Account of the Main Nuclear-Structure Properties in the Optical Potential for Neutron Elastic Scattering

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In this work an attempt is made to establish the correspondence between the nonlocal optical potential and simple structural properties of nuclei. The presentation of the potential as a superposition of separable partial potentials permits one to solve the Schrödinger equation analytically rather than numerically, as is usually done. Additional conditions establish an evident correspondence between the potential and the simple Fermi-gas model of nuclei. It allows one to keep some potential parameters fixed, so that the number of actually variable parameters is usually less than that for a local potential. A simple expression for the S-matrix elements is found, using the asymptotic properties of the partial equation and its solution. For calculation of the cross section the computation of only ten relatively simple integrals was necessary, instead of the numerical solution of a complicated integrodifferential equation. The neutron elastic-scattering cross sections were calculated for this model for Pb, Ca, and Fe nuclei. The main conclusions are as follows: The scattering and absorption of neutrons are connected with the outer part of the potential, corresponding to the outer layer of the nucleus and to the density fluctuations in this layer. The inner part of the potential influences the above processes only slightly and might be described as corresponding to the "inert core" of the nucleus.

I. INTRODUCTION

FOR a description of scattering and reactions between nuclei and different nuclear particles, use is often made of an optical potential such as that described by Preston,[1] with the following basic properties: (a) It contains an imaginary part, and (b) it is nonlocal. But this potential is usually replaced by the local potential, which simplifies the numerical solution of the corresponding Schrödinger equation and allows one to find the cross section of the process under study as a function of the potential parameters. The values of such parameters are usually not determined in a unique manner, so that the physical characteristics of the process are dubious. Perey and Buck[2] have shown that the use of a certain physically reasonable nonlocal potential allows the elimination of this nonuniqueness. The theoretically derived optical potential is nonlocal even for a local nucleon-nucleon interaction. But according to Gilitan and Thaler,[3] the interaction is in fact nonlocal. Because of that nonlocality, the optical potential deserves further study.

In the present work the specific form of a nonlocal potential is considered as a *superposition* of partial separable potentials (for different l) connected with a simple Fermi-gas nuclear model, which allows an analytic solution of the Schrödinger equation. From the physical point of view, such a presentation—according to Neilson[4]—corresponds to the replacement of a single optical refracting body by a set of several optical bodies with different refractive characteristics. This replacement permits qualitative estimation of the connection between the main structural properties of a nucleus and its interaction with outer particles. By assuming some connection between the shape of the potential and electron-nucleus scattering data (see Hofstadter[5]), the number of actually variable parameters is decreased. Sitenko[6] has shown that the imaginary part of the potential is due to the density fluctuactions. Analysis of the scattering in terms of the proposed potential allows one to draw some conclusions about the importance of the fluctuations in the interaction.

II. POTENTIAL

The Schrödinger equation with general nonlocal potential $V(\mathbf{r}, \mathbf{r}')$ has the following form as given by Wheeler[7] and Yamaguchi[8]:

$$[-(\hbar^2/2M)\Delta+E]\psi(\mathbf{r})=\int V(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}')d\mathbf{r}'. \quad (1)$$

The expression $V(\mathbf{r}, \mathbf{r}')$ could also have, besides the usual terms, the spin-orbit term $\propto(\mathbf{l}\cdot\mathbf{s})$; but for elastic scattering processes we can drop it. After the usual separation of the radial and angular parts of the potential and of the wave function,

$$V(\mathbf{r},\mathbf{r}')=\sum_{l,m} g_l(r,r')[i^l y_l^m(\theta,\varphi)][i^l y_l^m(\theta',\varphi')]^*, \quad (2)$$

$$\psi(\mathbf{r})=\sum_{l,\nu} R_l(r)C_l{}^\nu i^l Y_l{}^\nu(\theta,\varphi), \quad (3)$$

the set of the integrodifferential equations for the

* Research supported in part by the Atomic Energy Control Board of Canada.

[1] M. A. Preston, *Physics of the Nucleus* (Addison-Wesley Publishing Co. Inc., Reading, Mass. 1962), p. 561.
[2] F. Perey and B. Buck, Nucl. Phys. **32**, 353 (1962).
[3] D. A. Gilitan and R. M. Thaler, Phys. Rev. **131**, 805 (1963).
[4] G. C. Neilson (private communication).
[5] R. Hofstadter, Rev. Mod. Phys. **28**, 214 (1956).
[6] A. G. Sitenko, Zh. Eksperim. i Teor. Fiz. **43**, 319 (1962) [English transl.: Soviet Phys.—JETP **17**, 228 (1962)].
[7] J. A. Wheeler, Phys. Rev. **50**, 643 (1936).
[8] Y. Yamaguchi, Phys. Rev. **94**, 1628 (1954).

radial partial wave functions $R_l(r)$ is obtained:

$$\{(d^2/dr^2)+(2/r)(d/dr)+[k^2-l(l+1)/r^2]\}R_l(r) = (2M/\hbar^2)\int_0^\infty g_l(r,r')R_l(r')r'^2dr', \quad (4)$$

in which $k^2=2ME/\hbar^2$, $E=$energy, $M=$reduced mass, $g_l(r, r')=$radial partial component of $V(\mathbf{r}, \mathbf{r}')$, and $l=$angular-momentum quantum number. The standard method of solving this problem is to construct the $g_l(r, r')$, taking into account some physical ideas, and then, using these $g_l(r, r')$ in Eq. (4), to obtain the corresponding solution numerically on a computer. But the solution of the integrodifferential equation is a rather complicated problem even for computers. So usually the $g_l(r, r')$ is taken to be local in form, that is, $g_l(r, r')=v(r)\delta(r-r')$, and the corresponding differential equations are solved. The local potential parameters are defined by comparison with experimental data, not in a unique manner, and have only formal meaning.

