpha\beta}e^{i\lambda_{\alpha\beta}} \rightarrow \frac{1}{3}A_{\alpha\beta}e^{i\lambda_{\alpha\beta}} \quad (\alpha\beta=03),$$

where the B 's and the ν 's are the magnitudes and phases, respectively, of the $I=\frac{1}{2}$ complex amplitudes. Using these modified $C_i(M^2)$ and D_i expressions and using the A 's and λ 's of Eq. (11), we fitted the π^- data in the same manner as we had fitted the π^+ data, and obtained

$$\begin{aligned} \text{Re}\{B_{00}e^{i\nu_{00}}\} &= (38_{-17}^{+10}) (\mu\text{b}/\text{BeV}^2)^{1/2}, \\ \text{Im}\{B_{00}e^{i\nu_{00}}\} &= (-85_{-9}^{+10}) (\mu\text{b}/\text{BeV}^2)^{1/2}, \\ \text{Re}\{B_{10}e^{i\nu_{10}}\} &= (-68_{-113}^{+80}) (\mu\text{b}/\text{BeV}^2)^{1/2}, \\ \text{Im}\{B_{10}e^{i\nu_{10}}\} &= (77_{-99}^{+71}) (\mu\text{b}/\text{BeV}^2)^{1/2}, \\ \text{Re}\{B_{30}e^{i\nu_{30}}\} &= (151_{-140}^{+147}) (\mu\text{b}/\text{BeV}^6)^{1/2}, \\ \text{Im}\{B_{30}e^{i\nu_{30}}\} &= (-136_{-158}^{+262}) (\mu\text{b}/\text{BeV}^6)^{1/2}, \end{aligned}$$

with $\chi^2=34$ and $\langle\chi^2\rangle=33$.

We see that B_{10} and B_{30} are consistent with zero and that therefore $(sS_{1/2})_{1/2}$ is the only $I=\frac{1}{2}$ amplitude needed to fit the data. [Had we limited ourselves at the beginning of this subsection to only one $I=\frac{1}{2}$ amplitude, the simplest hypothesis would have been $(sS_{1/2})_{1/2}$; all other amplitudes are suppressed owing to angular momentum barriers with no strong forces acting, such as in the $\Delta(1238)$, to enhance any of them.] Setting $B_{10}=0$ and $B_{30}=0$, we obtain²⁵

$$B_{00}=(93_{-18}^{+11}) (\mu\text{b}/\text{BeV}^2)^{1/2}, \quad \nu_{00}=(5.12_{-0.17}^{+0.11}) \text{ rad},$$

with $\chi^2=37$ and $\langle\chi^2\rangle=37$.

The $D_i(\pi^-)$ values as given by the fit are given in column 3b of Table IV. The $D_i(\pi^-)$, $i=4$ to 7, are of course $\frac{1}{9}$ of the corresponding $D_i(\pi^+)$ value, as given by the π^+ fit, because B_{00} and ν_{00} do not enter into the expressions for these D 's. The π^- angular distributions are shown in Figs. 9–11; the curves are the distributions as given by Eqs. (12)–(14), using these fitted $D_i(\pi^-)$ values. Agreement is good.

Because the $(sP_{3/2})_{3/2}$ and $(pS_{1/2})_{1/2,3/2}$ amplitudes are assumed to be present only in the $I=\frac{3}{2}$ state, their relative contribution to the π^-p reaction can be computed as follows:

$$\begin{aligned} &(\text{Relative contribution to } \pi^-p \text{ reaction})\sigma(\pi^-p \rightarrow \pi^-p\eta) \\ &= \frac{1}{9}(\text{relative contribution to } \pi^+p \text{ reaction}) \\ &\quad \times \sigma(\pi^+p \rightarrow \pi^+p\eta). \end{aligned}$$

Thus, as the reaction $\pi^+p \rightarrow \eta\Delta(1238)$ accounts for 73% of η production in the π^+p reaction at 1170 MeV/c, the reaction $\pi^-p \rightarrow \eta\Delta(1238)$ accounts for $\frac{1}{9}$ (73%) (53.3 $\mu\text{b}/15.9 \mu\text{b}$) = 27% of η production in the π^-p reaction at this momentum. Similarly, 6% of the π^-p reaction proceeds via $I=\frac{3}{2}$ (pS) amplitudes. The remainder of the reaction, 67%, is accounted for by the total sS intensity, which is composed of both $I=\frac{3}{2}$ and $I=\frac{1}{2}$ amplitudes. We cannot say how much of the reaction proceeds via the $I=\frac{1}{2}$ sS amplitude alone or

via the $I=\frac{3}{2}$ sS amplitude alone, because in any expression involving these two amplitudes there is always present an interference term between them.²⁶ This interference term is present in $C_1(M^2)$, and hence in $d\sigma/dM_{\pi p^2}$ and in $d\sigma/dM_{\eta 0^2}$. Thus only the total sS contribution can be given. The relative contributions and the partial cross sections for the π^-p reaction at 1170 MeV/c are listed in Table VII.

The distributions in $M_{\pi p^2}$ and $M_{\eta 0^2}$, as given by the values of the π^- parameters, are shown in Fig. 6. Again, agreement is good.

If we extrapolate the π^- parameters, assuming phases which are constant with beam momentum and an amplitude dependence as prescribed in Sec. V C with $u=67$, we obtain the dashed curve of Fig. 6. The curve is in good agreement with the π^- cross-section points.

There are no mass or angular distributions published for points h^8 and i^9 . For points j^{10} there are mass distributions which indicate that the $N^*(1688)$ and the $A_2(1300)$ are produced in addition to the $\Delta(1238)$. Thus more amplitudes are necessary at high energy.

E. $N^*(1550) \rightarrow \eta p$ Resonance

In the last few years the existence of a resonance, the $N^*(1550)$, which decays mainly into ηp has been inferred on the basis of phase-shift analysis of π^-p elastic scattering²⁷ and on the basis of experiments which exhibit a cross-section enhancement near threshold in the reactions $\gamma p \rightarrow \eta n$ and $\pi^-p \rightarrow \eta n$.²⁸ There has been one experiment reported in which the resonance has been seen as an enhancement in the η - N mass distribution in a reaction involving the final state of three or more particles.²⁹

We see from our Dalitz plots in Fig. 8 that the mass and width values of the $N^*(1550)$ are such as to fill up the entire region of the plots; thus it is difficult to

²⁷ P. Auvil, C. Lovelace, A. Donnachie, and A. T. Lea, Phys. Letters **12**, 76 (1964); **19**, 148 (1965); P. Bareyre, C. Brickman, A. V. Sterling, and G. Villet, *ibid.* **18**, 342 (1965); R. J. Cence, *ibid.* **20**, 306 (1966); L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. **138**, B190 (1965); B. H. Bransden, P. J. O'Donnell, and R. G. Moorhouse, *ibid.* **139**, B1566 (1965); Proc. Roy. Soc. (London) **A289**, 538 (1966); Phys. Letters **19**, 420 (1965).

