APPENDIX B

The following integrals used in the text are tabulated in Ref. 9:

$$\int_0^\infty \cos(px)\frac{\sin[b(x^2+a^2)^{1/2}]}{(x^2+c^2)(x^2+a^2)^{1/2}}dx = \pi e^{-|cp|}\frac{\sin[b(a^2-c^2)^{1/2}]}{2c(a^2-c^2)^{1/2}}, \quad \text{(B1)}$$

where $c \neq a$, $b \leqslant p$;

$$\int_0^\infty \cos(px)\cos[b(a^2+x^2)^{1/2}]\frac{c}{x^2+c^2}dx = \tfrac{1}{2}\pi e^{-|cp|}\cos[b(a^2-c^2)^{1/2}], \quad \text{(B2)}$$

where $b \leqslant p$;

$$\int_0^\infty \sin(px)x\frac{\sin[b(a^2+x^2)^{1/2}]}{(a^2+x^2)^{1/2}}\frac{c}{x^2+c^2}dx = \tfrac{1}{2}\pi\frac{c}{(a^2-c^2)^{1/2}}e^{-|cp|}\sin[b(a^2-c^2)^{1/2}], \quad \text{(B3)}$$

where $b < p < \infty$.

[9] A. Erdélyi, *Tables of Integral Transforms* (McGraw-Hill Book Co., New York, 1954).

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Quantum Mechanics of Paraparticles*

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We show that it is possible to formulate a consistent first-quantized theory of paraparticles, i.e., particles which are neither bosons nor fermions. We examine a number of properties of the theory and show that the formulation of Messiah and Greenberg in terms of generalized rays can be replaced by an equivalent formulation in which states are represented by rays in the usual way. We use this alternative formulation to establish some results of ordinary quantum mechanics. We examine in detail the consistency of the theory with the cluster law and show that paraparticles must have states associated with whole families of different permutation symmetries, according to the following rule: If a given particle has N-particle states associated with a given Young diagram, then it must have $(N-1)$-, $(N-2)$-, $\cdots$, two-particle states associated with *all* Young diagrams which can be obtained from the first by successively removing squares. This gives rise to infinitely many different kinds of paraparticle, all with rather complicated properties.

I. INTRODUCTION

THERE is no evidence that particles other than bosons or fermions exist in nature. There is also no evidence that any but the most stable known particles actually are either bosons or fermions.[1] The name "paraparticle" has been introduced for a particle which is neither boson nor fermion, and the possible existence of such particles has been discussed by several authors.[1–5]

Theories of paraparticles have been constructed in two ways. The first approach considers the allowed permutation symmetries of the multiparticle wave functions and is formulated within the framework of first-quantized quantum mechanics. (See Refs. 1–3.) The second approach is a second-quantized theory in which commutation relations more general than the customary Bose or Fermi type are considered. (See Refs. 2–5.) In this paper we discuss only the former (first-quantized) approach; we do not discuss the as-yet unexplained connection between the two types of theory.

Our aim is to examine the consistency of the first-quantized theory of paraparticles. In particular, we reexamine the argument of Steinmann[2] that parastatistics are incompatible with the important cluster law, which requires that two well-separated groups of particles can be treated as separate isolated systems.

* Work supported in part by the National Science Foundation and the U. S. Air Force Office of Scientific Research.

[1] The experimental situation with regard to the first-quantized theory discussed here is reviewed by A. M. L. Messiah and O. W. Greenberg, Phys. Rev. **136**, B248 (1964).

[2] O. Steinmann, Nuovo Cimento **44**, A755 (1966).

[3] P. V. Landshoff and H. P. Stapp, Ann. Phys. (N. Y.) **43**, 72 (1967).

[4] O. W. Greenberg and A. M. L. Messiah, Phys. Rev. **138**, B1155 (1965).

[5] Y. Ohnuki and S. Kamefuchi, Phys. Rev. **170**, 1279 (1968).

We show that a consistent first-quantized theory of paraparticles *is* possible.[6] It can be formulated in terms of the Hilbert space $\mathcal{H}^N$—the tensor product of the one-particle space $\mathcal{H}^1$ to the Nth power—as proposed by Messiah and Greenberg. In this formulation the conventional rule which associates every state with a unique ray (i.e., one-dimensional subspace) is relaxed, and the states are instead represented by any irreducible subspace invariant under the permutation group S_N. (See Messiah and Greenberg,[1] where such subspaces are called generalized rays.) We emphasize an alternative and equivalent formulation on a smaller space. This version of the theory eliminates the former's redundancy (which arises from the symmetry under permutations) and restores the connection between states and rays. It allows one to establish a number of results of ordinary quantum mechanics whose usual proofs depend on this connection.

Having thoroughly explored its formulation, we can now show that the theory is consistent with the cluster law. We find that the requirement of consistency restricts the possible symmetry types of the multiparticle wave functions as follows: If a particle has N-particle states associated with a given Young diagram, then it must have $(N-1)$-, $(N-2)$-, $\cdots$, two-particle states associated with *all* Young diagrams which can be obtained from the first by successively removing squares. This means, for example, that all of the infinitely many possible kinds of paraparticle would have symmetric *and* antisymmetric two-particle states; i.e., they would behave in pairs like bosons and fermions. Our conclusion is therefore that, although paraparticles appear to be consistent with all reasonable requirements, their properties are disagreeably more complicated than those of the conventional boson or fermion.

II. STATES OF SEVERAL IDENTICAL PARTICLES

Generalized Rays

Theoretical considerations alone cannot determine the correct formalism for discussing systems of identical particles. Experiment must always be the final test. Nonetheless, it is possible to argue the reasonableness of a theoretical framework and we shall begin this section by presenting what seems a reasonable framework for the discussion of identical particles. Our arguments lead to the formalism of Messiah and Greenberg, which we review briefly. The formalism includes the conventional theories of bosons and fermions, but does not single them out for a preferred role.

