Elastic and Inelastic Electron Scattering from Nuclear Multipole Moments in the First-Order Born Approximation

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We consider elastic and inelastic scattering of electrons from deformed light nuclei in the Born approximation. For the elastic scattering case, we develop the most general expression for the cross section in terms of the electric and magnetic multipole moments of the nuclear ground state and the corresponding form factors, which are expressed by the (spectroscopic) moments of charge and magnetization distributions. For the inelastic scattering case, the transition form factors are similarly expressed by transition multipole densities of charge, current, and magnetization in the most general fashion. We also develop a model of collective vibrations of deformed ground-state charge and magnetization densities which, in the case of charge vibrations, explains present data and predicts the filling in of inelastic diffraction minima; in the case of vibration of magnetization, corresponding transitions are shown to appear at large momentum transfers only, where they may be identified as such by the action of selection rules. In the Appendices, we consider relations between spectroscopic and intrinsic quantities, consider static deformed and Helm-type models, and discuss charge and magnetic radii. In addition, the previously derived generalized Helm model is rederived in a simple fashion.

I. INTRODUCTION

ELECTRON scattering has proven itself one of the most effective methods of studying the properties of the energy levels of atomic nuclei. Using elastic scattering of electrons, one can study the ground-state properties such as static distributions of charge and magnetization. For determining these quantities completely, electrons with energies of several hundreds of MeV must be used; electrons of $\lesssim 50$ MeV suffice for a determination of an rms "charge radius" and "magnetization radius" only. Using inelastic electron scattering, one may similarly determine "transition densities," corresponding to the initial and final nuclear state in question, for the three quantities in the nucleus that interact with the passing electron, namely, the distributions of charge, current, and magnetization.[1] Since the ground states of atomic nuclei in general possess no spherical symmetry, and since the transitions are best described in terms of multipolarities, a multipole expansion of the density operators is necessary. This expansion leads to the static multipole densities of the nuclear ground state in elastic scattering or of transition multipole densities in inelastic scattering (both defined as reduced matrix elements of the multipole density operators). The conventional static multipole moments (and the analogously defined static charge and magnetization density functions), given as usual by expectation values in the stretched configuration, may be expressed by the static multipole densities. Since in electron scattering the momentum transfer q is variable, the static moments are multiplied by form factors which are functions of q when one calculates the elastic electron scattering cross section in the Born approximation. Similarly, the inelastic Born cross section is expressed in terms of the "transition form factors" which contain the transition multipole densities. The development thus outlined is carried out in Secs. II and III. Since the Born approximation is used, the resulting expressions are valid for the light nuclei only, up to the s-d shell. The static or transition multipole moments of densities which will appear in the theoretical expressions of the form factors will then have to be calculated from a nuclear model or else they will have to be given by a phenomenological expression that produces a fit to the experimental cross section. The simplest example of the latter method is the fit of the elastic monopole form factor[2]

$$F(q)=(4\pi/q)\int_0^\infty r\rho(r)\,\sin qr dr \tag{1}$$

for a spherically symmetric ground-state charge density $\rho(r)$; but phenomenological fits may in principle be made for all (nonvanishing) multipoles of charge, current, and magnetization densities, both elastic and inelastic. Examples of fits will be shown for the elastic scattering in ^{10}B.

For the case of inelastic scattering, with the present accuracy of the experiments, an excessive amount of ambiguity would appear if completely arbitrary phenomenological expressions could be chosen for the multipole moments of the transition densities. In Secs. IV and V, therefore, we introduce a hydrodynamic collec-

* Supported in part by a grant of the National Science Foundation.
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[1] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Appendix B.
[2] R. Hofstadter, Rev. Mod. Phys. **28**, 214 (1956).