²⁸ F. Bulos, R. E. Lanou, A. E. Pifer, A. M. Shapiro, M. Widgoff, R. Panvini, A. E. Brenner, C. A. Bordner, M. E. Law, E. E. Ronat, K. Strauch, J. J. Szymanski, P. Bastien, B. B. Brabson, Y. Eisenberg, B. T. Feld, V. K. Fisher, I. A. Pless, L. Rosenson, R. K. Yamamoto, G. Calvelli, L. Guerriero, G. A. Salandin, A. Tomasin, L. Ventura, C. Voci, and F. Waldner, Phys. Rev. Letters **13**, 486 (1964); W. Bruce Richards, Charles B. Chiu, Richard D. Eandi, A. Carl Helmholtz, Robert W. Kenny, Burton J. Moyer, John A. Poirier, Robert J. Cence, Vincent Z. Peterson, Narendar K. Sehgal, and Victor J. Stenger, *ibid.* **16**, 1221 (1966); F. Uchiyama-Campbell and R. K. Logan, Phys. Rev. **149**, 1220 (1966); C. Bacci, G. Penso, G. Salvini, C. Mencuccini, and V. Silvestrini, Nuovo Cimento **45**, 983 (1966); W. G. Jones, D. M. Binnie, A. Duane, J. P. Horsey, D. C. Mason, J. A. Newth, I. U. Rahman, J. Walters, N. Horwitz, and P. Palit, Phys. Letters **23**, 597 (1966); R. Prepost, D. Lundquist, and D. Quinn, *ibid.* **18**, 82 (1967).

²⁹ G. Bassompierre, Y. Goldschmidt-Clermont, A. Grant, V. P. Henri, B. Jongejans, R. L. Lander, D. Linglin, F. Muller, J. M. Perreau, R. Sekulin, W. De Baere, J. Debaisieux, P. DuFour, F. Grard, J. Heughebaert, L. Pape, P. Peeters, F. Verbeure, and R. Windmolders, Phys. Letters **25B**, 440 (1967).

deduce the presence or absence of this resonance in our experiment. In order to make use of all four independent variables and all the correlations which exist among the variables, we have added amplitudes for the production of the $N^*(1550)$ resonance to the matrix element. (See Appendix B.) The $C_i(M_{\pi p^2})$ expressions derived in Appendix A are now replaced by more complicated expressions, $C_i(M_{\pi p^2}, M_{\eta p^2})$, and these new expressions are used to fit the data.

For the π^+ data we included in the fit $I=\frac{3}{2}$ amplitudes for s - and p -wave production of the $N^*(1550)$ along with the original four $I=\frac{3}{2}$ amplitudes of Sec. V B. At 1050 MeV/ c , addition of the $N^*(1550)$ amplitudes does not change the χ^2 minimum from that obtained with just the original four amplitudes. At 1170 MeV/ c , we find a χ^2 minimum at $\text{Re}\{s\text{-wave amplitude}\} = (-4_{-13}^{+7}) (\mu\text{b}/\text{BeV})^{1/2}$, $\text{Im}\{s\text{-wave amplitude}\} = (-8_{-15}^{+8}) (\mu\text{b}/\text{BeV})^{1/2}$, and p -wave amplitudes equal zero, with $\chi^2=41.0$, $\langle\chi^2\rangle=38$; setting both s - and p -wave $N^*(1550)$ amplitudes equal to zero, we obtain $\chi^2=42.3$ and $\langle\chi^2\rangle=40$. Thus, production of the $N^*(1550) \rightarrow \eta p$ is undetectable in our π^+ data.

In fitting the π^- data, we set the $I=\frac{3}{2}$ amplitudes for $N^*(1550)$ production equal to zero because the π^- data show that none are required. We find a χ^2 minimum at $\text{Re}\{s\text{-wave amplitude}\} = 1.1 \pm 2.1 (\mu\text{b}/\text{BeV})^{1/2}$, $\text{Im}\{s\text{-wave amplitude}\} = 0.3 \pm 1.7 (\mu\text{b}/\text{BeV})^{1/2}$, and p -wave amplitudes equal zero, with $\chi^2=37.7$ and $\langle\chi^2\rangle=43$; setting both s - and p -wave $N^*(1550)$ amplitudes equal to zero, we obtain $\chi^2=37.8$ and $\langle\chi^2\rangle=45$. Thus the decay $N^*(1550) \rightarrow \eta p$ is undetectable in our π^- data.

We next ask: What are the upper limits for the reaction $\pi^\pm p \rightarrow \pi^\pm N^*(1550)$; $N^*(1550) \rightarrow \eta p$? When the matrix element contains amplitudes for $N^*(1550)$ production, the cross section is not composed of the sum of the partial cross sections due to each amplitude. There are always present interference terms involving the $N^*(1550)$.²⁶ Thus, in obtaining an upper limit, we cannot quote a cross section due to the $N^*(1550)$ alone. If, however, we attribute all interference terms involving the $N^*(1550)$, as well as the $N^*(1550)$ intensity terms, to the formation of the resonance, we obtain at the 90% confidence level

$$\begin{aligned} \sigma[\pi^+ p \rightarrow \pi^+ N^*(1550); N^*(1550) \rightarrow \eta p] \\ \sigma(\pi^+ p \rightarrow \pi^+ p \eta) \\ \leq 0.08 (0.8 \mu\text{b}), \text{ at } 1050 \text{ MeV}/c \\ \leq 0.19 (10.1 \mu\text{b}), \text{ at } 1170 \text{ MeV}/c. \end{aligned}$$

Assuming the $I=\frac{3}{2}$ amplitudes for $N^*(1550)$ production are zero at 1170 MeV/ c , we obtain at the 90% confidence level an upper limit on $I=\frac{1}{2} N^*(1550)$ cross section of 7.6 μb , or

$$\begin{aligned} \sigma[\pi^- p \rightarrow \pi^- N^*(1550); N^*(1550) \rightarrow \eta p] \\ \sigma(\pi^- p \rightarrow \pi^- p \eta) \leq 0.48. \end{aligned}$$

VI. SUMMARY

A simple phenomenological model has been presented which describes η production in the reaction $\pi p \rightarrow \pi p \eta$ near threshold. From a partial-wave analysis, using our π^+ data at 1050 and 1170 MeV/ c and π^- data at 1170 MeV/ c , we found that four $I=\frac{3}{2}$ amplitudes, $(sS_{1/2})_{1/2}$, $(sP_{3/2})_{3/2}$, and $(pS_{1/2})_{1/2, 3/2}$, are required by the π^+ data and that an additional $I=\frac{1}{2}$ amplitude, $(sS_{1/2})_{1/2}$, is required by the π^- data. The magnitudes and phases of these amplitudes have been determined.