The fundamental elements in any quantum-mechanical theory are the Hilbert space-of-state vectors and the operators corresponding to observables, whose expectation values are the measurable numbers. The structure of the Hilbert space depends on the number of different types of particles and their spins. We shall for simplicity consider only a single kind of spinless particle and suppose that the number of particles is fixed.[7]

We take for granted that the one-particle states are in one-to-one correspondence with the rays of the space $\mathcal{H}^1$ made up of all L^2 wave functions of one coordinate. The states of a system of N *nonidentical* particles, distinguished by a label $i=1, \cdots, N$ but otherwise identical, would correspond to the vectors of the space $\mathcal{H}^N$ which is the tensor product of N single-particle spaces,

$$\mathcal{H}^N=\mathcal{H}^1\otimes\mathcal{H}^1\otimes\cdots\otimes\mathcal{H}^1 \quad (N \text{ times}).$$

This is the space of all L^2 functions of N coordinates. It can be spanned by product functions

$$f(\mathbf{x}_1,\cdots,\mathbf{x}_N)=a_1(\mathbf{x}_1)\cdots a_N(\mathbf{x}_N),$$

where $a_1, \cdots, a_N$ are any one-particle wave functions; or, in abstract notation,

$$|f\rangle=|a_1\cdots a_N\rangle\equiv|a_1\rangle\otimes\cdots\otimes|a_N\rangle.$$

Notice that the space $\mathcal{H}^N$ is defined in such a way that the order of the labels in a product vector *is* relevant; the vectors $|ab\rangle$ and $|ba\rangle$ are quite distinct unless $a=b$.

The starting point of the theory of identical particles is the assumption that the physical observables for N *identical* particles are exactly those observables for N *labeled* particles *which do not distinguish the labels.* In order to identify this sublcass of observables we introduce the unitary permutation operators. For any P in the group S_N of permutations of N objects,

$$P: (1,2,\cdots,N)\to(P1,P2,\cdots,PN),$$

one defines U_P by the relation

$$U_P|a_1\cdots a_N\rangle=|a_{P1}\cdots a_{PN}\rangle,$$

for any $a_1, \cdots, a_N$; or in terms of concrete wave functions,

$$(U_P{}^\dagger f)(\mathbf{x}_1,\cdots,\mathbf{x}_N)=f(\mathbf{x}_{P1},\cdots,\mathbf{x}_{PN}).$$

The operators U_P arise from permuting the particle labels. We can therefore identify the observables which do not distinguish the labels as those observables whose expectation value for any vector $|f\rangle$ in $\mathcal{H}^N$ is the same as that for $U_P|f\rangle$, for any P; that is,

$$\langle f|A|f\rangle=\langle f|U_P{}^\dagger AU_P|f\rangle \quad (P\in S_N). \qquad (2.1)$$

Since this relation must hold with $|f\rangle$ replaced by any linear combination of vectors,[8] it holds also for off-

[6] That is, we disagree with the conclusion of Steinmann (Ref. 2). A similar opinion is given by Landshoff and Stapp (Ref. 3); but without spelling out the details of the argument. We discuss Steinmann's paper in Sec. III below.

[7] The restrictions to one kind of particle and to spin zero are quite inessential. The complications which arise when particle production is possible are described in Ref. 1.

[8] The argument here is not affected by superselection rules. If $|g\rangle$ and $|h\rangle$ belong to different superselection sectors, we can use $|f\rangle=|g\rangle+|h\rangle$ to represent a statistical mixture of the states $|g\rangle$ and $|h\rangle$ and compute expectation values in the ordinary way (after correctly normalizing).

diagonal matrix elements and so is an operator identity. The observables for N *identical* particles are therefore that subset of observables for N *labeled* particles which commute with the permutation operators

$$[A,U_P]=0. \tag{2.2}$$

Having identified the observables of our system of N identical particles we must next establish the correspondence between the states and the vectors of $\mathcal{H}^N$. Specifically, we must find which vectors of $\mathcal{H}^N$ correspond to states and the nature of this correspondence. We first note that two vectors which give the same expectation values for all (identical-particle) observables must represent the same state. Thus Eq. (2.1) implies that whenever a vector $|f\rangle$ in $\mathcal{H}^N$ corresponds to a physical state then the vector $U_P|f\rangle$, for any P, must correspond to the same state.

In the conventional theory of identical particles it is assumed that every physically distinct state of N identical particles must correspond to some unique ray in $\mathcal{H}^N$. If the vector $|f\rangle$ represents some state, then $|f\rangle$ and $U_P|f\rangle$ must lie in the same ray and hence must be proportional. This immediately leads to the conclusion that only vectors which are symmetric or antisymmetric under permutations can actually represent states.[9]

There is, however, no *a priori* reason to insist that every state of N identical particles correspond to a unique ray in $\mathcal{H}^N$. The possibility that a single state of N identical particles could correspond to some larger collection of vectors in $\mathcal{H}^N$ must be considered, and it is this possibility which leads to the notion of paraparticles.

To decide what collections of vectors in $\mathcal{H}^N$ can correspond to physical states we note that any state can be completely characterized by specifying the expectation values of *all* observables. Thus the sets of vectors which represent states must have the following two properties:

(i) Two vectors $|g\rangle$ and $|h\rangle$ representing the same state must give the same expectation value for all observables:

$$\langle g|A|g\rangle=\langle h|A|h\rangle \quad (\text{all } A \text{ with} [A,U_P]=0). \tag{2.3}$$

(ii) Conversely two vectors $|g\rangle$ and $|h\rangle$ representing distinct states must give different expectation values for some observable:

$$\langle g|A|g\rangle\neq\langle h|A|h\rangle \quad (\text{some } A \text{ with } [A,U_P]=0). \tag{2.4}$$

The apparatus of group theory enables one to determine the collections of vectors with these properties as follows:

We consider an arbitrary vector $|f\rangle$ in $\mathcal{H}^N$ and the subspace spanned by the $N!$ vectors $U_P|f\rangle$. Since this subspace is invariant under the permutations we can choose any basis $\{|f_i\rangle\}$ and then

$$U_P|f_i\rangle=\sum_j |f_j\rangle D_{ji}(P),$$

where the matrices $D(P)$ form a unitary representation of S_N. In general the representation will be reducible and the subspace can be decomposed as the direct sum of smaller subspaces each supporting an irreducible representation.