tive model which is described by incompressible and irrotational quantized vibrations of the nuclear ground-state charge and magnetization densities and thus relates the transition densities to the ground-state densities. This procedure was first used for a spherically symmetric charge distribution by Tassie.[3,4] One may even go one step further and describe the densities of a nonspherical nuclear ground state as a collective vibration of this kind, being "frozen" into the permanently deformed state. This model is outlined in Appendix B (together with a further simplification using a "Helm model"[5,6] distribution for the spherically symmetric density, Appendix C), but it turns out that such a model does not produce good fits to the experiments. The collective model for the transition densities, however, does give a reasonable result for charge-type transitions, as exemplified by the $3^+ \to 4^+$ $C2$ transition to the 6.02-MeV level in ^{10}B. The vibration of the quadrupole part of the charge density of the deformed ground state adds here a $C4$ part to the transition form factor which fills in the first diffraction minimum, just as the quadrupole part of the ground-state density itself fills in the diffraction minimum of the elastic form factor in a well-known fashion.[7,8] As to the model of collective vibrations of the ground-state magnetization density, it is found that the strong $M1$ transitions seen in 180° electron scattering (at relatively low momentum transfer[9,10]) can probably not be described this way (in spite of having been called "collective magnetic transitions" by Kurath[11]) but should rather be viewed as single-particle excitations, because the collective vibration of the dominant, spherically symmetric part of the ground-state magnetization density does not lead to an $M1$ form factor proportional to $q\langle j_0(qr)\rangle$ as observed but only to $q\langle j_2(qr)\rangle$. This absence of collective magnetic transitions at low values of q seems to be analogous to the absence of collective $M1$ photon transitions for even nuclei as observed by Lipas.[12] However, vibrations of the deformed part of the magnetization density may give a (smaller) contribution with a lower power of q, so that deformed nuclei may indeed show some collective ML excitation even at low q. More interestingly, collective magnetic excitations could appear at large values of momentum transfer and should thus be looked for in high-energy (several hundred MeV) electron scattering at 180°. They may in some cases be identified uniquely from the selection rules.

All the formalism of multipole densities presented in this paper has been developed in the laboratory system, so that only projections of intrinsic densities appear (e.g., one has vanishing projections of quadrupole moments for spin-0 or spin-$\frac{1}{2}$ ground states). A short discussion is given, however, of the relation between our vibrating collective model (with laboratory densities) and the well-known β and γ vibrations of intrinsic densities of even nuclei in Appendix A. Appendix D discusses charge and magnetization radii.

II. FORMALISM OF SCATTERING IN TERMS OF MULTIPOLE DENSITIES

In the following the general formalism of scattering from nuclear moments will be developed using the Born approximation. It should be noted, however, that many equations, in particular the definitions of the moments, will be valid in general rather than in the Born approximation, and all such equations will be indicated by a prime on the equation number.

The general Born-approximation differential cross section of electron scattering, elastic as well as inelastic, is given by[13,14]

$$\frac{d\sigma}{d\Omega}=4\pi Z^2\sigma_M\frac{\Delta^4}{q^4}\frac{1}{\hat{J}_i^2}\Bigg\{\sum_{L=0}|\langle J_f\|\tilde{M}_L(q)\|J_i\rangle|^2+\frac{q^2}{\Delta^2}\left(\frac{1}{2}+\frac{q^2}{\Delta^2}\tan^2\tfrac{1}{2}\vartheta\right)\times\sum_{L=1}\left[|\langle J_f\|\tilde{T}_L^e(q)\|J_i\rangle|^2+|\langle J_f\|\tilde{T}_L^m(q)\|J_i\rangle|^2\right]\Bigg\},\tag{2}$$

where the values of L are limited by the triangle condition $\Delta(J_iLJ_f)$. The incident and scattered electron momenta are $\mathbf{k}_1$ and $\mathbf{k}_2$ (we assume $k_1, k_2 \gg m_e$, the electron mass), and the electron scattering angle is $\vartheta = \sphericalangle(\mathbf{k}_1, \mathbf{k}_2)$. The momentum and energy transfer are

$$\mathbf{q}=\mathbf{k}_1-\mathbf{k}_2,\tag{3a'}$$

$$\omega=k_1-k_2\tag{3b'}$$

(ω being the energy of the nuclear level excited in electron scattering, since we neglect nuclear recoil; $\omega=0$ for elastic scattering); the squared four-momentum transfer is given by