We found that the $\pi^+ p$ reaction is dominated by s -wave production of the $\Delta(1238) \rightarrow \pi^+ p$ resonance, while the $\pi^- p$ reaction is dominated by the $I=\frac{1}{2}$ amplitude in which all particles are in relative S -states. The fitted parameters produce a cross-section variation with energy, and mass and angular distributions which are in good agreement with our data.

The parameters were then extrapolated to higher energies where we obtain good agreement with the energy variation of the cross section and with some of the published mass and angular distributions. It appears, however, that as the beam momentum increases, an energy variation of the phases and/or more partial waves probably become necessary.

We further find that the reaction $\pi p \rightarrow \pi N^*(1550)$, $N^*(1550) \rightarrow \eta p$ is undetectable in both our π^+ and π^- data; upper limits are obtained.

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APPENDIX A: DERIVATION OF MATRIX ELEMENT

We want to derive the absolute square of the matrix element, Eq. (6), arising from the 10 complex amplitudes considered in Sec. V. The ten amplitudes written

TABLE VIII. The 10 possible states resulting from s - and p -wave η and π - p production and S - and P -wave decay of the π - p system.

Final state (LL) g	Initial state $\mathcal{L}g$
$(sS_{1/2})_{1/2}$	$\mathcal{P}_{1/2}$
$(sP_{1/2})_{1/2}$	$\mathcal{S}_{1/2}$
$(sP_{3/2})_{3/2}$	$\mathcal{D}_{3/2}$
$(pS_{1/2})_{1/2}$	$\mathcal{S}_{1/2}$
$(pS_{1/2})_{3/2}$	$\mathcal{D}_{3/2}$
$(pP_{1/2})_{1/2}$	$\mathcal{P}_{1/2}$
$(pP_{1/2})_{3/2}$	$\mathcal{P}_{3/2}$
$(pP_{3/2})_{1/2}$	$\mathcal{P}_{1/2}$
$(pP_{3/2})_{3/2}$	$\mathcal{P}_{3/2}$
$(pP_{3/2})_{5/2}$	$\mathcal{F}_{5/2}$

in terms of $\mathcal{L}_J \rightarrow (lL_J)_J$ are listed in Table VIII. (See Sec. V for notation.)

We take the beam direction as the quantization axis. (See Sec. IV.) Along this axis the initial π - p state has $g_z = \pm \frac{1}{2}$, the spin projection of the proton, because there can be no component of the initial orbital angular momentum \mathcal{L} along the beam. We expand the states $(lL_J)_J$ in terms of spherical harmonics. In general,

$$\begin{aligned} (lL_J)_J &= a(M_{\pi p}^2) |lL_J, g_z\rangle \\ &= a(M_{\pi p}^2) \sum_{m=g_z-M_J} \langle lm; JM_J | g_z \rangle |lm\rangle |JM_J\rangle \\ &= a(M_{\pi p}^2) \sum_m \mu_m \mathcal{Y}_l^m(\xi, \omega) |JM_J\rangle, \end{aligned}$$

where the μ_m are Clebsch-Gordan coefficients, the $\mathcal{Y}_l^m(\xi, \omega)$ are spherical harmonics in the variables ξ and ω , $-l \leq m \leq l$, and $-J \leq M_J \leq J$. Expanding further,

we get

$$\begin{aligned} |JM_J\rangle &= \sum_{M=M_J-M_S} \langle LM; \frac{1}{2}M_S | JM_J \rangle |LM\rangle |\frac{1}{2}M_S\rangle \\ &= \sum_M \nu_M Y_L^M(\theta, \phi) \chi^{M_S}, \end{aligned}$$

where the ν_M are Clebsch-Gordan coefficients, the $Y_L^M(\theta, \phi)$ are spherical harmonics in the variables θ and ϕ , $-L \leq m \leq L$, χ^{M_S} are the decay-proton spin projectors, and $M_S = \pm \frac{1}{2}$. Combining the two expansions, we have

$$(lL_J)_J = a_{\alpha\beta}(M_{\pi p}^2) \sum_m \sum_M \mu_m \nu_M \mathcal{Y}_l^m(\xi, \omega) Y_L^M(\theta, \phi) \chi^{M_S},$$

where $\alpha=0$ for $l=0$ and $\alpha=2g$ for $l=1$; $\beta=0$ for $L=0$ and $\beta=2J$ for $L=1$.