Another way of treating the problem is by imposing on the potential $g_l(r, r')$ more definite formal and physical conditions, which do not contradict the main idea of the optical potential, but allow one to get more information about the nucleus involved. These conditions are formulated as follows: From the theory of approximate solutions of integral equations as given by Kantorovich and Krylov,[9] it is known that an effective method of solution is the replacement of the kernel $K(r, r')$ of the equation with the degenerate one[10]

$$K(r,r')\rightarrow\sum_{j=1}^{n}\alpha_j(r)\beta_j(r'). \quad (5)$$

For the potential $g_l(r, r')$, which should be defined on the basis of practical applicability and physical expediency, we can put quite formally

$$g_l(r,r')=\sum_{l'}[v_{ll'}(r)f_{ll'}(r')+iw_{ll'}(r)u_{ll'}(r')], \quad (6)$$

and, in the general case with some definite number of l', consider this formula as practically exact. The solution of Eq. (4) with such a potential can be obtained formally in a simple manner. The physical meaning of such an expression is that it describes the interaction of the partial wave $R_l(r)$, having a given l, with all nucleons in the nucleus, which have all possible values l' of angular momentum. This solution is still rather complicated from the practical point of view.

For further simplification of the problem, the fact that the interaction between two nucleons is especially effective when both of them have the same value of l is of use. This follows (a) from the validity of the shell model, and (b) from the existence of an energy gap in the spectrum of even-even nuclei, connected with "pairing correlations" of nucleons with the same l, etc. Because of the effectiveness of s interactions, we can suppose that if an incident nucleon has a definite value of l, then because the nucleons in a nucleus target move nearly independently (according to the simple shell model), the incident nucleon should interact mainly with the inner nucleons having the same l. But the incident nucleon is described by a plane wave, which contains all values of l. Generalizing a previous supposition, one can imagine that *each* partial wave $R_l(r)$ $(l=l_i)$ of the incident nucleon should interact with the inner nucleons with the same $l=l_i$. Taking into account only the main part of the interaction—as accords with the spirit of an optical-potential model—one can write approximately the real part of the potential

$$\mathrm{Re}g_l(r,r')\simeq v_{ll}(r)f_{ll}(r')=v_l(r)v_l(r'),$$

where $f_l(r)=v_l(r)$ because of symmetry considerations, as shown by Yamaguchi.[8] To find an explicit expression for $v_l(r)$, the main feature of the optical potential is used: the correspondence of the form of the real part of the potential to the shape of the nuclear density distribution. So if the potential $v_l(r)v_l(r')$ is replaced in Eq. (4) by the local potential $v_l(r)v_l(r')\delta(r-r')$, then, after integration over r', one should obtain $v_l^2(r)R_l(r)=\kappa\rho_l(r)R_l(r)$, the usual "potential" part of the equation, with the local partial potential $\rho_l(r)$ equal to the partial density distribution of the inner nucleons with given l. Then

$$v_l(r)=\kappa^{1/2}\rho_l^{1/2}(r), \quad (7)$$

in which κ is a proportionality factor.

For the definition of the imaginary part of the potential, use is made of the fact that it is due to the fluctuations of a nucleon's density (see Sitenko, Ref. 6). These fluctuations are small inside the nucleus, where all possible states are filled according to the Pauli principle. They can be intense only in the outer layer of the nucleus, where the nucleons have $l=l_{\max}\equiv L$, where the nucleon vacancies (holes) are concentrated, and where virtual transitions to the next nonfilled shell are possible. The most preferred energetically are the fluctuations connected with the exchange processes between nucleons and holes in the same (outer) shell. Because the fluctuations are connected with the outer shell $(l=L)$, their distribution should correspond to the distribution of nucleons on this L shell, and the shape of the imaginary part of the potential should reproduce the shape of the nucleon

[9] A. V. Kantorovich and V. I. Krylov, *Approximate Methods of Higher Analysis* (John Wiley & Sons, Inc., New York, 1964), Chap. 2.

[10] The method of finding the functions α_l and β_l is given by Kantorovich and Krylov (Ref. 9) when $K(r, r')$ is known. In our case, with unknown $g_l(r, r')$, physical conditions are imposed instead on α_j and β_j, and then $g_l(r, r')$ is found, the index j being replaced by the index l' according to the significance of l' as an angular-momentum characteristic of the nuclear nucleons.

distribution on this shell:

$$w_l(r)=\gamma^{1/2}\rho_L{}^{1/2}(r). \tag{8}$$

Besides absorbing nucleons, the fluctuations also scatter them. This part of the potential is

$$\kappa_L\rho_L{}^{1/2}(r)\rho_L{}^{1/2}(r'). \tag{9}$$

We define $\rho_l(r)$ according to the Thomas-Fermi method[11] whose applicability to finite nuclei follows from the work of Zel'dovich and Rabinovich[12]:

$$\rho_l(r)=[(2l+1)/(2\pi r)]\int_0^{K_\mu(r)} kJ_{l+1/2}{}^2(kr)\,dk, \tag{10}$$

where $J_{l+1/2}(kr)$ is a Bessel function of half-integer index, and $K_\mu(r)=[3\pi^2\rho(r)]^{1/3}$ is the Fermi momentum, with

