

be in disagreement but the differences may be a manifestation of differing amounts of 1S , 3S , 1P , and 3P waves in the final-state p - ^3H system.

These results are consistent with a simple model in which the relative strength of the various configurations in the final-state p - ^3H system is dependent on the initial-state origin of the proton and triton. In the case of inelastic scattering the final-state p - ^3H system is formed from the highly correlated ^4He target nucleus, within which the proton and triton are in a relative 1S state. No such initial-state correlation exists in the other reactions.

The one remaining inconsistency in Table III is the narrow width reported for this state by Parker *et al.*⁵ Meyerhof²⁰ has analyzed this reaction in terms of the p - ^3H phase shifts and finds fair agreement, but some difficulty with the energy scale. In our opinion the errors

²⁰ W. E. Meyerhof, *Rev. Mod. Phys.* **37**, 512 (1965).

quoted in Ref. 5 are probably too low. The smallness of the errors arises from the "kinematic amplifier" effect discussed by Donovan.²¹ The gain in accuracy as a result of this effect is, however, real only if the angles of the detectors are correspondingly well known. In the experiment under discussion any error in angle would result in an apparent change of both the energy and width of the state. A reexamination of the data in Ref. 5 would therefore be useful.

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²¹ P. F. Donovan, *Rev. Mod. Phys.* **37**, 501 (1965).

Nucleon Correlation Effect in the Shell-Model Description of the Deuteron*

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We have assumed that an effective central potential in place of actual N - N interaction can be used in general for finite-nucleus calculations. By introducing, however, a class of residual two-body interactions, we have studied the nucleon correlation effect in the independent-particle-model (IPM) wave functions of nuclei. As an illustrative example, this method has been applied to the deuteron. A shell-model description of the deuteron is proposed that takes proper account of the nucleon correlation and the residual potential. We have obtained an adequate agreement with the elastic electron-scattering data even for the deuteron without using the tensor force. This method should be applicable to many other nuclear problems and result in interesting predictions.

ASSUMING the nucleon-nucleon interaction as that of two free nucleons, one can make calculations to explain nuclear properties, following the work of Bruckner, Bethe, and others.¹ Since this approach is extremely difficult, some approximations have to be used to make the calculations tractable, and these cause many uncertainties. In this work we shall examine the behavior of a nuclear system with a closed shell plus two particles (or holes). Essentially, we assume nucleons moving in an averaged central field. This oversimplified field has been corrected with a residual two-body interaction which leads to the correlation of the independent-particle wave functions for the system.

* Part of this work was done at the University of Wyoming during the author's visit in the summer of 1968.

¹ M. A. Preston, *Physics of the Nucleus* (Addison-Wesley Publishing Co., Reading, Mass., 1966), p. 266, and references therein; G. Breit, *Proc. Nat. Acad. Sci. U.S.A.* **46**, 746 (1960).

The actual Hamiltonian for a nucleus of mass number A

$$H = \sum_{i=1}^A \left(-\frac{\hbar^2}{2m} \nabla_i^2 \right) + \frac{1}{2} \sum_{i \neq j}^A V(\mathbf{r}_i - \mathbf{r}_j) \quad (1)$$

can be written as

$$H = H_0 + V_{\text{int}}, \quad (2)$$

where

$$H_0 = \sum_{i=1}^A \left(-\frac{\hbar^2}{2m} \nabla_i^2 + U_i \right) \quad (3)$$

and

$$V_{\text{int}} = \frac{1}{2} \sum_{i \neq j}^A V(\mathbf{r}_{ij}) - \sum_{i=1}^A U_i, \quad (4)$$

U_i being the central potential. Since the two-body interaction $V(\mathbf{r}_{ij})$ depends only on the relative coordinates, the choice of U_i here as the harmonic-

oscillator potential has a particular advantage² because the eigenfunctions of two nucleons in the nucleus due to this potential are separable in their relative and c.m. coordinates.³ We have

$$\begin{aligned} \sum_{i=1}^A U_i &= \frac{1}{2}k \sum_{i=1}^A |\mathbf{r}_i|^2 \\ &= \frac{1}{4}(k/A) \left[\sum_{i \neq j}^A |\mathbf{r}_i - \mathbf{r}_j|^2 + 2 \left(\sum_{i=1}^A \mathbf{r}_i \right)^2 \right] \quad (5) \\ &= \frac{1}{4}(k/A) \sum_{i \neq j}^A |\mathbf{r}_i - \mathbf{r}_j|^2 \end{aligned}$$

in the c.m. system.