Since every observable commutes with all permutation operators, $[A,U_P]=0$, Schur's lemma implies that every observable is a multiple of the identity on each irreducible subspace. This means that any two vectors in one irreducible subspace satisfy Eq. (2.3) and must correspond to the same state.

By contrast if $|g\rangle$ and $|h\rangle$ belong to different irreducible subspaces then there exist self-adjoint operators A satisfying $[A,U_P]=0$ with different expectation values for $|g\rangle$ and $|h\rangle$; i.e., Eq. (2.4). (The projector onto either subspace is an example of such an observable; we shall discuss others in Sec. III.) Different irreducible subspaces must therefore correspond to physically distinct states.

Vectors which do not lie in any irreducible subspace cannot correspond to pure states. To see this, note that each such vector $|v\rangle$ can be written as a sum of vectors $|v_\mu\rangle$ which lie in different irreducible subspaces

$$|v\rangle=\sum_\mu a_\mu|v_\mu\rangle.$$

A consequence of Shur's lemma is that matrix elements of all physical quantities A (ones for which $[A,U_P]=0$ for all P) which connect states of different irreducible representations vanish. For any physical quantity the vector $|v\rangle$ is then equivalent in its consequences to the density matrix

$$\rho=\sum_\mu |v_\mu\rangle|a_\mu|^2\langle v_\mu|.$$

This cannot be a pure state unless all but one of the a_μ vanishes, which would mean that the vector $|v\rangle$ does lie in an irreducible subspace.

Messiah and Greenberg have introduced the name *generalized ray* for the irreducible subspaces of S_N. The situation may then be summarized by their generalized ray postulate: *Every state of a system of N identical particles corresponds to some generalized ray in* $\mathcal{H}^N$.

We have not yet discussed *which* irreducible subspaces correspond to states. For example, in the conventional theory with just bosons and fermions, only the rays of the totally symmetric or totally antisymmetric representations actually represent states. Before discussing what happens in general, it will be useful to discuss in detail the example of three-particle states.

Three-Particle States

We consider first those three-particle vectors made up from three orthogonal one-particle vectors $|a\rangle$, $|b\rangle$, $|c\rangle$.

[9] Here, and throughout this paper, we use the word "state" to mean "pure state" except when explicitly noted to the contrary.

If we write

$$|f\rangle=|abc\rangle\equiv|a\rangle\otimes|b\rangle\otimes|c\rangle,$$

then the six independent vectors $U_P|f\rangle$ span a six-dimensional subspace $\mathcal{S}$ of $\mathcal{H}^3$. This can be decomposed into irreducible subspaces invariant under S_3 in the well-known way:

$$\mathcal{S}=\mathcal{E}_s\oplus(\mathcal{E}_t'\oplus\mathcal{E}_t'')\oplus\mathcal{E}_a,$$

where $\mathcal{E}_s$ and $\mathcal{E}_a$ are the one-dimensional subspaces defined by the symmetric and antisymmetric combinations

$$|f_s\rangle=\sum_P U_P|f\rangle$$

and

$$|f_a\rangle=\sum_P \epsilon_P U_P|f\rangle.$$

The remaining four-dimensional space splits into two irreducible subspaces transforming under the same representation of S_3, namely, that of the triangular Young diagram. (The subscript t stands for "triangular.") One particular choice for these two subspaces is given by the basis vectors

$$|f_1'\rangle=\tfrac{1}{2}(|abc\rangle-|cab\rangle+|bac\rangle-|cba\rangle);\tag{2.5}$$

$$|f_2'\rangle=(|abc\rangle-2|bca\rangle+|cab\rangle-2|acb\rangle+|bac\rangle+|cba\rangle)/2\sqrt{3}\tag{2.6}$$

spanning $\mathcal{E}_t'$, and

$$|f_1''\rangle=(|abc\rangle-2|bca\rangle+|cab\rangle+2|acb\rangle-|bac\rangle-|cba\rangle)/2\sqrt{3},\tag{2.7}$$

$$|f_2''\rangle=-\tfrac{1}{2}(|abc\rangle-|cab\rangle-|bac\rangle+|cba\rangle)\tag{2.8}$$

spanning $\mathcal{E}_t''$. These bases are chosen so that the two spaces define identically the same representation of S_3. As $|f\rangle$ runs over the whole of $\mathcal{H}^3$ we can, in this way, decompose $\mathcal{H}^3$ into three sectors

$$\mathcal{H}^3=\mathcal{H}_s\oplus\mathcal{H}_t\oplus\mathcal{H}_a.$$

According to our general discussion, pure states may be represented by any irreducible subspace in any of the three sectors $\mathcal{H}_s$, $\mathcal{H}_t$, and $\mathcal{H}_a$; the irreducible subspaces of $\mathcal{H}_s$ and $\mathcal{H}_a$ being one dimensional, those of $\mathcal{H}_t$ being two dimensional. Two states represented by subspaces of different representations (e.g., one from $\mathcal{H}_s$ and one from $\mathcal{H}_t$) are physically distinct (if they occur). Similarly, states represented by different subspaces of the same representation (e.g., $\mathcal{E}_t'$ and $\mathcal{E}_t''$ above) are also distinct.

The situation in the general N-particle case is quite analogous. In particular, the space $\mathcal{H}^N$ decomposes as

$$\mathcal{H}^N=\oplus_\mu\mathcal{H}_\mu{}^N,$$

where $\mathcal{H}_\mu{}^N$ contains all irreducible subspaces transforming according to the irreducible representation $D^{(N)\mu}$ of S_N.

Superselection Rules

As pointed out by Messiah and Greenberg, if there exist states corresponding to subspaces of different representations (e.g., symmetric and triangular in our three-particle example), they must be separated by a superselection rule. This follows from Schur's lemma which guarantees that if $[A,U_P]=0$ then all matrix elements of A connecting different representations are zero.