$$\rho(r)=\sum_{l=0}^{\infty}\rho_l(r)=\rho_0\{1+\exp[(r-C)/\beta]\}^{-1} \tag{11}$$

equal to the density distribution of protons evaluated experimentally by Hofstadter.[5] The numbers of nucleons on the shells with $l<L$ calculated by this method approximately agree with the predictions of the shell model.[11] All $\rho_l(r)$ with $l\geq L$ are included in the last shell of the nucleus. The normalizing constant ρ_0 is $\rho_0=0.68\times10^{38}$ cm^{-3}; the other constants are $C=r_0\times A^{1/3}$, $r_0=(1.07\pm0.02)\times10^{-13}$ cm, $\beta=(0.5455\pm0.0682)\times10^{-13}$ cm. Summing up all the previous proposals, the potential can be written in the form

$$g_l(r,r')=\kappa_0 p_l{}^{1/2}(r)\,p_l{}^{1/2}(r')+(\kappa_L+i\gamma)\,p_L{}^{1/2}(r)\,p_L{}^{1/2}(r'). \tag{12}$$

Here κ_0 describes the interaction of the incident particle with the nuclear matter of the nucleus—a homogeneous unperturbed medium of nucleons in nonexcited states when all neighboring states are filled up. κ_L represents the interaction with the fluctuations, which consists of the averaged interactions with the nucleons in the excited virtual states, with the nucleons in the states with vacancies (holes) as neighbors, and with the holes themselves. This interaction corresponds to the elastic scattering by the fluctuations. The symbol γ corresponds to the absorption of the incident nucleons by the fluctuations.

Cofactors p_l have the same analytical expression as ρ_l, the only differences being that (i) the parameter C is replaced by $R=C+d$, in which d is the distance by which the potential extends out of the range of the density distribution; and (ii) β is replaced by two parameters: for the inner shells $l<L=l_{\max}$, the p_l have diffuseness $\alpha_i=\beta$; for the outer shells ($l\geq L$) the diffuseness should be larger. There is no confining shell that restricts the increase in the spreading of the density of fluctuations and of the potential distributions, so α_0, the diffuseness of the outer potential component, is bigger than β.

Altogether there are six parameters: three dynamic (κ_0, κ_L, and γ), and three geometric (d, α_i, and α_0). The parameter r_0 is fixed by the electron scattering measurements.[5]

It is evident that the parameters κ_0 and α_i are connected with the inner part of the nucleus [the contribution of κ_0 in the outer shell is negligibly small: $\rho_L(r)\ll\sum\rho_l(r)$, $(l<L)$]. They can be called "inner-core parameters."

Because it is accepted that the inner part of every nucleus consists of closed shells with no holes, these parameters are characteristic of nuclear matter and should be kept fixed for nearly all nuclei except for light ones and very heavy ones. The parameters that may vary significantly are then κ_L, γ, α_0, and d. Actually, d also is rather slowly varying or even a constant parameter, as follows from practical computations. So the real individual nuclear characteristics are connected with the parameters κ_L, γ, and α_0 describing the interaction of the incident particle with the outer layer of a nucleus, which outer layer changes from nucleus to nucleus.

III. SOLUTION OF THE SCHRÖDINGER EQUATION

Equation (4) can be rewritten as follows, introducing $\xi=kr$:

$$R_l''(\xi)+(2/\xi)R_l(\xi)+[1-l(l+1)/\xi^2]R_l(\xi)$$
$$=\kappa_0' p_l{}^{1/2}(\xi)\int_0^\infty p_l{}^{1/2}(\xi')R_l(\xi')\xi'^2d\xi'+(\kappa_L'+i\gamma')\,p_L{}^{1/2}(\xi)$$
$$\times\int_0^\infty p_l{}^{1/2}(\xi')R_l(\xi')\xi'^2d\xi',$$
$$\kappa_0'=\kappa_0/E,\qquad \kappa_L'=\kappa_L/E. \tag{13}$$

The p_l, having the same form as ρ_l, can be written as in Ref. 13:

$$p_l(r)=[(2l+1)/2\pi r][K(r)]^3$$
$$\times\{j_l{}^2[rK(r)]-j_{l-1}[rK(r)]\,j_{l+1}[rK(r)]\}, \tag{14}$$
$$K(r)\equiv K_\mu(r)=[3\pi^2\rho(r)]^{1/3}.$$

Here C is replaced by $R=C+d$; β, by α_i $(l<L)$ and α_0 $(l\geq L)$.

Because the integrals on the right-hand side of (13) are taken from 0 to ∞, they are constants, depending only on parameters and k, but not on r. So

$$R_l''(\xi)+(2/\xi)R_l'(\xi)+[1-l(l+1)/\xi^2]R_l(\xi)$$
$$=\kappa_0' p_l{}^{1/2}(\xi)A_l+(\kappa_L'+i\gamma')B_l p_l{}^{1/2}(\xi), \tag{13'}$$

[11] L. P. Rappoport and S. G. Kadmenskiy, Zh. Eksperim. i Teor. Fiz. **37**, 1303 (1959) [English transl.: Soviet Phys.—JETP **10**, 929 (1959)].

[12] Ya. B. Zel'dovich and V. M. Rabinovich, Zh. Eksperim. i Teor. Fiz. **37**, 1296 (1959) [English transl.: Soviet Phys.—JETP **10**, 924 (1959)].