Hence,

$$V_{\text{int}} = \frac{1}{2} \sum_{i \neq j}^A [V(\mathbf{r}_{ij}) - \frac{1}{2}(k/A) |\mathbf{r}_{ij}|^2]. \quad (6)$$

More realistic forms of $V(\mathbf{r}_{ij})$ based on meson theory are available, but they are quite involved.⁴ However, the interaction potential (6) can be approximated by some well-known potential functions. As an illustrative application, we use the form⁵ for $V_{\text{int}} (\equiv V_{12})$ given by

$$-V_{12} = V_{0c} f_{12}^c(\mathbf{r}) + V_{0\sigma} f_{12}^\sigma(\mathbf{r}) \delta_1 \cdot \delta_2, \quad (7)$$

where $\mathbf{r} = \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, and f_{12}^c and f_{12}^σ are central and spin-dependent parts of the potential, respectively. If f_{12} is assumed to be the same for both dependences, then

$$-V_{12} = \left(\frac{3}{4}V_c + \frac{1}{4}V_\sigma \right) + \left(\frac{1}{4}V_c - \frac{1}{4}V_\sigma \right) \delta_1 \cdot \delta_2, \quad (8)$$

where

$$V_m = V_{0m} \exp(-r^2/r_{0m}^2) [a - b(r/r_{0m})^{2\nu}], \quad (9)$$

with m denoting the spin multiplet—triplet (t) or singlet (s); and V_{0m} are adjustable parameters not exceeding the free value of V_0 for given a , b , and ν . The range parameters are chosen to be⁶ $r_{0t} = 1.36$ fm and $r_{0s} = 1.76$ fm.

The usual shell-model wave function of a closed-shell nucleus is a Slater determinant of single-particle functions determined from a central-field model. Here we use the method of second quantization⁷ by taking the closed shell to be the vacuum state, $|0\rangle$. This method is particularly favorable for the correct description of holes. Thus, the correlated wave function in L - S coupling that we have chosen for this work can be written in this formalism as

$$\Psi_{LM} = (1/N) \psi_{LM}^0(1+c(r/r_0)) |0\rangle, \quad (10)$$

² One could also use some other potential, e.g., A. E. S. Green, G. Darewych, and R. Brezdvin, *Phys. Rev.* **157**, 929 (1967).

³ I. Talmi, *Helv. Phys. Acta.* **25**, 185 (1952).

⁴ N - N Interaction Conference, *Rev. Mod. Phys.* **39**, Session D (1967).

⁵ L. R. B. Elton and M. A. K. Lodhi, *Nucl. Phys.* **66**, 209 (1965).

⁶ L. R. B. Elton, *Introductory Nuclear Theory* (W. B. Saunders Co., Philadelphia, 1966).

⁷ G. E. Brown *et al.*, *Nucl. Phys.* **22**, 1 (1961).

where c is the correlation coefficient to be determined, N is the normalization constant given by

$$N^2 = 1 + 2c \langle \mathbf{r}/r_0 \rangle + c^2 \langle r^2/r_0^2 \rangle, \quad (11)$$

and ψ_{LM}^0 is the single-particle coupled wave function of the two extra core particles (or holes) given by

$$\psi_{LM}^0 = N_0 \sum_m \text{c.g. coeff. } a^\dagger(nl m_i m_s m_t) a^\dagger(nl m'_i m'_s m'_t). \quad (12)$$