In terms of this expansion, the 10 final-state complex amplitudes are

$$\begin{aligned} (sS_{1/2})_{1/2} &= a_{00}(\mathcal{Y}_0^0 Y_0^0 \chi^{\pm 1/2}), \\ (sP_{1/2})_{1/2} &= a_{01}[\mp(\frac{1}{3})^{1/2} \mathcal{Y}_0^0 Y_1^0 \chi^{\pm 1/2} \pm (\frac{2}{3})^{1/2} \mathcal{Y}_0^0 Y_1^{\pm 1} \chi^{\mp 1/2}], \\ (sP_{3/2})_{3/2} &= a_{03}[(\frac{2}{3})^{1/2} \mathcal{Y}_0^0 Y_1^0 \chi^{\pm 1/2} + (\frac{1}{3})^{1/2} \mathcal{Y}_0^0 Y_1^{\pm 1} \chi^{\mp 1/2}], \\ (pS_{1/2})_{1/2} &= a_{10}[\mp(\frac{1}{3})^{1/2} \mathcal{Y}_1^0 Y_0^0 \chi^{\pm 1/2} \pm (\frac{2}{3})^{1/2} \mathcal{Y}_1^{\pm 1} Y_0^0 \chi^{\mp 1/2}], \\ (pS_{1/2})_{3/2} &= a_{30}[(\frac{2}{3})^{1/2} \mathcal{Y}_1^0 Y_0^0 \chi^{\pm 1/2} + (\frac{1}{3})^{1/2} \mathcal{Y}_1^{\pm 1} Y_0^0 \chi^{\mp 1/2}], \\ (pP_{1/2})_{1/2} &= a_{11}[(-\frac{2}{3} \mathcal{Y}_1^{\pm 1} Y_1^{\mp 1} + \frac{1}{3} \mathcal{Y}_1^0 Y_1^0) \chi^{\pm 1/2} + (\frac{1}{3} \sqrt{2} \mathcal{Y}_1^{\pm 1} Y_1^0 - \frac{1}{3} \sqrt{2} \mathcal{Y}_1^0 Y_1^{\pm 1}) \chi^{\mp 1/2}], \\ (pP_{1/2})_{3/2} &= a_{31}[(\mp \frac{1}{3} \sqrt{2} \mathcal{Y}_1^{\pm 1} Y_1^{\mp 1} \mp \frac{1}{3} \sqrt{2} \mathcal{Y}_1^0 Y_1^0) \chi^{\pm 1/2} + (\pm \frac{1}{3} \mathcal{Y}_1^{\pm 1} Y_1^0 \pm \frac{2}{3} \mathcal{Y}_1^0 Y_1^{\pm 1}) \chi^{\mp 1/2}], \\ (pP_{3/2})_{1/2} &= a_{13}\{[(\frac{1}{2})^{1/2} \mathcal{Y}_1^{\mp 1} Y_1^{\pm 1} - \frac{1}{3} \sqrt{2} \mathcal{Y}_1^0 Y_1^0 - \frac{1}{6} \sqrt{2} \mathcal{Y}_1^{\pm 1} Y_1^{\mp 1}] \chi^{1/2} + (-\frac{1}{3} \mathcal{Y}_1^0 Y_1^{\pm 1} + \frac{1}{3} \mathcal{Y}_1^{\pm 1} Y_1^0) \chi^{\mp 1/2}\}, \\ (pP_{3/2})_{3/2} &= a_{33}\{[\pm(\frac{2}{3})^{1/2} \mathcal{Y}_1^{\mp 1} Y_1^{\pm 1} \pm \frac{1}{3}(\frac{2}{3})^{1/2} \mathcal{Y}_1^0 Y_1^0 \mp \frac{2}{3}(\frac{2}{3})^{1/2} \mathcal{Y}_1^{\pm 1} Y_1^{\mp 1}] \chi^{\pm 1/2} + [\pm \frac{1}{3}(\frac{1}{3})^{1/2} \mathcal{Y}_1^0 Y_1^{\pm 1} \\ &\quad \mp \frac{4}{3}(\frac{1}{3})^{1/2} \mathcal{Y}_1^{\pm 1} Y_1^0] \chi^{\mp 1/2}\}, \\ (pP_{3/2})_{5/2} &= a_{53}\{[(\frac{1}{10})^{1/2} \mathcal{Y}_1^{\mp 1} Y_1^{\pm 1} + (\frac{2}{3})^{1/2} \mathcal{Y}_1^0 Y_1^0 + (\frac{1}{10})^{1/2} \mathcal{Y}_1^{\pm 1} Y_1^{\mp 1}] \chi^{\pm 1/2} + [(\frac{1}{5})^{1/2} \mathcal{Y}_1^0 Y_1^{\pm 1} + (\frac{1}{5})^{1/2} \mathcal{Y}_1^{\pm 1} Y_1^0] \chi^{\mp 1/2}\}, \end{aligned}$$

where the upper sign is used when the spin projection of the *initial* proton is $+\frac{1}{2}$, and the lower sign is used when the spin projection is $-\frac{1}{2}$. (All arguments have been suppressed.)

The matrix element is now the sum of these expanded complex amplitudes,

$$\mathfrak{M}^{\pm} = \sum_{l=0,1} \sum_{L=0,1} (g + \frac{1}{2})^{1/2} \begin{pmatrix} -1 \\ +1 \end{pmatrix}^{g+\mathcal{L}+\frac{1}{2}} a_{\alpha\beta}(M_{\pi p}^2) \sum_m \sum_M \mu_m \nu_M \mathcal{Y}_l^m(\xi, \omega) Y_L^M(\theta, \phi) \chi^{\pm 1/2},$$

where \mathfrak{M}^+ is that portion of the matrix element for $g_z = +\frac{1}{2}$ and \mathfrak{M}^- is for $g_z = -\frac{1}{2}$ and the factors dependent on \mathcal{L} and g result from the expansion of an incoming plane wave into an outgoing spherical wave. The desired expression, Eq. (6), is $|\mathfrak{M}|^2 = (\frac{1}{2}) |\mathfrak{M}^+|^2 + (\frac{1}{2}) |\mathfrak{M}^-|^2$.

We can now gather terms and write

$$\mathfrak{M}^{\pm} = G^{\pm} \chi^{\pm 1/2} + H^{\pm} \chi^{\mp 1/2},$$

where

$$\begin{aligned} G^{\pm} &= a_{00} \mathcal{Y}_0^0 Y_0^0 + (\frac{1}{3})^{1/2} (a_{10} + 2a_{30}) \mathcal{Y}_1^0 Y_0^0 + (\frac{1}{3})^{1/2} (a_{01} + 2a_{03}) \mathcal{Y}_0^0 Y_1^0 + \frac{1}{3} \{ (a_{11} + 2a_{31}) - \sqrt{2} [a_{13} + (\frac{2}{3})^{1/2} a_{33} - 3(\frac{3}{5})^{1/2} a_{53}] \} \\ &\quad \times \mathcal{Y}_1^0 Y_1^0 - \frac{2}{3} \{ (a_{11} - a_{31}) - \frac{1}{4} \sqrt{2} [a_{13} + 4(\frac{2}{3})^{1/2} a_{33} + 3(\frac{3}{5})^{1/2} a_{53}] \} \mathcal{Y}_1^{\pm 1} Y_1^{\mp 1} \\ &\quad + \frac{1}{2} \sqrt{2} [a_{13} - 2(\frac{2}{3})^{1/2} a_{33} + (\frac{3}{5})^{1/2} a_{53}] \mathcal{Y}_1^{\mp 1} Y_1^{\pm 1}, \\ H^{\pm} &= -(\frac{2}{3})^{1/2} (a_{10} - a_{30}) \mathcal{Y}_1^{\pm 1} Y_0^0 - (\frac{2}{3})^{1/2} (a_{01} - a_{03}) \mathcal{Y}_0^0 Y_1^{\pm 1} \\ &\quad - \frac{1}{3} \sqrt{2} \{ (a_{11} + 2a_{31}) + \frac{1}{2} \sqrt{2} [a_{13} + (\frac{2}{3})^{1/2} a_{33} - 3(\frac{3}{5})^{1/2} a_{53}] \} \mathcal{Y}_1^0 Y_1^{\pm 1} \\ &\quad + \frac{1}{3} \sqrt{2} \{ (a_{11} - a_{31}) + \frac{1}{2} \sqrt{2} [a_{13} + 4(\frac{2}{3})^{1/2} a_{33} + 3(\frac{3}{5})^{1/2} a_{53}] \} \mathcal{Y}_1^{\pm 1} Y_1^0. \end{aligned}$$

TABLE IX. Definition of the $C_i(M^2)$'s and D_i 's in terms of the a 's. The terms within heavy brackets are zero under the choice of amplitudes made in Sec. V.