[13] G. N. Watson, *The Theory of Bessel Functions* (Cambridge University Press, New York, 1966).

where

$$A_l=\int_0^\infty p_l^{1/2}(\xi')R_l(\xi')\xi'^2d\xi';$$

$$B_l=\int_0^\infty p_L^{1/2}(\xi')R_l(\xi')\xi'^2d\xi'.$$

Having in mind that R_l is complex, $R_l(\xi)=P_l(\xi)+iQ_l(\xi)$, we obtain from (13′) the equations for $P_l(\xi)$ and $Q_l(\xi)$

$$P_l''(\xi)+(2/\xi)P_l'(\xi)+[1-l(l+1)/\xi^2]P_l(\xi)$$
$$=\kappa_0'\alpha_{1l}p_l^{1/2}(\xi)+(\kappa_L'\alpha_{2l}-\gamma'\beta_{2l})p_l^{1/2}(\xi);$$
$$P_l'(\xi)\equiv dP_l(\xi)/d\xi, \tag{15}$$
$$Q_l''(\xi)+(2/\xi)Q_l'(\xi)+[1-l(l+1)/\xi^2]Q_l(\xi)$$
$$=\kappa_0'\beta_{1l}p_l^{1/2}(\xi)+(\kappa_L'\beta_{2l}+\gamma'\alpha_{2l})p_l^{1/2}(\xi).$$

The solution of the system (15) can be found easily[14] because the homogeneous equations corresponding to (15) have as solutions the Bessel functions of half-integer index. The result is

$$P_l(\xi)=\eta_l(\xi)\int j_l(\xi)[\kappa_0'\alpha_{1l}p_l^{1/2}(\xi)+\kappa_L'\alpha_{2l}p_L^{1/2}(\xi)$$
$$+\gamma'\beta_{2l}p_L^{1/2}(\xi)]\xi^2d\xi-j_l(\xi)\int\eta_l(\xi)[\kappa_0'\alpha_{1l}p_l^{1/2}(\xi)$$
$$+\kappa_L'\alpha_{2l}p_L^{1/2}(\xi)+\gamma'\beta_{2l}p_L^{1/2}(\xi)]\xi^2d\xi$$
$$+C_1j_l(\xi)+C_2\eta_l(\xi); \tag{16a}$$
$$Q_l(\xi)=\eta_l(\xi)\int j_l(\xi)[\kappa_0'\beta_{1l}p_l^{1/2}(\xi)+\kappa_L'\beta_{2l}p_L^{1/2}(\xi)$$
$$+\gamma'\alpha_{2l}p_L^{1/2}(\xi)]\xi^2d\xi-j_l(\xi)\int\eta_l(\xi)[\kappa_0'\beta_{1l}p_l^{1/2}(\xi)$$
$$+\gamma'\alpha_{2l}p_L^{1/2}(\xi)+\kappa_L'\beta_{2l}p_L^{1/2}(\xi)]\xi^2d\xi$$
$$+D_1j_l(\xi)+D_2\eta_l(\xi). \tag{16b}$$

The last two terms in P_l and Q_l, which contain the arbitrary constants C_1, C_2, D_1, and D_2, are solutions of the homogeneous equations. The integrals in P_l and Q_l are indefinite, and consist of variable parts $[U_l(\xi)$ and $W_l(\xi)]$ and arbitrary constants (say C_1' and C_2' for P_l, D_1' and D_2' for Q_l). Putting $C_1+C_1'=\tilde{C}_1$, $C_2+C_2'=\tilde{C}_2$, $D_1+D_1'=\tilde{D}_1$, and $D_2+D_2'=\tilde{D}_2$, and taking into account the initial condition $P_l(\xi)|_{\xi=0}=j_l(\xi)$, $Q_l(\xi)|_{\xi=0}=0$ (Ref. 15.), we obtain $\tilde{C}_1=1$; $\tilde{C}_2=\tilde{D}_1=\tilde{D}_2=0$. Thereby the solutions of the Eqs. (15), and hence the solution of Eq. (13), are specified in analytical form. The quantities α_{1l}, α_{2l}, β_{1l}, β_{2l} are

$$\alpha_{1l}\equiv\mathrm{Re}A_l=\int_0^\infty p_l^{1/2}(\xi')P_l(\xi')\xi'^2d\xi', \tag{17}$$
$$\beta_{1l}\equiv\mathrm{Im}A_l=\int_0^\infty p_l^{1/2}(\xi')Q_l(\xi')\xi'^2d\xi;$$
$$\alpha_{2l}\equiv\mathrm{Re}B_l=\int_0^\infty p_L^{1/2}(\xi')P_l(\xi')\xi'^2d\xi; \tag{18}$$
$$\beta_{2l}\equiv\mathrm{Im}B_l=\int_0^\infty p_L^{1/2}(\xi')Q_l(\xi')\xi'^2d\xi'.$$

On inserting in these expressions the functions $P_l(\xi)$ and $Q_l(\xi)$, there results a system of algebraic equations for α_{1l}, β_{1l}, α_{2l}, and β_{2l}, the solution of which is the following (index l is omitted):

$$\alpha_1=[a_1-b_{42}(\kappa_L'\alpha_2-\gamma'\beta_2)]/d_{31};$$
$$\beta_1=-b_{42}(\kappa_L'\beta_2-\gamma'\alpha_2)/d_{31};$$
$$\alpha_2=\frac{[a_2+\kappa'(a_2b_{31}+a_1b_{57})](d_{31}d_{L86}+C_{42}b_{L57})}{(d_{31}d_{L86}+C_{42}C_{L57})^2+\gamma'^2(d_{31}b_{86}+C_{42}b_{57})^2}; \tag{19}$$
$$\beta_2=-\gamma'\alpha_2(C_{42}b_{57}+d_{31}b_{86})/(d_{31}d_{L86}+C_{42}C_{L57});$$

where

$\kappa^1\equiv\kappa_0'$, $b_{nm}=b_n-b_m$; $C_{hi}=\kappa'C_{hi}$; $C_{L57}=\kappa_L'b_{57}$;