The creation operator $a^\dagger(nl m_i m_s m_t)$ [the corresponding annihilation operator $a(nl m_i m_s m_t)$ appears in the case of a hole] creates a particle in the nl harmonic-oscillator state with the orbital, spin, and isospin magnetic quantum numbers m_i , m_s , and m_t , respectively. The factor N_0 is unity unless the particles are in the same orbit, in which case it is $\frac{1}{2}\sqrt{2}$.⁸ Also, the two-body interaction operator V_{12} is written as

$$V_{12} = \frac{1}{2} \sum_k \phi_{k1}(1) \phi_{k2}(2) |V(12)| \phi_{k3}(1) \phi_{k4}(2) \times a_{k2}^\dagger a_{k1}^\dagger a_{k3} a_{k4}, \quad (13)$$

where the ϕ_k are single-particle wave function of H_0 belonging to the eigenvalue ϵ given by

$$H_0 \phi_k(i) = \epsilon \phi_k(i). \quad (14)$$

It can be easily shown that the wave function (10) is still an eigenfunction of L^2 , L_z , S^2 , and S_z . It is obviously an eigenfunction of S^2 and S_z , since the correlation factor does not alter the spin part in the wave function ψ^0 . For L^2 and L_z , we know that the angular dependence of \mathbf{r}_{ij} is the same as for $1/r_{ij}$, which is diagonal in L and L_z . To maintain the shell model, we may use the variational principle that the expectation value of the energy is a minimum when we have the correct wave function,⁹ given by (10), which is properly antisymmetrized. With this wave function the energy of the system is given by

$$\begin{aligned} W &= (1/N^2) \langle \psi_{LM}^0(1+c(r/r_0)) | H | \psi_{LM}^0(1+c(r/r_0)) \rangle \\ &= (1/N^2) \langle \psi_{LM}^0 | \tilde{H} | \psi_{LM}^0 \rangle, \end{aligned} \quad (15)$$

where

$$\tilde{H} = H + c((\mathbf{r}/r_0)H + H(\mathbf{r}/r_0)) + c^2(r/r_0)H(r/r_0). \quad (16)$$

The total energy W of the system is obtained by minimizing with respect to c , using the uncorrelated wave function (12) for a given L as a basis.

The correlated wave function can be modified to simulate the collective motion of the nuclear matter through its correlated part, and the resulting correction to the energy in the independent-particle model is of the order of several MeV. The wave function (10), in

⁸ D. M. Brink and G. R. Satchler, *Nuovo Cimento* **4**, 549 (1956).

⁹ K. A. Brueckner *et al.*, *Phys. Rev.* **118**, 1442 (1960).

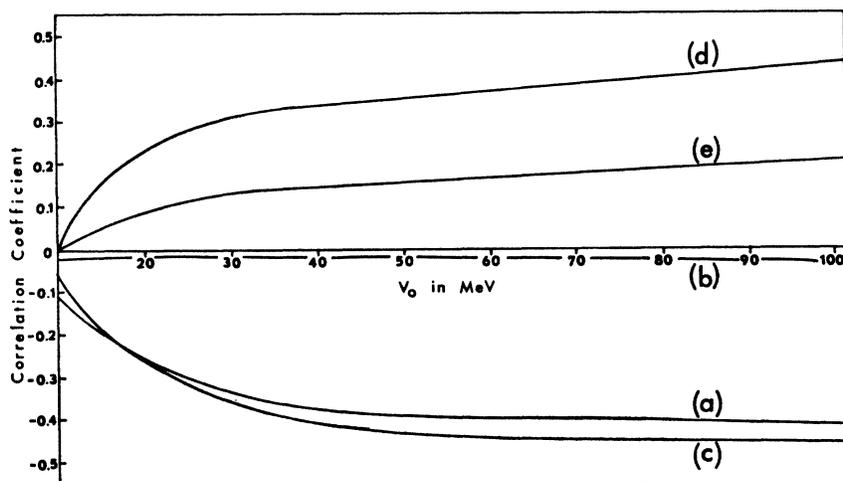


FIG. 1. Dependence of correlation coefficient c on V_0 for the values of λ given in Table I as determined by the potentials (18). The variation in c for a variation of 0.05 in λ is indistinguishable on this scale.

general, can be used for any number of particles, provided a suitable correlation between the extra core particles (or holes) is introduced. In this paper, however, the purpose is simply to describe a system with a doubly closed shell plus or minus two particles—in particular, the deuteron, for which this scheme with the wave function (10) is particularly convenient.