C_1	$ a_{00} ^2 + [a_{01} ^2] + 2 a_{03} ^2 + a_{10} ^2 + 2 a_{30} ^2 + [a_{11} ^2 + 2 a_{31} ^2 + a_{13} ^2 + 2 a_{33} ^2 + 3 a_{53} ^2]$
C_2	$\frac{2}{3}\sqrt{3}\langle a_{00} (a_{10} + 2a_{30}) \rangle + (2/3\sqrt{3})\langle ([a_{01}] + 2a_{03}) [(a_{11} + 2a_{31}) - \sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$ $+ (4/9)\langle ([a_{01}] - a_{03}) [(a_{11} + 2a_{31}) + \frac{1}{2}\sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$
C_3	$\frac{2}{3}\sqrt{3}\langle a_{00} ([a_{01}] + 2a_{03}) \rangle + (2/3\sqrt{3})\langle (a_{10} + 2a_{30}) [(a_{11} + 2a_{31}) - \sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$ $+ (4/9)\langle (a_{10} - a_{30}) [(a_{11} - a_{31}) + \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$
C_4	$\frac{2}{3}\langle (a_{10} + 2a_{30}) ([a_{01}] + 2a_{03}) \rangle + \frac{2}{3}\langle a_{00} [(a_{11} + 2a_{31}) - \sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$
C_5	$-\frac{2}{3}\langle (a_{10} - a_{30}) ([a_{01}] - a_{03}) \rangle + \frac{2}{3}\langle a_{00} [(a_{11} - a_{31}) - \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$ $-\sqrt{2}\langle a_{00} [[a_{13} - 2(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$
C_6	$(2/3(5)^{1/2}) (a_{10} + 2a_{30}) ^2 - (2/3(5)^{1/2}) (a_{10} - a_{30}) ^2$ $+ [(2/9(5)^{1/2}) (a_{11} + 2a_{31}) - \sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] ^2$ $- (4/9(5)^{1/2}) (a_{11} - a_{31}) - \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] ^2$ $+ (4/9(5)^{1/2}) (a_{11} + 2a_{31}) + \frac{1}{2}\sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] ^2$ $- (2/9(5)^{1/2}) (a_{11} - a_{31}) + \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] ^2$ $- (1/2(5)^{1/2}) [a_{13} - 2(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}] ^2]$
C_7	$(2/3(5)^{1/2}) ([a_{01}] + 2a_{03}) ^2 - (2/3(5)^{1/2}) ([a_{01}] - a_{03}) ^2$ $+ [(2/9(5)^{1/2}) (a_{11} + 2a_{31}) - \sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] ^2$ $- (4/9(5)^{1/2}) (a_{11} - a_{31}) - \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] ^2$ $- (2/9(5)^{1/2}) (a_{11} + 2a_{31}) + \frac{1}{2}\sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] ^2$ $+ (4/9(5)^{1/2}) (a_{11} - a_{31}) + \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] ^2$ $- (1/2(5)^{1/2}) [a_{13} - 2(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}] ^2]$
C_8	$(4/3(15)^{1/2})\langle (a_{10} + 2a_{30}) [(a_{11} + 2a_{31}) - \sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$ $- (4/9(5)^{1/2})\langle (a_{10} - a_{30}) [(a_{11} - a_{31}) + \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$
C_9	$(4/3(5)^{1/2})\langle (a_{10} + 2a_{30}) [(a_{11} - a_{31}) - \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$ $- (4/9)(\frac{2}{3})^{1/2}\langle (a_{10} - a_{30}) [(a_{11} + 2a_{31}) + \frac{1}{2}\sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$ $- (2/10^{1/2})\langle (a_{10} + 2a_{30}) [[a_{13} - 2(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$
C_{10}	$(4/3(15)^{1/2})\langle ([a_{01}] + 2a_{03}) [(a_{11} + 2a_{31}) - \sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$ $- (4/9(5)^{1/2})\langle ([a_{01}] - a_{03}) [(a_{11} + 2a_{31}) + \frac{1}{2}\sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$
C_{11}	$(4/3(5)^{1/2})\langle ([a_{01}] + 2a_{03}) [(a_{11} - a_{31}) - \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$ $- (4/9)(\frac{2}{3})^{1/2}\langle ([a_{01}] - a_{03}) [(a_{11} - a_{31}) + \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$ $- (2/10^{1/2})\langle ([a_{01}] + 2a_{03}) [[a_{13} - 2(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$
C_{12}	$[(4/45) (a_{11} + 2a_{31}) - \sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] ^2$ $+ (4/45) (a_{11} + 2a_{31}) + \frac{1}{2}\sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] ^2$ $+ (4/45) (a_{11} - a_{31}) - \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] ^2$ $+ (4/45) (a_{11} - a_{31}) + \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] ^2$ $+ \frac{1}{16}[a_{13} - 2(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}] ^2]$
C_{13}	$[(4/15)\langle (a_{11} + 2a_{31}) - \sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] (a_{11} - a_{31}) - \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$ $- (4/15)\langle (a_{11} + 2a_{31}) + \frac{1}{2}\sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] (a_{11} - a_{31}) + \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$ $- \frac{1}{2}\sqrt{2}\langle (a_{11} + 2a_{31}) - \sqrt{2}[a_{13} + (\frac{2}{3})^{1/2}a_{33} - 3(\frac{2}{3})^{1/2}a_{53}]] [a_{13} - 2(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] \rangle$
C_{14}	$[-\frac{2}{3}\sqrt{2}\langle (a_{11} - a_{31}) - \frac{1}{2}\sqrt{2}[a_{13} + 4(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] [a_{13} - 2(\frac{2}{3})^{1/2}a_{33} + 3(\frac{2}{3})^{1/2}a_{53}]] \rangle]$

We note that upon taking the absolute square of the amplitudes G^\pm and H^\pm , the variables ϕ and ω always appear in the combination $\phi - \omega$. As defined by the spherical harmonics, ϕ and ω are measured with respect to some co-ordinate system fixed in space. The angle ω is the azimuthal angle of the production plane in space; it is of no interest in this experiment. What is of interest is the angle $\phi - \omega$, the azimuthal angle of the decay proton with respect to the production plane. We therefore redefine the angle $\phi - \omega$ as ϕ .