$d_{31}=[1+\kappa'b_{31}]$;

$d_{L86}=[1+\kappa_L'b_{86}]$; $(n, m, h, i=1-8)$

$$a_1=\int_0^\infty p_l^{1/2}(r)j_l(kr)r^2dr;$$

$$a_2=\int_0^\infty p_L^{1/2}(r)j_l(kr)r^2dr;$$

$$b_1=k^3\int_0^\infty p_l^{1/2}(r)\eta_l(kr)\int^{r'=r}j_l(kr')p_l^{1/2}(r')r'^2dr'r^2dr;$$

$$b_2=k^3\int_0^\infty p_l^{1/2}(r)\eta_l(kr)\int^{r'=r}j_l(kr')p_L^{1/2}(r')r'^2dr'r^2dr;$$

$$b_3=k^3\int_0^\infty p_l^{1/2}(r)j_l(kr)\int^{r'=r}\eta_l(kr')p_l^{1/2}(r')r'^2dr'r^2dr;$$

$$b_4=k^3\int_0^\infty p_l^{1/2}(r)j_l(kr)\int^{r'=r}\eta_l(kr')p_L^{1/2}(r')r'^2dr'r^2dr;$$

$$b_5=k^3\int_0^\infty p_L^{1/2}(r)\eta_l(kr)\int^{r'=r}j_l(kr')p_l^{1/2}(r')r'^2dr'r^2dr;$$

$$b_6=k^3\int_0^\infty p_L^{1/2}j_l\int^{r'=r}\eta_lp_l^{1/2}r'^2dr'r^2dr;$$

$$b_7=k^3\int_0^\infty p_L^{1/2}j_l\int^{r'=r}\eta_lp_l^{1/2}r'^2dr'r^2dr;$$

[14] E. Kamke, *Differentialgleihungen* (Chelsea Publishing Company, New York, 1964), Chap. 5.

[15] P. E. Hodgson, *The Optical Model of Elastic Scattering* (Clarendon Press, Oxford, England, 1963).

$$b_8=k^3\int_0^\infty p_L^{1/2}j_l\int^{r'=r}p_L^{1/2}\eta_l r'^2dr'r^2dr,$$

$$p_L\equiv p_L(r),\text{ etc.}\quad(20)$$

It is necessary to stress again that the indefinite integrals in the expressions for the b's do not have any arbitrary constants, since these were eliminated by the initial conditions [Eq. (17)]. This means that the solutions P_l and Q_l are expressed by the known functions and the integrals of the known functions. All quantities are defined completely and the solution is complete.

IV. S-MATRIX ELEMENTS AND THE DIFFERENTIAL CROSS SECTION

To obtain the expression for the S-matrix elements the asymptotic expressions of the Eqs. (4) and their solutions are considered. Let us write: $u_l(kr)=krR_l(kr)$. From Eq. (4) is obtained the equation for u_l ($u_l''\equiv d^2u_l/dr^2$)

$$u_l''+\{k^2-[l(l+1)/r^2]\}u_l=(2M/\hbar^2)kr\int_0^\infty g_l(r,r')\times R_l(r')r'^2dr',\qquad[u_l(0)=0].\quad(21)$$

The corresponding equation for the free motion is

$$\varphi_l''+\{k^2-[l(l+1)/r^2]\}\varphi_l=0,\qquad[\varphi_l(0)=0]\quad(22)$$

with the solution

$$\varphi_l(kr)=krj_l(kr).\quad(23)$$

Multiplying (21) by $\varphi_l(kr)$ and (22) by $u_l(kr)$ and subtracting the second product from the first, we obtain

$$\varphi_l u_l''-u_l\varphi''=(2M/\hbar^2)kr\varphi_l(kr)\times\int_0^\infty g_l(r,r')R_l(kr')r'^2dr'.\quad(24)$$

Integrating the left- and right-hand sides of this equation by parts from 0 to r_a (the value of r where u_l and φ_l begin to behave asymptotically), we then obtain

$$[\varphi_l(kr)u_l'(kr)-u_l(kr)\varphi_l'(kr)]_{r=r_a}=(2M/\hbar^2)k\int_0^{r_a}r\varphi_l(kr)\times\int_0^\infty g_l(r,r')R_l(kr')r'^2dr'dr.\quad(25)$$

But the asymptotic values of the functions $u_l(kr)$ and $\varphi_l(kr)$ are as follows (the factor $\frac{1}{2}$ should be used in Ref. 16):

$$u_l(kr)\,|_{r\geq r_a}=\tfrac{1}{2}\{krj_l(kr)-ikr\eta_l(kr)+S_l[krj_l(kr)+ikr\eta_l(kr)]\}_{r\geq r_a},\quad(26)$$

$$\varphi_l(kr)=krj_l(kr).\quad(27)$$

[16] B. Buck, R. M. Maddison, and P. E. Hodgson, Phil. Mag. 5, 1181 (1960).