To demonstrate the correlation effect, the above technique has been applied to the deuteron by using the independent-particle wave function as the basis. This is the only nuclide which can be treated equally well by either of the two aforesaid descriptions, viz., the closed shell with two particles or two holes. From either approach we get the same result identically, as expected. (Perhaps for those who frown on the independent-particle description of deuteron, the α core as vacuum state with two $1s$ holes will be acceptable.) The energy of the deuteron, using the usual shell-model wave function for the state $(1s)^2$ is thus obtained from (15) as

$$W = 2\epsilon + \frac{1}{N^2} \left[\langle V_{12} \rangle + 2c \left\langle \frac{r}{r_0} V_{12} \right\rangle + c^2 \left\{ \frac{\hbar^2}{r_0^2 m} + \left\langle \frac{r^2}{r_0^2} V_{12} \right\rangle \right\} \right], \quad (17)$$

TABLE I. The correlation coefficient c for the optimum value of V_0 and λ . No information can be obtained about the singlet parameters in this way.

Potential	V_0 (MeV)	λ	c
a	90.0	0.9	-0.435
b	90.0	0.9	-0.067
c	90.0	0.9	-0.458
d	10.0	0.9	-0.012
e	10.0	0.9	-0.007

where m is the nucleon mass. Clearly, $W \rightarrow 2\epsilon$ as $V_{12} \rightarrow 0$, and hence c is also zero.

In order to evaluate the matrix elements in (17), the following form factors have been used in the potential given by (8) and (9):

$$(r_0/r) \exp(-r^2/r_0^2), \quad (18a)$$

$$\exp(-r^2/r_0^2), \quad (18b)$$

$$(r/r_0) \exp(-r^2/r_0^2), \quad (18c)$$

$$(r/r_0)^2 \exp(-r^2/r_0^2), \quad (18d)$$

$$[1 - 0.35(r/r_0)] \exp(-r^2/r_0^2). \quad (18e)$$

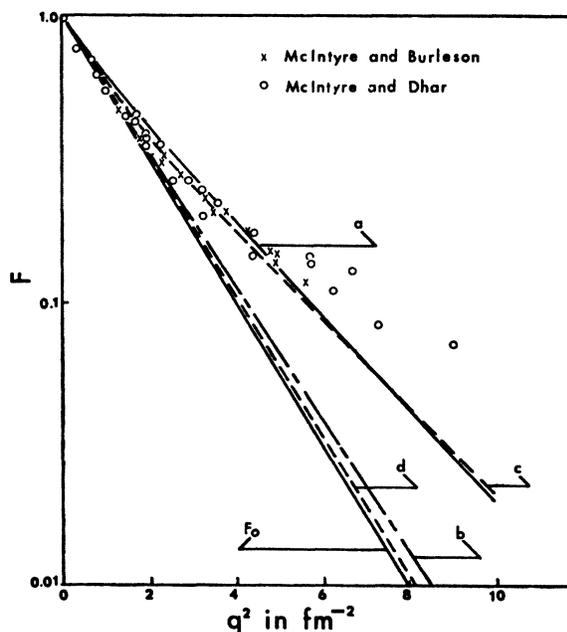


FIG. 2. Correlation effects on the elastic electron scattering form factor. F_0 is the form factor with uncorrelated wave function. Curve (e) coincides with F_0 on this scale.

The variational calculations have been performed for each of these potentials to determine the correlation coefficient c belonging to the minimum energy value. The parameters $\lambda = r_0\alpha$ (α being the harmonic-oscillator parameter) and V_0 were varied in the ranges $0.5 \leq \lambda \leq 1.0$ and $10 \text{ MeV} \leq V_0 \leq 100 \text{ MeV}$. For the optimum values of λ and V_0 , the coefficient c is given in Table I. The value of c is rather insensitive to large V_0 in each case. (See Fig. 1.)