The absolute square of the matrix element, $|\mathfrak{M}|^2$, is now obtained by taking the absolute square of G^\pm and H^\pm . That is,

$$|\mathfrak{M}|^2 = \frac{1}{2}|\mathfrak{M}^+|^2 + \frac{1}{2}|\mathfrak{M}^-|^2 = \frac{1}{2}(|G^+|^2 + |H^+|^2 + |G^-|^2 + |H^-|^2). \quad (A1)$$

Gathering terms that have the same angular dependence, we have

$$\begin{aligned}
 |\mathfrak{M}|^2 &= \sum_{i=1}^{14} C_i (M_{\pi p^2})^{\frac{1}{2}} [\mathfrak{Y}_i^m(\xi, 0) Y_L^{-M}(\theta, \phi) + \mathfrak{Y}_i^{-m}(\xi, 0) Y_L^M(\theta, \phi)], \\
 &= C_1 \mathfrak{Y}_0^0 Y_0^0 + C_2 \mathfrak{Y}_1^0 Y_0^0 + C_3 \mathfrak{Y}_0^0 Y_1^0 + C_4 \mathfrak{Y}_1^0 Y_1^0 + C_5 \frac{1}{2} (\mathfrak{Y}_1^1 Y_1^{-1} + \mathfrak{Y}_1^{-1} Y_1^1) + C_6 \mathfrak{Y}_2^0 Y_0^0 + C_7 \mathfrak{Y}_0^0 Y_2^0 \\
 &\quad + C_8 \mathfrak{Y}_2^0 Y_1^0 + C_9 \frac{1}{2} (\mathfrak{Y}_2^1 Y_1^{-1} + \mathfrak{Y}_2^{-1} Y_1^1) + C_{10} \mathfrak{Y}_1^0 Y_2^0 + C_{11} \frac{1}{2} (\mathfrak{Y}_1^1 Y_2^{-1} + \mathfrak{Y}_1^{-1} Y_2^1) \\
 &\quad + C_{12} \mathfrak{Y}_2^0 Y_2^0 + C_{13} \frac{1}{2} (\mathfrak{Y}_2^1 Y_2^{-1} + \mathfrak{Y}_2^{-1} Y_2^1) + C_{14} \frac{1}{2} (\mathfrak{Y}_2^2 Y_2^{-2} + \mathfrak{Y}_2^{-2} Y_2^2), \quad (6)
 \end{aligned}$$

where the 14 C 's are real functions of $M_{\pi p^2}$ resulting from combinations of the various complex $a_{\alpha\beta}$'s. The dependence of each $C_i(M_{\pi p^2})$ on the $a_{\alpha\beta}(M_{\pi p^2})$'s is given in Table IX. If we are not interested specifically in the values of \mathfrak{g} (the total angular momentum of the state) and J (the total angular momentum of the final π - p state), we can look at the *type* of terms, $\langle LL' | \mathfrak{Y}^m \rangle$, which comprise each $C_i(M_{\pi p^2})$. These terms are listed in Table V for each $C_i(M_{\pi p^2})$. For example, from Table IX we see that $C_4 = \frac{2}{3} \langle (a_{10} + 2a_{30}) | (a_{01} + 2a_{03}) \rangle + \frac{2}{3} \langle a_{00} | (a_{11} + 2a_{31}) - \sqrt{2} [a_{13} + (\frac{2}{3})^{1/2} a_{33} - 3(\frac{2}{3})^{1/2} a_{53}] \rangle$. If we are not interested in the values of \mathfrak{g} and J , we see that C_4 consists of terms of the type $\langle pS | sP \rangle$ and $\langle sS | pP \rangle$.

In Sec. V, we showed that the only complex amplitudes which need to be considered are $(sS_{1/2})_{1/2}$, $(sP_{3/2})_{3/2}$, and $(pS_{1/2})_{1/2, 3/2}$. Thus G^\pm and H^\pm simplify to

$$\begin{aligned}
 G^\pm &= a_{00} + (a_{10} + 2a_{30}) \cos \xi + 2a_{03} \cos \theta, \\
 H^\pm &= (a_{10} - a_{30}) \sin \xi e^{\pm i\omega} - a_{03} \sin \theta e^{\pm i\phi}, \quad (A2)
 \end{aligned}$$

and each $C_i(M_{\pi p^2})$ simplifies considerably; all terms in Tables V and IX, which are within heavy brackets become zero. We therefore have

$$\begin{aligned}
 C_1 &= |a_{00}|^2 + 2|a_{03}|^2 + |a_{10}|^2 + 2|a_{30}|^2, \\
 C_2 &= \frac{2}{3}\sqrt{3} \operatorname{Re}(a_{00}^* a_{10} + 2a_{00}^* a_{30}), \\
 C_3 &= \frac{4}{3}\sqrt{3} \operatorname{Re}(a_{00}^* a_{03}), \\
 C_4 &= \frac{4}{3} \operatorname{Re}(a_{10}^* a_{03} + 2a_{30}^* a_{03}),
 \end{aligned}$$

$$\begin{aligned}
 G^\pm &= (a_{00} + a_{00}') + (a_{10} + 2a_{30}) \cos \xi + (a_{10}' + 2a_{30}') \cos \xi' + 2a_{03} \cos \theta, \\
 H^\pm &= (a_{10} - a_{30}) \sin \xi e^{\pm i\omega} + (a_{10}' - a_{30}') \sin \xi' e^{\pm i\omega'} - a_{03} \sin \theta e^{\pm i\phi}.
 \end{aligned}$$

