Cluster-Model Theory of the Ground-State Rotational Band of Ne^{20}

F. C. Chang
Physics Department, St. John's University, Jamaica, New York

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The excitation energies of the ground-state rotational band as well as the reduced transition rate for the transition from the first excited state to the ground state in Ne^{20} have been calculated with the use of cluster wave functions. The agreement between the calculated and observed values suggests that the properties of the ground-state rotational band in Ne^{20} can also be explained in terms of cluster motion.

I. INTRODUCTION

The low-lying states of positive parity of Ne^{20} have been analyzed by many investigators.[1–5] Their methods include deformed Hartree-Fock calculation and diagonalization of the matrix of a shell-model Hamiltonian with a residual interaction within the s-d shell. It has been pointed out[6] that an O^{16}-α structure can result in 0+, 2+, 4+, 6+, and 8+ states. Also, the breakup energy of an α cluster in Ne^{20} was estimated to be 10.3 MeV.[7] Thus it appears that an appropriate cluster wave function should be capable of describing the properties of the ground-state rotational band in Ne^{20}. In this paper we propose to study these states in terms of cluster motion and in the framework of the shell model, and show that the cluster interpretation of these states has validity.

II. METHOD OF CALCULATION

We consider Ne^{20} as an inert O^{16} core plus four nucleons in the s-d shell. The Hamiltonian of the system is assumed to be

$$H=\sum_i H_i+\tfrac{1}{2}\sum_{i\neq j}V_{ij}+H_c. \tag{1}$$

Here i and j run over the nucleons outside the core. H_i is a one-body Hamiltonian generated by the O^{16} core, and is of the form

$$H_i=(p_i^2/2m)+\tfrac{1}{2}m\omega^2r_i^2, \tag{2}$$

where m is the nucleon mass, ω is the classical oscillator frequency, and $\mathbf{r}_i$ and $\mathbf{p}_i$ are the coordinate and momentum of the ith nucleon. The last term in (2) describes an average interaction between the ith nucleon and the core. For the odd nucleons the spin-orbit interaction is perhaps negligible, as long as we are dealing with states in which they form an α cluster. V_{ij} is the following two-body interaction, used by Wildermuth and Kanellopoulos[8]:

$$V_{ij}=V_0\exp[-(r_{ij}/r_0)^2]\{0.41[1-\tfrac{1}{4}(1+\sigma_i\cdot\sigma_j)(1+\tau_i\cdot\tau_j)]+0.09[\tfrac{1}{2}(1+\sigma_i\cdot\sigma_j)-\tfrac{1}{2}(1+\tau_i\cdot\tau_j)]\};$$
$$V_0=-68.6\text{ MeV},\quad r_0=1.55\text{ fm},\quad r_{ij}=|\mathbf{r}_i-\mathbf{r}_j|. \tag{3}$$

H_c is the Hamiltonian for the core, which can be chosen so that the core nucleons are moving in the same oscillator potential as the outside nucleons.

Four nucleons in the s-d shell in an oscillator potential have a total of eight oscillator quanta. If they form an α cluster in its ground state, the radial quantum number N and angular-momentum quantum number J for the c.m. motion of the cluster can have values as follows:

N	J
4	0
3	2
2	4
1	6
0	8

It then follows that the structure of an inert O^{16} core

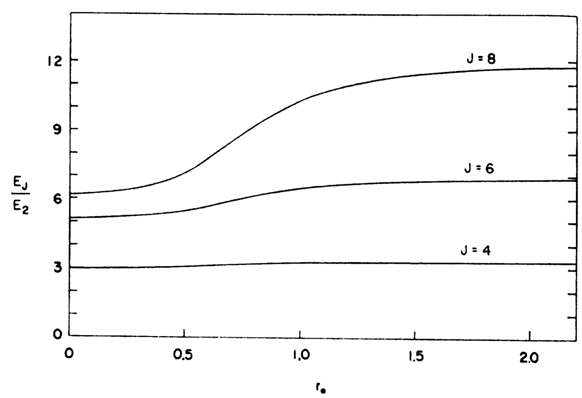

FIG. 1. Some ratios of the excitation energies of the ground-state rotational band of Ne^{20}, as a function of r_0. E_J denotes the excitation energy of the state with angular momentum J. r_0 is in units of $(2\hbar/m\omega)^{1/2}$.