But asymptotically for $r\geqq r_a$

$$krj_l(kr)\sim\cos[kr-(l+1)\pi/2];\quad(28)$$

$$kr\eta_l(kr)\sim\sin[kr-(l+1)\pi/2].\quad(29)$$

Replacing j_l and η_l in (26) and (27) with their asymptotic values (28) and (29), and inserting these values in the left-hand side of (25), we obtain, after obvious calculations,

$$ik[S_l-1]=(2M/\hbar^2)\times k\int_0^{r_a}r\varphi_l(kr)g_l(r,r')R_l(kr')r'^2dr'dr.\quad(30)$$

Using the expressions for $R_l=P_l+iQ_l$ given above, the expression for the S-matrix element is

$$[S_l-1]=(2M/\hbar^2)k\Big[\kappa_0(\beta_{1l}-i\alpha_{1l})\times\int_0^{r_a}p_l^{1/2}(r)j_l(kr)r^2dr+(\kappa_L+i\gamma)(\alpha_{2l}+i\beta_{2l})\times\int_0^{r_a}p_L^{1/2}(r)j_l(kr)r^2dr\Big].\quad(31)$$

To obtain the differential cross section is trivial. The scattering amplitude is

$$f(\theta)=(i/2k)\sum_{l=0}^{\infty}(2l+1)[1-S_l]P_l(\cos\theta),\quad(32)$$

and the differential cross section

$$d\sigma/d\Omega=[\mathrm{Re}f(\theta)]^2+[\mathrm{Im}f(\theta)]^2.\quad(33)$$

V. COMPUTATION AND DISCUSSION OF RESULTS

Computations of the differential cross section for the elastic scattering of neutrons were performed for the following nuclei:

Pb^{208} ($A=208$, $L=6$, $r_0=1.09\times10^{-13}$ cm, $\beta=0.528\times10^{-13}$ cm),

Ca^{40} ($A=40$, $L=2$, $r_0=1.06\times10^{-13}$ cm, $\beta=0.546\times10^{-13}$ cm),

Fe^{56} ($A=56$, $L=4$, $r_0=1.09\times10^{-13}$ cm, $\beta\simeq0.58\times10^{-13}$ cm).

The numerical evaluation of the integrals b_i and the computational program were carried out by J. F. Easton, engineer-analyst of the Nuclear Research Centre, University of Alberta.

The experimental data on the scattering of neutrons by these nuclei were taken from the work of McDonald and Robson[17] for Ca^{40}, and from the work of Perey and Buck[2] for Pb and Fe.

[17] W. J. McDonald and J. M. Robson, Nucl. Phys. 59, 321 (1964).

The fitting of the experimental points with the theoretical curves was done both for the absolute magnitude of $d\sigma/d\Omega$ and for the shape of the curves. The computed curves together with the experimental results are presented in Figs. 1–3.

The important feature of this computation is that each curve resulted in a unique set of parameters—no other values of parameters could give similar curves. The values of these parameters are presented in Table I. The parameters d (the range of extension of the potential out of the density distribution), α_i [the

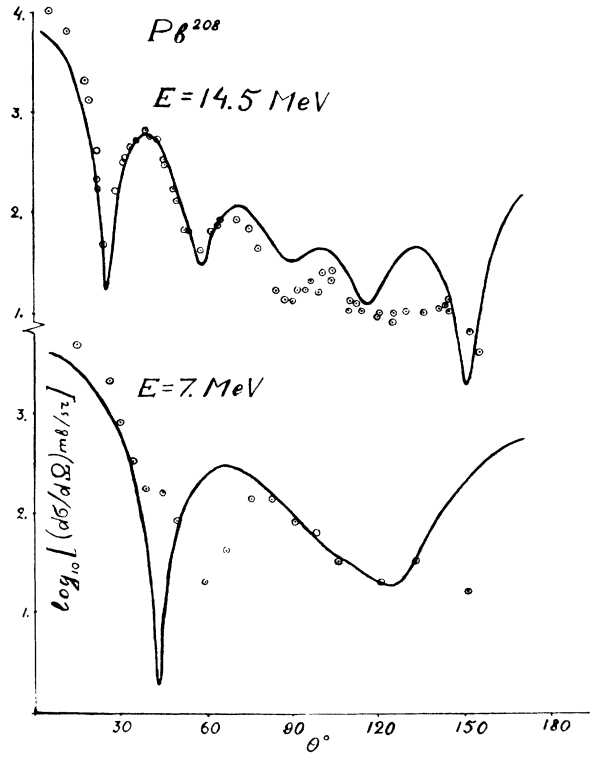

FIG. 1. The differential cross section $d\sigma/d\Omega$ for the elastic scattering of neutrons by Pb^{208} at energies 7 and 14.5 MeV. The values of the parameters are given in Table I. Experimental data are taken from Ref. 2.

diffuseness of the inner partial potentials ($l<L$), i.e., the diffuseness of the inner potential core], and κ_0 (the coefficient of interaction of the incident partial waves with the inner nuclear matter or core of the nucleus) could be called "inner-nuclear-core" parameters, because κ_0 and d are actually the same for all three different nuclei, and α_i is practically equal to the density diffuseness parameter β, which has been determined from electron scattering by nuclei.[5]

The calculated cross section is most sensitive to the values of α_0, κ_L, and γ, since they characterize the surface layer of the nucleus, which contains the nucleons with $l=l_{\max}\equiv L$, and where fluctuations of the lowest possible energy are concentrated. (High-energy fluctu-