By Eq. (A1),

$$\begin{aligned}
 |\mathfrak{M}|^2 &= (a_{00}^2 + 2a_{00}a_{00}' + a_{00}'^2) + (a_{10}^2 - 2a_{10}a_{30} + a_{30}^2) + (a_{10}'^2 - 2a_{10}'a_{30}' + a_{30}'^2) + a_{03}^2(1 + 3 \cos^2 \theta) \\
 &\quad + (6a_{10}a_{30} + 3a_{30}^2) \cos^2 \xi + (6a_{10}'a_{30}' + 3a_{30}'^2) \cos^2 \xi' + (2a_{10}a_{10}' + 4a_{10}a_{30}' + 4a_{30}a_{10}' + 8a_{30}a_{30}') \cos \xi \cos \xi' \\
 &\quad + (2a_{10}a_{10}' - 2a_{10}a_{30}' - 2a_{30}a_{10}' + 2a_{30}a_{30}') \sin \xi \sin \xi' \cos(\omega - \omega') + (2a_{00}a_{10} + 4a_{00}a_{30} + 2a_{00}'a_{10} + 4a_{00}'a_{30}) \cos \xi \\
 &\quad + (2a_{00}a_{10}' + 4a_{00}a_{30}' + 2a_{00}'a_{10}' + 4a_{00}'a_{30}') \cos \xi' + (4a_{00}a_{03} + 4a_{00}'a_{03}) \cos \theta + (4a_{10}a_{03} + 8a_{30}a_{03}) \cos \xi \cos \theta \\
 &\quad + (4a_{10}'a_{03} + 8a_{30}'a_{03}) \cos \xi' \cos \theta + (-2a_{10}a_{03} + 2a_{30}a_{03}) \sin \xi \sin \theta \cos(\omega - \phi) \\
 &\quad + (-2a_{10}'a_{03} + 2a_{30}'a_{03}) \sin \xi' \sin \theta \cos(\omega' - \phi), \quad (B1)
 \end{aligned}$$

where we use the shorthand notation $a_{\alpha\beta}^2 = |a_{\alpha\beta}|^2$ and

$$a_{\alpha\beta} a_{\alpha'\beta'} = \operatorname{Re}\{a_{\alpha\beta}^* a_{\alpha'\beta'}\}.$$

$$\begin{aligned}
 C_5 &= \frac{4}{3} \operatorname{Re}(a_{10}^* a_{03} - a_{30}^* a_{03}), \\
 C_6 &= (2/5^{1/2}) |a_{30}|^2 + (4/5^{1/2}) \operatorname{Re}(a_{10}^* a_{30}), \\
 C_7 &= (2/5^{1/2}) |a_{03}|^2.
 \end{aligned}$$

Expressing each complex $a_{\alpha\beta}(M_{\pi p^2})$ in terms of $[A_{\alpha\beta}]$ [kinematical factor] $[e^{i\lambda_{\alpha\beta}}]$, as prescribed in Sec. V, we obtain Eq. (9).

APPENDIX B: INCLUSION OF $N^*(1550) \rightarrow \eta p$ RESONANCE AMPLITUDES IN THE MATRIX ELEMENT

In Appendix A we showed that

$$\begin{aligned}
 G^\pm &= a_{00} + (a_{10} + 2a_{30}) \cos \xi + 2a_{03} \cos \theta, \\
 H^\pm &= (a_{10} - a_{30}) \sin \xi e^{\pm i\omega} - a_{03} \sin \theta e^{\pm i\phi}, \quad (A2)
 \end{aligned}$$

if we consider only the complex amplitudes $(sS_{1/2})_{1/2}$, $(sP_{3/2})_{3/2}$, and $(pS_{1/2})_{1/2, 3/2}$. All quantities are referred to the π - p rest frame and were defined in Appendix A.

In addition to the above amplitudes we also want to consider s' - and p' -wave production of the S' -wave resonance, the $N^*(1550) \rightarrow \eta p$. (We will use primes to indicate quantities referred to the η - p rest frame in the same way that the unprimed quantities are referred to the π - p rest frame.) Production via d' wave and higher need not be considered since these waves produce terms in $\cos^3 \xi$ and higher, $\cos^3 \theta$ and higher, and $\cos^2 \phi$ and higher; these higher-order terms are not required by the data. Adding amplitudes $(s'S'_{1/2})_{1/2}$ and $(p'S'_{1/2})_{1/2, 3/2}$ to Eq. (A2), we have

In Eq. (B1) we have a mixed set of variables. So that we may know which quantities belong in which C_i , the variables $\cos \xi'$, $\sin \xi' \cos \omega'$, and $\sin \xi' \sin \omega'$ must be

expressed in terms of the independent variables $\cos\xi$, $\cos\theta$, and ϕ . As in Appendix A, we define the azimuthal angle ω to be zero, thereby defining the plane of reference.

In order to express $\cos\xi'$ in terms of the independent angles, we note that by Lorentz transformation

$$\mathbf{q} = \mathbf{p}' + \mathfrak{G}[(\mathbf{p}' \cdot \mathfrak{G})(\gamma - 1)/\beta^2 - \gamma \epsilon_\pi], \quad (\text{B2})$$

where \mathbf{p} and ϵ_π are the momentum and energy, respectively, of the pion in the c.m. system, and

$$\begin{aligned} \mathfrak{G} &= -\mathbf{p}/(E - \epsilon_\eta), \\ \gamma &= (E - \epsilon_\eta)/M, \\ \epsilon_\eta &= (p^2 + m_\eta^2)^{1/2}. \end{aligned} \quad (\text{B3})$$

Using Eq. (B3) and the fact that $p^2 = (E - \epsilon_\eta)^2 - M^2 = (E - \epsilon_\eta + M)(E - \epsilon_\eta - M)$, $E_\pi = \gamma(\epsilon_\pi - \mathfrak{G} \cdot \mathbf{p}')$, we get $\mathbf{q} = \mathbf{p}' + \tau \mathbf{p}$, where $\tau = (\epsilon_\pi + E_\pi)/(E - \epsilon_\eta + M)$. Therefore,

$$\hat{p}' = (q/p')\hat{q} - (\tau p/p')\hat{p}. \quad (\text{B4})$$

The z axis, the direction of the beam in the c.m. system, is common to both the primed and unprimed coordinate

systems. Therefore, taking the z component of Eq. (B4), we get

$$\cos\xi' = (q/p') \cos\theta - (\tau p/p') \cos\xi. \quad (\text{B5})$$

To obtain the other two angular quantities we need the following space-angle relationships:

$$\hat{p} \cdot \hat{q} = \cos\xi \cos\theta + \sin\xi \sin\theta \cos\phi, \quad (\text{B6})$$

$$\hat{p}' \cdot \hat{q} = \cos\xi' \cos\theta + \sin\xi' \sin\theta \cos(\phi - \omega'), \quad (\text{B7})$$

$$\hat{p} \cdot \hat{p}' = \cos\xi \cos\xi' + \sin\xi \sin\xi' \cos\omega'. \quad (\text{B8})$$