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TABLE I. Coefficients $C(l_1l_2L_{12}l_3l_4L_{34}J)$ in the expansion of the space part of $\Psi_\alpha'(JM)$ in terms of single-particle wave functions.

J	l_1	l_2	L_{12}	l_3	l_4	L_{34}	$C(l_1l_2L_{12}l_3l_4L_{34}J)$
0	2	2	4	2	2	4	0.228
	2	2	2	2	2	2	0.170
	2	2	0	2	2	0	0.267
	2	2	2	0	2	2	0.225
	2	2	2	2	0	2	0.225
	0	2	2	2	2	2	0.225
	2	0	2	2	2	2	0.225
	0	2	2	0	2	2	0.298
	0	2	2	2	0	2	0.298
	2	0	2	0	2	2	0.298
	2	0	2	2	0	2	0.298
	2	2	0	0	0	0	0.298
	0	0	0	2	2	0	0.298
	0	0	0	0	0	0	0.334
2	2	2	4	2	2	4	0.137
	2	2	4	2	2	2	0.143
	2	2	2	2	2	4	0.143
	2	2	2	2	2	2	0.107
	2	2	2	2	2	0	0.167
	2	2	0	2	2	2	0.167
	2	2	4	0	2	2	0.190
	2	2	4	2	0	2	0.190
	0	2	2	2	2	4	0.190
	2	0	2	2	2	4	0.190
	2	2	2	0	2	2	0.141
	2	2	2	2	0	2	0.141
	0	2	2	2	2	2	0.141
	2	0	2	2	2	2	0.141
	2	2	0	0	2	2	0.221
	2	2	0	2	0	2	0.221
	0	2	2	2	2	0	0.221
	2	0	2	2	2	0	0.221
	0	2	2	0	2	2	0.187
	0	2	2	2	0	2	0.187
	2	0	2	0	2	2	0.187
	2	0	2	2	0	2	0.187
	2	2	2	0	0	0	0.187
	0	0	0	2	2	2	0.187
	0	0	0	0	2	2	0.247
	0	0	0	2	0	2	0.247
	0	2	2	0	0	0	0.247
	2	0	2	0	0	0	0.247
4	2	2	4	2	2	4	0.159
	2	2	4	2	2	2	0.150
	2	2	2	2	2	4	0.150
	2	2	4	2	2	0	0.246
	2	2	0	2	2	4	0.246
	2	2	2	2	2	2	0.157
	2	2	4	0	2	2	0.198
	2	2	4	2	0	2	0.198
	0	2	2	2	2	4	0.198
	2	0	2	2	2	4	0.198
	2	2	2	0	2	2	0.208
	2	2	2	2	0	2	0.208
	0	2	2	2	2	2	0.208
	2	0	2	2	2	2	0.208
	2	2	4	0	0	0	0.275
	0	0	0	2	2	4	0.275
	0	2	2	0	2	2	0.275
	0	2	2	2	0	2	0.275
	2	0	2	0	2	2	0.275
	2	0	2	2	0	2	0.275
6	2	2	4	2	2	4	0.286
	2	2	4	2	2	2	0.319
	2	2	2	2	2	4	0.319
	2	2	4	0	2	2	0.423
	2	2	4	2	0	2	0.423
	0	2	2	2	2	4	0.423
	2	0	2	2	2	4	0.423
8	2	2	4	2	2	4	1.000

TABLE II. Excitation energies E_J of the ground-state rotational band of Ne^{20}.