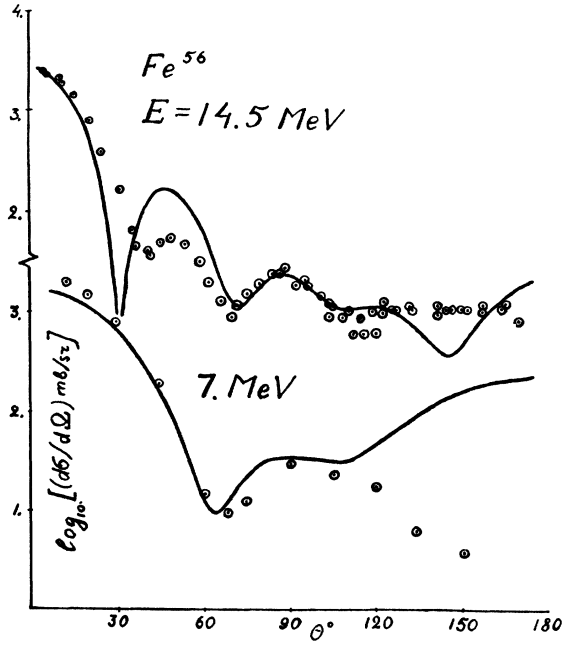

FIG. 2. $d\sigma/d\Omega$ for neutron scattering by Fe^{56}, $E=7$ and 14.5 MeV. Experimental data are taken from Ref. 2.

ations of nuclear density are possible in all parts of the nucleus, but statistically they are much less probable.) This follows from an analysis of the curves of Figs. 1–3, and is illustrated in Table II, in which the comparison of the dependence of $d\sigma/d\Omega$ on κ_0, κ_L, and γ for Pb^{208} is presented. It is shown there that a change of the values of κ_0, even over a wide range, affects the values of the cross section less strongly than a change of κ_L and γ over narrow ranges. Variation of κ_L and γ leads to much more significant variations of $d\sigma/d\Omega$, illustrating that κ_0 is really an "inert-core" parameter; that the most active part of the interaction is concentrated in the surface layer of the nucleus; and that it is the

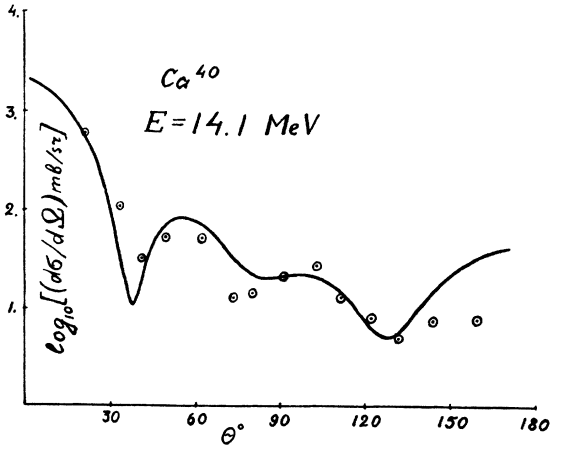

FIG. 3. $d\sigma/d\Omega$ for neutron scattering by Ca^{40}, $E=14.1$ MeV. Experimental data are taken from Ref. 17.

TABLE I. Calculated values of the potential parameters: d, the extension of the potential out of the density distribution; α_i, the diffuseness of the inner partial potential for $l<L$ (very close to the density diffuseness); κ_0, the coefficient of the interaction of the incident particle with the inner nuclear matter; α_0, the diffuseness of the outer layer of the potential ($l\geq L$), corresponding to the low-energy surface density fluctuations; κ_L, the real, and γ, the imaginary, coefficient of the interaction of the projectile with the fluctuating surface layer of the nucleus. This interaction is more intense and more effective than the inner (κ_0) part of interaction (see also Table II).

	Energy (MeV)	d (F)	α_i (F)	κ_0 (MeV)	α_0 (F)	κ_L (MeV)	γ (MeV)
Pb^{208}	14.5	0.05	0.53	−0.28	1.24	−1.15	−1.15
	7	0.05	0.53	−0.28	1.04	−1.18	−1.18
Ca^{40}	14.1	0.05	0.55	−0.28	0.88	−3.38	−5.75
Fe^{56}	14.5	0.05	0.58	−0.28	1.24	−2.97	−3.95
	7	0.05	0.58	−0.28	0.957	−4.47	−5.97

fluctuations, characterized by κ_L and γ, which are responsible for the interaction. In discussing the quality of fit of the computed curves to the experimental data, it is necessary to note that our possibilities in fitting were limited because no proper search routine was available. Everything was done by trial and error, and the author was surprised by the agreement achieved.

To simplify the computation of the indefinite integrals in the expressions for the b_i's, it was assumed that the limiting values of these integrals,

$$\lim F_{2l}(\xi)=\lim\int \eta_l p_L^{1/2}\xi^2 d\xi, \quad \text{for} \quad r\to\infty$$

were equal to

$$\lim F_{1l}(\xi)=\int j_l p_L^{1/2}\xi^2 d\xi, \quad \text{for} \quad r\to\infty.$$

It is possible to show that this assumption caused very minor errors in the computed values of the b_i's.

A correction should also be made of a numerical error in previous work,[18] in which during the computation of numerical constants the value of $d\sigma/d\Omega$ for Pb^{208} was accidentally made 100 times too small. It is interesting to note, though, that the value of κ_0 in that wrong result was practically the same as in the present work; this is another illustration of the "inertness" of the inner core.

VI. CONCLUSIONS

1. The proposed nonlocal optical potential qualitatively corresponds to the basic properties of a simple Fermi-gas model of a nucleus. Formal presentation of this potential as a sum of partial separable nonlocal potentials permits solution of the Schrödinger equation, analytically.