Now equate Eq. (B8) with p dotted with Eq. (B4) and use Eqs. (B6) and (B5) to get

$$\sin\xi' \cos\omega' = (q/p') \sin\theta \cos\phi - (\tau p/p') \sin\xi. \quad (\text{B9})$$

To obtain the other angular quantity, equate Eq. (B7) with q dotted with Eq. (B4) and use Eqs. (B5), (B7), and (B9) to get

$$\sin\xi \sin\omega' = (q/p') \sin\theta \sin\phi. \quad (\text{B10})$$

We now put Eqs. (B5), (B9), and (B10) into Eq. (B1) and—converting to spherical harmonics—we obtain the quantities

$$\begin{aligned} C_1(M^2, M'^2) &= \{ (a_{00}^2 + 2a_{03}^2 + a_{10}^2 + 2a_{30}^2) + (a_{00}'^2 + 2a_{00}a_{00}') + (4a_{03}a_{30}')q/p' - (2a_{10}a_{10}' + 4a_{30}a_{30}')\tau p/p' \\ &\quad + (a_{10}'^2 + 2a_{30}'^2)[(q/p')^2 + (\tau p/p')^2] \}, \\ C_2(M^2, M'^2) &= [(2a_{00}a_{10} + 4a_{00}a_{30}) + (2a_{00}'a_{10} + 4a_{00}'a_{30}) - (2a_{00}a_{10}' + 4a_{00}a_{30}' + 2a_{00}'a_{10}' + 4a_{00}'a_{30}')\tau p/p']\frac{1}{3}\sqrt{3}, \\ C_3(M^2, M'^2) &= [(4a_{00}a_{03}) + (4a_{00}'a_{03}) + (2a_{00}a_{10}' + 4a_{00}a_{30}' + 2a_{00}'a_{10}' + 4a_{00}'a_{30}')q/p']\frac{1}{3}\sqrt{3}, \\ C_4(M^2, M'^2) &= [(4a_{03}a_{10} + 8a_{03}a_{30}) - (4a_{03}a_{10}' + 8a_{03}a_{30}')\tau p/p' + (2a_{10}a_{10}' + 4a_{10}a_{30}' + 4a_{10}'a_{30} + 8a_{30}a_{30}')q/p' \\ &\quad - (2a_{10}'^2 + 8a_{10}'a_{30}' + 8a_{30}'^2)\tau p/p'](\frac{1}{3}), \quad (\text{B11}) \\ C_5(M^2, M'^2) &= [(4a_{03}a_{10} - 4a_{03}a_{30}) - (4a_{03}a_{10}' - 4a_{03}a_{30}')\tau p/p' + (-4a_{10}a_{10}' + 4a_{10}a_{30}' + 4a_{10}'a_{30} - 4a_{30}a_{30}')q/p' \\ &\quad - (-4a_{10}'^2 + 8a_{10}'a_{30}' - 4a_{30}'^2)\tau p/p'](\frac{1}{3}), \\ C_6(M^2, M'^2) &= [(2a_{10}a_{30} + a_{30}^2) + (2a_{10}'a_{30}' + a_{30}'^2)\tau^2 p^2/p'^2 - (2a_{10}a_{30}' + 2a_{10}'a_{30} + 2a_{30}a_{30}')\tau p/p']2/5^{1/2}, \\ C_7(M^2, M'^2) &= [(a_{03}^2) + (2a_{03}a_{10}' + 2a_{03}a_{30}')q/p' + (2a_{10}'a_{30}' + a_{30}'^2)q^2/p'^2](2/5^{1/2}). \end{aligned}$$

We now need the M^2 dependence of the a factors, and the M'^2 dependence of the a' factors. For the π^+p data the M^2 dependence is given in Sec. V and the M'^2 dependence is as follows:

$$\begin{aligned} a_{00}'(M'^2) &= A_{00}'e^{i\lambda_{00}'}k(M'/q'\Gamma')^{1/2}[(1-x)^{1/2}/(\epsilon' - i)], \\ a_{10}'(M'^2) &= A_{10}'e^{i\lambda_{10}'}p'(M'/q'\Gamma')^{1/2}[(1-x)^{1/2}/(\epsilon' - i)], \\ a_{30}'(M'^2) &= A_{30}'e^{i\lambda_{30}'}p'k^2(M'/q'\Gamma')^{1/2}[(1-x)^{1/2}/(\epsilon' - i)], \end{aligned} \quad (\text{B12})$$

where²³ $\epsilon' = (M_0^2 - M'^2)/\Gamma'M_0$, $\Gamma' = \Gamma_0(q'/q_0)(M'/M_0)^{-1}$, with (Ref. 14) $M_0 = 1550$ MeV, $\Gamma_0 = 130$ MeV, $x = 0.30$, and q_0 the value of q' at resonance.

Substituting Eqs. (11) and (B12) into Eq. (B11) gives $C_i(M^2, M'^2)$ explicitly in terms of the $A_{\alpha\beta}$, $A_{\alpha\beta}'$, $\lambda_{\alpha\beta}$, $\lambda_{\alpha\beta}'$, and various momentum dependences. Now, integration over M'^2 , one variable of the production Dalitz plot, gives the C_i 's in terms of the other variable, M^2 . Treating these new expressions as described in Sec. V, we can fit the π^+p data and determine the quantities A_{00}' , A_{10}' , A_{30}' , λ_{00}' , λ_{10}' , and λ_{30}' .

For the π^-p data we make the substitution $A_{\alpha\beta}'e^{i\lambda_{\alpha\beta}'} \rightarrow \frac{1}{3}A_{\alpha\beta}'e^{i\lambda_{\alpha\beta}'} + \frac{2}{3}B_{\alpha\beta}'e^{i\nu_{\alpha\beta}'}$ and $A_{\alpha\beta}e^{i\lambda_{\alpha\beta}} \rightarrow \frac{1}{3}A_{\alpha\beta}e^{i\lambda_{\alpha\beta}} + \frac{2}{3}B_{\alpha\beta}e^{i\nu_{\alpha\beta}}$ except that $A_{00}e^{i\lambda_{00}} \rightarrow \frac{1}{3}A_{00}e^{i\lambda_{00}} + \frac{2}{3}B_{00}e^{i\nu_{00}}$. Now fitting the π^-p data, we obtain the quantities $B_{\alpha\beta}'$ and $\nu_{\alpha\beta}'$.