	E_J (MeV)	
J	Experimental	Calculated
2	1.63	1.63
4	4.25	4.59
6	8.79	8.38
8	11.99	11.83

plus an α cluster in its ground state gives rise to 0+, 2+, 4+, 6+, and 8+ states. We shall treat the interaction (3) as a perturbation. The states just mentioned are all degenerate in the zeroth-order approximation. It is reasonable to expect that states in which the α cluster is broken up are not degenerate with these states. The reason is that a nucleon not in an α cluster should interact with the core more strongly than a nucleon in an α cluster does. Thus for the former nucleon a larger oscillator constant is required. This allows the possibility that states without α clustering outside the core may have higher (unperturbed) energies.

For simplification of calculations we shall use one oscillator constant for both the α cluster and the nucleons. This approximation is probably good, because harmonic-oscillator wave functions for low-energy states are not very sensitive to small variations in the oscillator constant.[9] Hence, we assume, for the ground-state rotational band of Ne^{20}, a cluster wave function of the form

$$Ca[\Psi(O^{16})\Psi_\alpha(JM)], \tag{4}$$

where C is a normalization constant, a is an antisymmetrizer, and the functions $\Psi(O^{16})$ and $\Psi_\alpha(JM)$ describe an inert O^{16} core and an α cluster with angular-momentum quantum numbers J and M, respectively.

TABLE III. Interaction energies for the $J=0$ and two states, E_0 and E_2, and the first excitation energy, E_2-E_0, for several values of r_0.

r_0 (fm)	E_0 (MeV)	E_2 (MeV)	E_2-E_0 (MeV)
0.82	−8.47	−7.90	0.57
1.19	−19.87	−18.80	1.07
1.55	−34.89	−33.47	1.42
2.38	−74.90	−73.19	1.71
3.52	−129.92	−128.51	1.41
5.08	−191.60	−190.84	0.76

[9] A. de-Shalit and I. Talmi, *Nuclear Shell Theory* (Academic Press Inc., New York, 1963), p. 41.

The space part of $\Psi_\alpha(JM)$ is taken to be

$$\left(\frac{m\omega}{\pi\hbar}\right)^{9/4} R_{NJ}(R)\,Y_J{}^M\left(\frac{\mathbf{R}}{R}\right)\exp\left(-\frac{m\omega}{2\hbar}(\xi^2+\eta^2+\zeta^2)\right), \quad (5)$$

with

$$\mathbf{R}=\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4), \qquad \boldsymbol{\xi}=(\mathbf{r}_1-\mathbf{r}_2)/\sqrt{2},$$

$$\boldsymbol{\eta}=(\mathbf{r}_3-\mathbf{r}_4)/\sqrt{2}, \qquad \boldsymbol{\zeta}=\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2-\mathbf{r}_3-\mathbf{r}_4).$$

Here $R_{NJ}(R)$ is the radial function of a harmonic oscillator and $Y_J{}^M(\mathbf{R}/R)$ is a spherical harmonic. The space part is totally symmetric (corresponding to the partition [4]). The spin-isospin part is totally antisymmetric, and has spin $S=0$ and isospin $T=0$.

The space part of $\Psi_\alpha(JM)$ can be expanded in terms of single-particle wave functions as follows:

$$\sum_{l_1l_2l_3l_4,L_{12}L_{34}} C(l_1l_2L_{12}l_3l_4L_{34}J)\,\psi(l_1l_2(L_{12})l_3l_4(L_{34})JM). \quad (6)$$

In the state $\psi(l_1l_2(L_{12})l_3l_4(L_{34})JM)$, orbital angular momenta l_1 and l_2 of nucleons 1 and 2 are coupled to L_{12}, l_3 and l_4 of nucleons 3 and 4 are coupled to L_{34}, and then L_{12} and L_{34} to J.