This result led Messiah and Greenberg to propose, in analogy with the conventional situation where bosons and fermions have state vectors of a unique symmetry type, that any paraparticle would have N-particle states corresponding to just one representation of S_N. We shall see in Sec. III that this proposal is incompatible with the cluster law. For the present we merely note that if there are states corresponding to different representations of S_N, then they must be separated by a superselection rule.

Superpositions

We now restrict attention to states all represented by subspaces of the same irreducible representation; for definiteness we consider states of three particles represented by subspaces of the triangular representation of S_3. Given two such states we ask the question: What pure states can be formed from superposition of these two pure states?

We denote the two subspaces by $\mathcal{E}$ and $\mathcal{F}$, with bases $\{|g_i\rangle\}$ and $\{|h_i\rangle\}$ $(i=1, 2)$ chosen to support identically the same representation of S_N. (For example, we could consider $\mathcal{E}_t'$ and $\mathcal{E}_t''$ above.) In general, a linear combination of any vector from $\mathcal{E}$ and any vector from $\mathcal{F}$ (e.g., $\alpha|g_1\rangle+\beta|h_2\rangle$) does not lie in any single irreducible subspace and so cannot represent a pure state. In fact such superpositions do lie in an irreducible subspace if and only if we superpose vectors of $\mathcal{E}$ and $\mathcal{F}$ transforming according to the same row of the irreducible representation (e.g., $\alpha|g_1\rangle+\beta|h_1\rangle$). Thus the states arising from superposition of the original two states form a two-real-parameter family, represented, for example, by $\alpha|g_1\rangle+\beta|h_1\rangle$, exactly as in the conventional situation where states are given by unique rays.

Elimination of the Generalized Ray

We are now in a position to see that the distinction between rays and generalized rays is a consequence of the particular framework we have chosen to use, and not a distinction of fundamental quantum mechanical significance. The point is that any "triangular" state represented by a generalized ray $\mathcal{E}$, with basis $\{|g_i\rangle\}$, can arbitrarily be represented by the number-one basis vector $|g_1\rangle$. According to what we have just found, the general result of superposing two such states can itself be represented by a vector $\alpha|g_1\rangle+\beta|h_1\rangle$, which is automatically the number-one basis vector of its generalized ray. If then we decompose $\mathcal{H}_t$, the space of all "triangular" vectors, as

$$\mathcal{H}_t = \mathcal{H}_{t1} \oplus \mathcal{H}_{t2},$$

where $\mathcal{H}_{t1}$ contains all number-one basis vectors, we find that every triangular state is represented by a unique ray in $\mathcal{H}_{t1}$ (or in $\mathcal{H}_{t2}$) and that every ray in $\mathcal{H}_{t1}$ represents a unique such state. Further, since the Hamiltonian commutes with all permutations, all representative vectors will remain in $\mathcal{H}_{t1}$ for all times if we choose them there initially.

By our arbitrary decision to label each triangular state by its number-one basis vector we have restored the connection between states and rays. Every triangular state is labeled by a unique ray in $\mathcal{H}_{t1}$ rather than a two-dimensional generalized ray in $\mathcal{H}_t$. It should be emphasized that we achieve this at the expense of symmetry under the permutation operators, since the latter carry vectors from $\mathcal{H}_{t1}$ to anywhere in $\mathcal{H}_t$; that is, once we restrict attention to $\mathcal{H}_{t1}$ the permutation operators are no longer defined. This is, of course, exactly as one would expect. The redundancy of the generalized ray is caused by the symmetry under permutations and to remove that redundancy we have chosen a smaller space, within which the permutation operators are not defined.

Some Results of Ordinary Quantum Mechanics

The theory formulated on the smaller space $\mathcal{H}_{t1}$ is not the only way to construct a theory of paraparticles with a unique connection between states and rays. Nor are such formulations necessarily more natural or convenient than ones using generalized rays. However, the existence of these formulations with the normal connection between states and rays does allow us to carry over to theories with generalized rays a number of results whose usual proofs depend on this connection. We mention as examples Wigner's theorem,[10] which states that any symmetry can be represented by a unitary or antiunitary operator, and the reduction of the ray representations of the rotation group to vector representations of $SU(2)$.

If we confine attention to $\mathcal{H}_{t1}$ then Wigner's theorem holds, and any symmetry can be represented by a unitary (or antiunitary) operator U_1 on $\mathcal{H}_{t1}$. The same is, of course, true on $\mathcal{H}_{t2}$ and it is easily seen that the operator $U = U_1 \oplus U_2$ represents the symmetry on the whole of $\mathcal{H}_t$ and commutes with the permutation operators, as one would expect. Thus Wigner's theorem can be extended to the formalism of generalized rays in the natural way.

Similarly, if we confine attention to $\mathcal{H}_{t1}$ then the well-known analysis of Wigner[11] establishes that we can adjust phases so that the rotation operators $U_1(R)$ give a vector representation of $SU(2)$. The same is true of $U_2(R)$ on $\mathcal{H}_{t2}$ and hence of $U(R)$ on $\mathcal{H}_t$. Since the rotation operators commute with the permutations the same is true of their generators, as one would expect.

Extension to N Particles

Finally, we note that the above considerations extend in an entirely straightforward manner to the general, N-particle, case. In this case there are many inequivalent irreducible representations $D^{(N)\mu}$ of S_N, with dimensions n_μ. Every state corresponding to $D^{(N)\mu}$ is represented by an n_μ-dimensional generalized ray $\mathcal{E}$ in $\mathcal{H}_\mu{}^N$. However, in each such $\mathcal{E}$ we can choose a basis $\{|g_i\rangle\}$, $i=1, \cdots n_\mu$ and, exactly as above, the state can then be represented by the unique ray defined by a number-one basis vector $|g_1\rangle$. Everything else goes through just as for three particles.