2. The hypothesis of the effectiveness of "the relative s interaction" for all values of l simplifies the case significantly. It permits one to represent the potential as a superposition of partial potentials, which corresponds to the representation of the nuclear density distribution (determined experimentally by Hofstadter[5]) as a sum of partial density distributions according to the statistical Thomas-Fermi method. It also permits one to obtain analytical expressions for S-matrix elements and for the differential cross section for the process.

3. Specification of the imaginary part of the inter-

TABLE II. The dependence of the relative differential cross section $\Theta=[(d\sigma/d\Omega)_{\text{theoret}}/(d\sigma/d\Omega)_{\text{expt}}]$ on the values of the dynamical parameters. Changing κ_0 (the inner-core parameter) from -10^{-4} MeV to -50 MeV changes Θ from 10^{-9} to 10^{-2}, or 100 times less than the experimental value, while it changes κ_L and γ from -10^{-4} MeV to -1.15 MeV, which is the experimental value. This means that κ_L and γ, the outer-layer parameters, are really responsible for the interaction of the projectiles with the target nuclei, and that κ_0 is really an inert-core parameter.

κ_0 (MeV)	κ_L (MeV)	γ (MeV)	$\Theta=\left(\frac{d\sigma}{d\Omega}\right)_{\text{theor}}\Big/\left(\frac{d\sigma}{d\Omega}\right)_{\text{expt}}$
-10^{-4}	-10^{-4}	-10^{-4}	10^{-9}
-10^{-4}	-10^{-4}	(−0.1)-(−1.)	(10^{-3})-(<1)
>−0.5	<−1	-10^{-4}	(10^{-2})-(<1)
-10^{-4}	−1.15	−1.15	≃1
−0.28	−1.15	−1.15	1
(−30)-(−50)	-10^{-4}	-10^{-4}	(10^{-3})-(10^{-2})
-10^{-4}	−6	−30	10^{2}

[18] B. B. Dotsenko, Nuclear Research Centre, University of Alberta, Edmonton, Canada (unpublished).

action as caused by fluctuations[6] concentrated in the surface shell ($l=l_{max}$) permits one to distinguish two main parts of nuclei: (i) the *inert core*, and (ii) the *strongly interacting nuclear surface* containing nucleons with $l=l_{max}$, holes, and fluctuations. These conclusions follow from the analysis of Figs. 1–3 and Tables I and II.

4. This analysis shows that the inert-core parameters are fixed constants for different nuclei, so that those of practical importance (for the specification of the interaction properties of different nuclei) are the three nuclear surface parameters α_0, κ_L, γ. This is distinctly less than the usual number of parameters of local optical potentials.

5. More extensive analysis of these parameters for different nuclei will supply material for possible correlation between the structure of the surface layer of a nucleus and its interaction with particles, and particularly for investigation of the role of the holes and density fluctuations in such interactions.

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Study of the J Dependence in the (d, He^3) Reaction*

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The J dependence of the angular distributions in the (d, He^3) reactions was experimentally observed at small angles in the $2p$ proton pickup. This J dependence is qualitatively reproduced by distorted-wave calculations which indicate that the effect arises mainly from the deuteron spin-orbit potential. The calculated angular distributions are strongly dependent on the Q value of the reactions and the incident deuteron energy. The calculations also show that the J dependence is localized at the surface of the nucleus, and is very sensitive to the volume of the real part of the deuteron optical potential. This sensitivity may be understood by considering the role of the spin-orbit term in the optical potential. The sensitivity of the calculated angular distributions to the other parameters is investigated.

I. INTRODUCTION

SINCE the first discovery[1] of the J dependence of the angular distributions in the (d, p) reaction, similar effects have been observed in many other direct reactions. The J dependence in the (d, He^3) reaction was first observed by Preedom, Newman, and Hiebert.[2] They found a difference between the angular distributions of $2p_{1/2}$ and $2p_{3/2}$ proton pickup in the region from 50° to 80° at an incident deuteron energy of 34.4 MeV. This difference was qualitatively reproduced by the ordinary distorted-wave Born-approximation (DWBA) theory, and was found to arise from the deuteron spin-orbit potential.

The J dependence of the $l=1$ transition at small angles (15°–30°) was observed in the course of our spectroscopic study of the (d, He^3) reaction on even-even Mo isotopes. This observation is important because at such small angles the direct interaction should predominate and distorted-wave calculations are expected to be reliable and rather insensitive to the choice of optical-potential parameters. In this paper the experimental evidence for such a J dependence is described. It is also shown that the J dependence is correctly given by the distorted-wave calculations. The changes in the shapes of the calculated curves in response to variations of the individual parameters are discussed in detail.

II. EXPERIMENTS AND COMPARISONS WITH THE DISTORTED-WAVE THEORY

The details of the experiments and analysis were described in a previous paper.[3] The 23-MeV deuteron beam from the Argonne cyclotron was used. The targets were isotopically enriched metallic foils, and

* Work performed under the auspices of the U.S. Atomic Energy Commission.

[1] L. L. Lee, Jr., and J. P. Schiffer, Phys. Rev. Letters **12**, 108 (1964); Phys. Rev. **136**, B405 (1964).

[2] B. M. Preedom, E. Newman, and J. C. Hiebert, Phys. Letters **22**, 657 (1966).

[3] H. Ohnuma and J. L. Yntema, Phys. Rev. **177**, 1695 (1969).