The nuclear form factor, as obtained from the high-energy electron scattering by the deuteron, has also been calculated. Using the wave function (10), we obtain

$$F(q) = \frac{1}{N^2} \exp(-q^2/4\alpha^2) \left\{ 1 + \frac{c}{(2\pi)^{1/2}\lambda} \exp(q^2/8\alpha^2) \right. \\ \times \left[1 + \sum_{n=0}^{\infty} \frac{1}{2^n} ({}_1F_1(-n; \frac{3}{2}; q^2/8\alpha^2) \right. \\ \left. \left. + \frac{1}{2}(2n+3) {}_1F_1(-n-1; \frac{3}{2}; q^2/8\alpha^2) \right) \right] + \frac{c^2}{\lambda^2} \left(3 - \frac{q^2}{4\alpha^2} \right) \left. \right\}, \quad (19)$$

where q is the momentum transfer, and ${}_1F_1$ is the usual confluent hypergeometric function. Clearly with $c=0$ the form factor reduces to $\exp(-q^2/4\alpha^2)$. Comparison of our result with the experiments¹⁰ removes the degeneracy in selecting the potential parameters available from the energy calculations. Figure 2 shows a great deal of discrepancy between the uncorrelated form factor F_0 and the observed points. However, fairly good agreement with experimental data has been obtained by using correlated wave function with a suitable residual potential. $F(q)$ is enhanced for any nonzero value of c , negative or positive. We find that the negative c enhances $F(q)$ more consistently with the data. That is what one might have foreseen, too, by looking at the expression (19). Of course, the right size of c , belonging to minimum energy, can be obtained from any one of the potentials (18) or otherwise if no restriction is imposed on the adjustable parameters V_0 and λ . However, since we have already fixed r_0 , we cannot vary V_0 and λ indiscriminately. Consequently,

¹⁰ J. A. McIntyre and G. Bursleson, *Phys. Rev.* **112**, 2077 (1958); R. M. Littauer *et al.*, *Phys. Rev. Letters* **7**, 141 (1961); J. A. McIntyre and S. Dhar, *Phys. Rev.* **106**, 1074 (1957).

only certain reasonable values of V_0 and λ are admissible for correcting $F_0(q)$. Under these conditions, the correction in $F(q)$ "prefers" the potentials given by (18) in the order (c, a, b, d) (see Figs. 1 and 2), rather than (a, b, c, d) as one might have expected naively.

It is well known that the central and spin-dependent forces which give rise to the S state of the deuteron cannot fit the form factor. An admixture of the D state, which is due to the inclusion of tensor forces, improves the calculations.¹¹ We have been able to show in a qualitative manner that potential form factors as simple as (18) without any tensor force are capable of fitting the data (see Fig. 2) when nucleon correlation is taken into account. Such agreement cannot otherwise be obtained without including the tensor force itself, in quite involved calculations.^{11,12}

To sum up, we have shown that the beneficial effect of nucleon correlation in the nuclear wave function in independent-particle-mode the (IPM) is rather significant if a suitable residual potential is used. We have also shown that the role of the tensor force, at least in the case of the deuteron, can be simulated by a local central potential¹³ (residual) which is manifested through the enhancement of the nuclear form factor as obtained from the elastic electron scattering. Besides, it seems to be indicated by this analysis that elastic electron scattering can also be useful for the study of the N - N correlation.¹⁴ This kind of information is generally believed to be available from inelastic electron scattering, and that has been shown to be true in some cases.¹⁵

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¹¹ R. Hofstadter, *Ann. Rev. Nucl. Sci.* **7**, 231 (1957).

¹² G. Breit and M. L. Rustgi, *Phys. Rev.* **165**, 1075 (1968).

¹³ See, for example, T. T. S. Kuo and G. E. Brown, *Phys. Letters* **18**, 54 (1965); D. W. L. Sprung *et al.*, *ibid.* **21**, 538 (1966); M. R. Manning and A. B. Volkov, *ibid.* **26B**, 60 (1967); H. A. Bethe, *Phys. Rev.* **167**, 879 (1968); R. K. Bhaduri and C. S. Warke, *Phys. Rev. Letters* **20**, 1379 (1968).

¹⁴ F. C. Khanna, *Phys. Rev. Letters* **20**, 871 (1968).

¹⁵ K. Gottfried, *Ann. Phys. (N.Y.)* **21**, 29 (1963); **21**, 47 (1963); F. C. Khanna, *Nucl. Phys.* **A97**, 417 (1967).