Because of the antisymmetrization, we can use, instead, a function $\Psi_\alpha'(JM)$, obtained from $\Psi_\alpha(JM)$ by deleting terms containing single-particle states occupied by the core nucleons and normalizing. The expansion coefficients $C(l_1l_2L_{12}l_3l_4L_{34}J)$ for the space part of $\Psi_\alpha'(JM)$ are listed in Table I. The spacings between the energy levels are determined by the interaction of the nucleons in the α cluster, which is given by

$$\int \Psi_\alpha'^*(JM)\left[\tfrac{1}{2}\sum_{i\neq j}V_{ij}\right]\Psi_\alpha'(JM), \quad (7)$$

where i and j run over the nucleons in the cluster.

III. RESULTS AND DISCUSSION

Some ratios of the excitation energies of the ground-state rotational band have been calculated, and are shown as a function of r_0 in Fig. 1. These ratios are larger than the corresponding experimental ratios for all values of r_0. It is a fact[10] that the oscillator potential is too strong near the nuclear surface for nucleon states of zero orbital angular momentum. On the basis of this fact, it is perhaps reasonable to use a smaller oscillator constant for the $J=0$ state of the α cluster. In this way a better agreement between the calculated and experimental ratios can be obtained; an estimate of the needed reduction in ω is given below. It should also be noted that, since the interaction energy given by Eq. (7) is proportional to V_0, the calculated ratios do not depend on V_0.

We have calculated the excitation energies of the ground-state rotational band, with an oscillator frequency taken as $(2\hbar/m\omega)^{1/2}=2.33$ fm. For the $J=0$ α cluster, ω is reduced by about 0.1%; this results in a lowering of the $J=0$ level by about 0.2 MeV. The agreement between the calculated and experimental values is satisfactory, as is shown in Table II.

[10] A. Faessler and R. K. Sheline, Phys. Rev. **148**, 1003 (1966).

TABLE IV. Comparison between the cluster wave function and the wave function based on the Nilsson model, for the ground state of Ne^{20}. η is Nilsson's deformation parameter.

Configuration	Probability Cluster	Nilsson $\eta=4$	$\eta=6$
d^4	0.152	0.321	0.264
d^3s	0.203	0.423	0.419
d^2s^2	0.534	0.208	0.246
ds^3	0.000	0.045	0.065
s^4	0.111	0.003	0.007

The interaction energies of the $J=0$, 2, 4, 6, and 8 states, calculated from Eq. (7), are -34.89, -33.47, -30.51, -26.71, and -23.27 MeV, respectively. To illustrate the dependence of the interaction energies on r_0, we give, in Table III, the interaction energies for the $J=0$ and 2 states and the first excitation energy, for several values of r_0.

The reduced transition rate $B(E2)$ for the $2+\rightarrow 0+$ (the first excited state to the ground state) transition in Ne^{20} is calculated to be $9.8e^2$ fm^4. Proton charge and neutron charge are taken as $1.5e$ and $0.5e$, respectively (i.e., an effective charge of $0.5e$ is added to each nucleon). As compared to the observed $B(E2)$ value of $57.3\pm10\%$ e^2 fm^4, the calculated one is too small by a factor of about 5.9; this discrepancy may indicate a sizeable amount of core excitation. An agreement within 10% of the calculated and observed $B(E2)$ values requires the assumption of an effective charge of $1.8e$ for each nucleon. Since in our calculations only one oscillator constant has been used for both the nucleons and the α cluster, configuration mixing is confined to the s-d shell. With the use of two unequal oscillator constants more configurations will be mixed, and the $B(E2)$ value may be enhanced.

For the $J=0$ state, a comparison between our wave function and a wave function based on the Nilsson model has been made. The latter is taken to be a product of four Nilsson single-particle functions, each of which has $N=2$, $\Omega=\frac{1}{2}$ and belongs to orbit 6 (using Nilsson's notation). The probabilities for d^4, d^3s, d^2s^2, ds^3, and s^4 configurations are given in Table IV. It is seen that a nucleon in the state described by our wave function is more likely to be in an s orbit, as compared to a nucleon in the state described by the wave function based on the Nilsson model.

In conclusion, our calculations suggest that the properties of the ground-state rotational band in Ne^{20} can also be explained in terms of α-cluster motion.