III. CONSEQUENCES OF THE CLUSTER LAW

Cluster Law

The presence or absence of particles on the moon should not affect the results of experiments performed on the earth. This is a special case of the cluster law, which states that systems sufficiently separated in space may be treated as isolated systems, and which may be regarded as an essential requirement of any reasonable physical theory.

The wave function for N *distinct* particles grouped into one cluster of particles on the earth and another on the moon will factor into two parts, each part describing one cluster. The cluster law then follows in an elementary way. By contrast, the permutation symmetries of the wave function for N *identical* particles mean that the function does *not* factor in this way, and the question naturally arises whether all theories of identical particles are consistent with the cluster law.

In the case of bosons or fermions the cluster law is easily verified since a symmetric or antisymmetric function is automatically symmetric or antisymmetric in any subset of the labels.

The object of this section is to discuss under what conditions a theory of paraparticles is compatible with the cluster law. In order to illustrate the situation we begin by considering the example of three-particle states.

Three-Particle States

We consider the most general triangular state formed from the three one-particle wave functions a, b, c. This state can be represented by a superposition of the vectors $|f_1'\rangle$ and $|f_1''\rangle$ of Eqs. (2.5) and (2.7):

$$|g\rangle = \alpha'|f_1'\rangle + \alpha''|f_1''\rangle. \tag{3.1}$$

We fix attention on a function c localized near the moon and functions a and b localized in the laboratory. The

[10] E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959), p. 233.

[11] E. P. Wigner, Ann. Math. **40**, 149 (1939).

cluster law requires that the resulting three-particle state, when observed by an experimeter confined to the laboratory, must precisely resemble some allowed two-particle state; in other words, no measurement made within the laboratory should be sensitive to the presence of the third particle on the moon.

We consider, in particular, the probability distribution $w^{(3)}(\mathbf{x},\mathbf{y},\mathbf{z})$ for finding one particle localized at $\mathbf{x}$, one at $\mathbf{y}$, and one at $\mathbf{z}$. This is given by

$$w^{(3)}(\mathbf{x},\mathbf{y},\mathbf{z})=\langle g|\Lambda(\mathbf{x},\mathbf{y},\mathbf{z})|g\rangle,$$

where the projection operator Λ is given by[1]

$$\Lambda(\mathbf{x},\mathbf{y},\mathbf{z})=\sum_P U_P|\mathbf{x},\mathbf{y},\mathbf{z}\rangle\langle\mathbf{x},\mathbf{y},\mathbf{z}|U_P^\dagger.$$

A straightforward calculation using the expressions (2.5), (2.7), and (3.1) gives

$$\begin{aligned}w^{(3)}(\mathbf{x},\mathbf{y},\mathbf{z})=&|c(\mathbf{z})|^2\{|a(\mathbf{x})b(\mathbf{y})|^2+|a(\mathbf{y})b(\mathbf{x})|^2\\&+2(|\alpha'|^2-|\alpha''|^2)\,\mathrm{Re}a(\mathbf{x})b(\mathbf{y})a^*(\mathbf{y})b^*(\mathbf{x})\}\\&+[\text{cyclic permutations in }\mathbf{x},\mathbf{y},\mathbf{z}]\\&+[\text{cross terms in }c(\mathbf{x})c^*(\mathbf{y}),\text{ etc.}].\end{aligned}\qquad(3.2)$$

We now consider the two-particle distribution seen by an observer confined to the laboratory. We choose for our coordinates in the laboratory the variables $\mathbf{x}$ and $\mathbf{y}$. (This choice is arbitrary since w is a symmetric function of its arguments.) With $\mathbf{x}$ and $\mathbf{y}$ in the laboratory, $c(\mathbf{x})=c(\mathbf{y})=0$ and the only term which contributes to w is that which is written explicitly in Eq. (3.2). Since the observer in the laboratory does not concern himself with the particle on the moon, we integrate over the variable $\mathbf{z}$, to give the observed distribution

$$\begin{aligned}\hat{w}^{(3)}(\mathbf{x},\mathbf{y})\equiv&\int d^3z\, w^{(3)}(\mathbf{x},\mathbf{y},\mathbf{z})\\=&|a(\mathbf{x})b(\mathbf{y})|^2+|a(\mathbf{y})b(\mathbf{x})|^2\\&+2(|\alpha'|^2-|\alpha''|^2)\,\mathrm{Re}a(\mathbf{x})b(\mathbf{y})a^*(\mathbf{y})b^*(\mathbf{x}).\end{aligned}\qquad(3.3)$$

This observed distribution $\hat{w}^{(3)}$ must precisely match some allowable two-particle distribution. The most general form of the latter, corresponding to a statistical mixture of symmetric and antisymmetric states, is

$$\begin{aligned}w^{(2)}(\mathbf{x},\mathbf{y})=&\lambda_s|a(\mathbf{x})b(\mathbf{y})+a(\mathbf{y})b(\mathbf{x})|^2\\&+\lambda_a|a(\mathbf{x})b(\mathbf{y})-a(\mathbf{y})b(\mathbf{x})|^2\\=&|a(\mathbf{x})b(\mathbf{y})|^2+|a(\mathbf{y})b(\mathbf{x})|^2\\&+2(\lambda_s-\lambda_a)\,\mathrm{Re}a(\mathbf{x})b(\mathbf{y})a^*(\mathbf{y})b^*(\mathbf{x})\end{aligned}\qquad(3.4)$$

with statistical weights λ_s and λ_a satisfying $\lambda_s+\lambda_a=1$. The distributions (3.3) and (3.4) match if and only if

$$\lambda_s=|\alpha'|^2,\quad\lambda_a=|\alpha''|^2.$$

With this choice of the weights the requirements of the cluster law are satisfied.

For particular choices of the coefficients α' and α'' in the three-particle vector $|g\rangle$ of Eq. (3.1) the observed distribution (3.3) is precisely that of a two-boson or two-fermion state. Specifically, if $\alpha'=1$, $\alpha''=0$ and the three-particle state is just $|f_1'\rangle$, then the observer sees a two-boson distribution in his laboratory; if $\alpha'=0$, $\alpha''=1$ and the three-particle state is just $|f_1''\rangle$, then the observer sees a two-fermion distribution. It follows that a particle which has three-particle states corresponding to the triangular Young diagram *must* have *both* boson- and fermion-type two-particle states.[12]

We must now consider the argument of Steinmann[2,3] which claims to find an inconsistency in this situation. He first states, in agreement with our conclusions, that the general triangular state made up from the one-particle wave functions a, b, c will appear to an earth-bound observer as a statistical mixture of boson and fermion two-particle states. He then argues that the observer can make a measurement on the two-particle system to determine its symmetry type. In making this two-particle measurement he changes the observed two-particle state to either pure symmetric or pure antisymmetric. Steinmann next claims that if we now consider the history of the whole three-particle system, the measurement has introduced a distinction between vectors which are supposed to be indistinguishable. However, this is not so. The three-particle state before the measurement is represented by

$$|g\rangle=\alpha'|f_1'\rangle+\alpha''|f_1''\rangle.$$

If the result of our earth-bound observer's experiment is a symmetric two-particle state, say, then the resulting three-particle state vector must, as we have seen above, be $|f_1'\rangle$. The state vectors before and after the measurement have the same three-particle symmetry type, namely, triangular (as they must do, since there is a superselection rule on symmetry type). But they belong to *different* irreducible subspaces and so *are* expected to be experimentally distinct, as indeed they are. The essential point is that the measurement carries the three-particle vector from one irreducible subspace *to another*. There is no question of its distinguishing between vectors in the same irreducible subspace (such as $|f_1'\rangle$ and $|f_2'\rangle$) which certainly are indistinguishable.

General Case

So far, we have established for three-particle states that parastatistics are compatible with the cluster law, at least as regards measurements of position. We must now extend our considerations to include arbitrary numbers of particles and all localized measurements.

Our procedure is as follows:

(i) Write down the most general form for an $(N+1)$-particle state vector of symmetry type $D^{(N+1)\mu}$ and made up from one-particle wave functions $a_1,\cdots,a_{N+1}$. The corresponding density matrix is denoted $W^{(N+1)}$.

[12] This and its generalization to the N-particle case was noted by A. Casher, G. Frieder, M. Glück, and A. Peres [Nucl. Phys. **66**, 632 (1965)].

(ii) Establish the form of the most general observable $M^{(N+1)}$ for an experiment made on the N particles in a laboratory, of an $(N+1)$-particle state with one particle far away. This will be expressed in terms of the general N-particle localized observable $L^{(N)}$, localized in the laboratory.

(iii) The result of the most general measurement made in the laboratory on the $(N+1)$-particle state is then a number of the form $\mathrm{Tr}(M^{(N+1)}W^{(N+1)})$. We show that this can be reduced to the form

$$\mathrm{Tr}(M^{(N+1)}W^{(N+1)})=\mathrm{Tr}(L^{(N)}\hat{W}^{(N+1)}), \qquad (3.5)$$

where $\hat{W}^{(N+1)}$ is a certain "reduced" density matrix.

(iv) The observed result (3.5) must precisely match the result $\mathrm{Tr}(L^{(N)}W^{(N)})$ of a measurement on a N-particle system in some allowed state given by a density matrix $W^{(N)}$. We show that such a state exists and is uniquely determined by this requirement. This completes the proof of consistency.

Multiparticle States

We shall for simplicity consider states made up from $(N+1)$ orthogonal, normalized, one-particle functions $a_1, \cdots, a_{N+1}$. The assumption of orthogonality is easily relaxed, but at the expense of some inconvenient normalization factors. States more general than these product states can also be treated, again at the expense of some inconvenience in notation.

The vectors in the irreducible subspaces of the representation $D^{(N+1)\mu}$ can be constructed in the familiar way.[13] If we define

$$|f_i^j\rangle=[n_\mu/(N+1)!]^{1/2} \sum_P D_{ij}^{(N+1)\mu}(P)^*U_P|a_1\cdots a_{N+1}\rangle \qquad (3.6)$$

then, as can be easily verified, the n_μ vectors

$$|f_1^j\rangle,\cdots,|f_{n_\mu}^j\rangle$$

are an orthonormal basis of an irreducible subspace transforming under $D^{(N+1)\mu}$; and as j runs from 1 to n_μ we obtain the n_μ equivalent orthogonal such subspaces. It follows that the most general state constructed from $a_1, \cdots, a_{N+1}$ and represented by a subspace of $D^{(N+1)\mu}$ can be represented by a vector

$$|g\rangle=\sum_{j=1}^{n_\mu}\alpha_j|f_1^j\rangle \qquad (3.7)$$

corresponding to the vector $|g\rangle$ of Eq. (3.1). (The choice of the lower index 1 is, of course, arbitrary; changing it to 2, 3, $\cdots$, n_μ gives distinct vectors all representing the same state.)

[13] See, for example, M. Hamermesh, *Group Theory and its Applications to Physical Problems* (Addison-Wesley Publishing Co., Inc., Reading, Mass., 1962), Chap. 7.

Local Observables

If we consider a system of N *distinct* particles, we can characterize an observable $L^{(N)}$ localized in the laboratory by the equation

$$L^{(N)}|\mathbf{x}_1\cdots\mathbf{x}_N\rangle=0$$

unless $\mathbf{x}_1, \cdots, \mathbf{x}_N$ are all in the laboratory. The prototype for such a localized observable has the form $L^{(N)}=|h\rangle\times\langle h|$, where $|h\rangle$ is a state completely localized in the laboratory.

The same measurement, considered as an observable for $N+1$ distinct particles, one of which is not observed, is given by the operator

$$M^{(N+1)}=L^{(N)}\otimes 1,$$

where M acts on $\mathcal{H}^{N+1}$, L acts on $\mathcal{H}^N$, and the unit operator on $\mathcal{H}^1$ expresses the fact that the state of the last particle is not measured.

Finally, the corresponding operator for $N+1$ identical particles is seen to be

$$M^{(N+1)}=\sum_{P\in S_{N+1}}U_P^\dagger(L^{(N)}\otimes 1)U_P. \qquad (3.8)$$

Reduced-Density Matrix

The expectation value of the laboratory observable $M^{(N+1)}$ in any state represented by a vector $|g\rangle$ is

$$\langle g|M^{(N+1)}|g\rangle=\mathrm{Tr}(M^{(N+1)}W^{(N+1)}),$$

where $W^{(N+1)}=|g\rangle\langle g|$ is the corresponding density matrix. Using the specific form (3.8) for M, this becomes

$$\mathrm{Tr}(M^{(N+1)}W^{(N+1)})=\mathrm{Tr}(L^{(N)}\hat{W}^{(N+1)}), \qquad (3.9)$$

where $\hat{W}^{(N+1)}$ is an operator on $\mathcal{H}^N$ which we call the reduced density matrix and is given by its matrix elements:

$$\langle\mathbf{x}_1'\cdots\mathbf{x}_N'|\hat{W}^{(N+1)}|\mathbf{x}_1\cdots\mathbf{x}_N\rangle =\int d^3x_{N+1}\sum_{P\in S_{N+1}}\langle\mathbf{x}_1'\cdots\mathbf{x}_N'\mathbf{x}_{N+1}|U_P|g\rangle \times\langle g|U_P^\dagger|\mathbf{x}_1\cdots\mathbf{x}_{N+1}\rangle.$$

Since $L^{(N)}$ is an observable localized in the laboratory, only those values of $\mathbf{x}_1'\cdots\mathbf{x}_N'$ and $\mathbf{x}_1\cdots\mathbf{x}_N$ in the laboratory are relevant for the computation of the expectation value (3.9).

We now take for our $(N+1)$-particle state the state represented by the vector $|g\rangle$ of Eqs. (3.7) and (3.6). We further take for a_{N+1} a function localized near the moon while $a_1, \cdots, a_N$ are localized in the laboratory. When we insert this explicit $|g\rangle$ into $\hat{W}$, the resulting rather complicated expression simplifies since $a_{N+1}(\mathbf{x}_1)=\cdots=a_{N+1}(\mathbf{x}_N)=0$ ($\mathbf{x}_1,\cdots,\mathbf{x}_N$ being fixed in the laboratory). Of all the permutations involved, only those which leave the last variable fixed actually contribute and this allows us to factor out a term $|a_{N+1}(\mathbf{x}_{N+1})|^2$, which disappears on integration. We finally obtain, for the reduced density matrix of the state $|g\rangle$,

$$\hat{W}^{(N+1)}=\sum_{P\in S_N}\sum_{i,j}\alpha_i D_{ij}^{(N+1)\mu}(P)\alpha_j^* \times\Big[\sum_{Q\in S_N} U_Q|a_1\cdots a_N\rangle\langle a_1\cdots a_N|U_{QP}^\dagger\Big], \quad (3.10)$$

where the sum over $P\in S_N$ includes all $P\in S_{N+1}$ which leave the last variable fixed.

Using the operator (3.10) in the expression (3.9) we can now calculate the observed result of any experiment localized in the laboratory.

Comparison with N-Particle States

According to the cluster law the observed result (3.9) of any measurement in the laboratory must coincide precisely with the result $\mathrm{Tr}(L^{(N)}W^{(N)})$ of a measurement on some allowed N-particle state,

$$\mathrm{Tr}(L^{(N)}\hat{W}^{(N+1)})=\mathrm{Tr}(L^{(N)}W^{(N)}).$$

This identity can hold for all localized measurements $L^{(N)}$ if and only if

$$\hat{W}^{(N+1)}=W^{(N)} \quad (3.11)$$

for some $W^{(N)}$.

The most general N-particle state is a statistical mixture, with weights λ_ν, of states corresponding to all possible symmetries $D^{(N)\nu}$. This has a density matrix

$$W^{(N)}=\sum_\nu \lambda_\nu|g_\nu\rangle\langle g_\nu|.$$

Here $\sum\lambda_\nu=1$ and the vectors $|g_\nu\rangle$ are given by Eqs. (3.7) and (3.6) with $N+1$ replaced by N and the coefficients α_j replaced by β_k^ν satisfying $\sum_k|\beta_k^\nu|^2=1$. After some algebra this gives

$$W^{(N)}=\sum_\nu\lambda_\nu\sum_{P\in S_N}\sum_{k,l}\beta_k^\nu D_{kl}^{(N)\nu}(P)\beta_l^{\nu*} \times\Big[\sum_{Q\in S_N} U_Q|a_1\cdots a_N\rangle\langle a_1\cdots a_N|U_{QP}^\dagger\Big]. \quad (3.12)$$

According to Eq. (3.11) our theory is consistent with the cluster law if and only if we can find numbers λ_ν, β_k^ν such that the two operators (3.10) and (3.12) are the same. Since the $N!$ operators in square brackets in each expression are linearly independent, we require that their coefficients are separately equal; i.e.,

$$\sum_\nu\lambda_\nu\sum_{k,l}\beta_k^\nu D_{kl}^{(N)\nu}(P)\beta_l^{\nu*} =\sum_{i,j}\alpha_i D_{ij}^{(N+1)\mu}(P)\alpha_j^* \quad (\text{all } P\in S_N). \quad (3.13)$$

On the right-hand side of Eq. (3.13) is the matrix $D^{(N+1)\mu}(P)$ of the μth irreducible representation of S_{N+1}. When P is restricted to S_N (last variable fixed), this gives a *reducible* representation of S_N which decomposes as[14]

$$D^{(N+1)\mu}(P)=\oplus_\nu\, a_\nu D^{(N)\nu}(P), \quad P\in S_N.$$

The representation $D^{(N+1)\mu}$ contains $D^{(N)\nu}$ either once or not at all (i.e., $a_\nu=1$ or 0) according as the Young diagram of the latter can be obtained from that of the former by removal of one square. If we choose a basis for $D^{(N+1)\mu}$ appropriate to this decomposition and relabel the α_j as α_k^ν, Eq. (3.13) becomes

$$\sum_\nu\lambda_\nu\sum_{k,l}\beta_k^\nu D_{kl}^{(N)\nu}(P)\beta_l^{\nu*} =\sum_\nu a_\nu\sum_{k,l}\alpha_k^\nu D_{kl}^{(N)\nu}(P)\alpha_l^{\nu*} \quad (\text{all } P\in S_N).$$

Using the orthogonality relations we can immediately project out the relation

$$\lambda_\nu\beta_k^\nu\beta_l^{\nu*}=a_\nu\alpha_k^\nu\alpha_l^{\nu*},$$

which has the unique solution (apart from some irrelevant phases)

$$\lambda_\nu=a_\nu\sum_k|\alpha_k^\nu|^2 \quad (3.14)$$

and

$$\beta_k^\nu=\alpha_k^\nu/\sqrt{\lambda_\nu} \quad \text{when} \quad a_\nu=1$$
$$=0 \quad \text{when} \quad a_\nu=0. \quad (3.15)$$

This completes the demonstration.

IV. CONCLUSION

From the solution given in Eqs. (3.14) and (3.15) it is clear that in general the observer sees a statistical mixture of all those N-particle symmetries $D^{(N)\nu}$ which can be obtained from $D^{(N+1)\mu}$. A particular choice of the coefficients α can give a pure state of any one of these symmetry types, and the occurrence of states of the original $(N+1)$-particle symmetry $D^{(N+1)\mu}$ therefore implies the existence of *all* of these N-particle symmetry types. This means that for a given kind of paraparticle there is a whole family of allowed symmetry types, with the property that whenever the family contains an $(N+1)$-particle symmetry type $D^{(N+1)\mu}$ it contains all N-, $(N-1)$-, $\cdots$, two-particle symmetries whose Young diagrams can be obtained from that of $D^{(N+1)\mu}$ by removal of successive blocks.

An illustration of this general rule is given by our three-particle example where both the boson and fermion two-particle diagrams can be obtained from the triangular three-particle diagram by removing a single square. (See Fig. 1.)

Special cases of the general rule are, at one extreme, the conventional situation where the family consists of single horizontal rows for bosons or vertical columns for fermions; and at the opposite extreme, a particle for which every Young diagram is an allowed symmetry type. In between, for example, one could have a paraparticle (generalizing the fermion) for which no more

[14] See Ref. 13, p. 214.

than two particles could occupy the same state. The symmetry types necessarily present in the general N-particle state would be just those Young diagrams with N blocks but no more than two columns. Once again both boson and fermion states must be admitted in the two-particle sector.

There seems to be no simple classification of the possible families of allowed symmetry types and the subject seems too academic to merit detailed examination at this point. We therefore content ourselves with a few disjoint concluding remarks

(i) There are clearly infinitely many distinct possibilities for the families of allowed symmetries of a paraparticle.

(ii) As can be easily seen, any paraparticle must include among its allowed N-particle symmetries either the totally symmetric or the totally antisymmetric type—plus, of course, some others.

(iii) Every paraparticle must have triangular three-particle states[15] and hence both symmetric and antisymmetric two-particle states.

(iv) The statistics of any paraparticle are different from Fermi-Dirac, Bose-Einstein, and Maxwell-Boltzmann. (For example, consider a paraparticle which has symmetric, antisymmetric, and triangular three-particle states. From three distinct one-particle states one can form four independent three-particle states; for a boson or fermion this number would be 1, for a Maxwell-Boltzmann particle 6.)

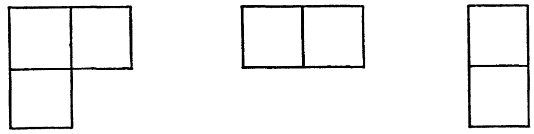

FIG. 1. Both boson and fermion two-particle diagrams can be obtained by removing one square from the triangular three-particle diagram.

We feel that these remarks justify us in the conclusion that, although there is no theoretical reason to exclude paraparticles, their properties are sufficiently disagreeable for one to hope sincerely that there will continue to be no evidence in their favor.

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[15] There is one exception, namely, a particle which has both symmetric and antisymmetric states, but no others.

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A "Backwards" Two-Body Problem of Classical Relativistic Electrodynamics*

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The two-body problem of classical electrodynamics can be formulated in terms of action at a distance by using the retarded Liénard-Wiechert potentials (or a combination of retarded and advanced potentials). The resulting equations of motion in the retarded case, for example, form a complicated functional delay-differential system. For such equations in the case of one-dimensional motion, as shown in an earlier paper, one can specify rather arbitrary past histories of the particles, and then solve for the future trajectories. Yet it is often assumed or asserted that unique trajectories would be determined by the specification of Newtonian initial data (the positions and velocities of the particles at some instant). Simple examples of delay-differential equations should lead one to doubt this. For example, even if the values of x and *all* its derivatives are given at t_0, the equation $x'(t)=ax(t)+bx(t-\tau)$, with $\tau>0$ and $b\neq 0$, still has infinitely many solutions valid for all t. Nevertheless, under certain special conditions for the electrodynamic equations it is found that instantaneous values of positions and velocities do indeed determine the solution uniquely. The case treated in this paper involves two charges of like sign moving on the x axis, assumed to have been subject only to their mutual retarded electrodynamic interaction for all time in the past. The similar questions of existence and uniqueness for a more general model, e.g., three-dimensional motion or half-retarded and half-advanced interactions, or even for charges with opposite signs remain open.

1. INTRODUCTION

FOR 60 years the two-body problem of classical electrodynamics remained completely unsolved. The equations of motion were expressible in terms of retarded action at a distance by use of the Liénard-Wiechert expressions. However, this led to a complicated functional delay-differential system which had not been treated mathematically. (Some authors also

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