$$\Delta^2=q^2-\omega^2=4k_1k_2\sin^2(\vartheta/2).\tag{3c'}$$

Z is the nuclear charge, and the Mott cross section of a

[3] L. J. Tassie, Australian J. Phys. **9**, 407 (1956); **9**, 481 (1956).
[4] See also A. M. Lane and E. D. Pendlebury, Nucl. Phys. **15**, 39 (1960).
[5] R. H. Helm, Phys. Rev. **104**, 1466 (1956).
[6] See also M. Rosen, R. Raphael, and H. Überall, Phys Rev. **163**, 927 (1967).
[7] See, e.g., B. W. Downs, D. G. Ravenhall, and D. R. Yennie, Phys. Rev. **106**, 1285 (1957).
[8] See also L. R. Suelzle, M. R. Yearian, and H. Crannell, Phys. Rev. **162**, 992 (1967).
[9] E. Spamer, Z. Physik **191**, 24 (1966), case of ^{10}B, ^{11}B.
[10] L. W. Fagg *et al.*, Bull. Am. Phys. Soc. **11**, 64 (1966); **12**, 664 (1967); Phys. Rev. **171**, 1250 (1968); **173**, 1103 (1968), and (to be published), case of ^{24}Mg, ^{25}Mg, ^{26}Mg.
[11] D. Kurath, Phys. Rev. **130**, 1525 (1963).
[12] P. O. Lipas, Phys. Letters **8**, 279 (1964).
[13] L. C. Maximon and D. B. Isabelle, Phys. Rev. **136**, B674 (1964), Eq. (28).
[14] T. De Forest and J. D. Walecka, Advan. Phys. **15**, 1 (1966).

fixed unit point charge is

$$\sigma_M = \frac{\alpha^2}{4k_1^2}\frac{\cos^2(\vartheta/2)}{\sin^4(\vartheta/2)}, \quad (4)$$

with $\alpha = 1/137$. The spins and spin projections of the initial and final nuclear states are J_i, M_i and J_f, M_f, respectively; our notation is $\hat{J}_i = (2J_i+1)^{1/2}$. The reduced matrix elements, or "form factors," are the matrix elements of the Coulomb, the transverse electric, and the transverse magnetic multipole operators (peculiar to the Born approximation):

$$eZ\tilde{M}_{LM}(q) = i^L \int j_L(qr)\, Y_{LM}(\hat{r})\, \rho^{\text{op}}(\mathbf{r})\, d^3r, \quad (5a)$$

$$eZ\tilde{T}_{LM}{}^{e}(q) = i^L \int d^3r [q^{-1}\mathbf{j}_c{}^{\text{op}}(\mathbf{r})\cdot \nabla\times j_L(qr)\mathbf{Y}_{LL}{}^{M}(\hat{r}) + q j_L(qr)\, \boldsymbol{\mu}_s{}^{\text{op}}(\mathbf{r})\cdot \mathbf{Y}_{LL}{}^{M}(\hat{r})], \quad (5b)$$

$$eZ\tilde{T}_{LM}{}^{m}(q) = i^L \int d^3r [j_L(qr)\, \mathbf{j}_c{}^{\text{op}}(\mathbf{r})\cdot \mathbf{Y}_{LL}{}^{M}(\hat{r}) + \boldsymbol{\mu}_s{}^{\text{op}}(\mathbf{r})\cdot \nabla\times j_L(qr)\mathbf{Y}_{LL}{}^{M}(\hat{r})], \quad (5c)$$

where unit vectors are designated by $\hat{r} = \mathbf{r}/r$. Note that we designate operators (operating on the nucleon coordinates in the nuclear wave function) by a tilde, or else by the superscript "op." The nuclear charge, (convection) current, and (spin) magnetization density operators are given by

$$\rho^{\text{op}}(\mathbf{r}) = e \sum_{i=1}^{A} \tfrac{1}{2}[1+\tau_3{}^{(i)}]\delta(\mathbf{r}-\mathbf{r}_i) \quad (6a')$$

(normalized to Ze),

$$\mathbf{j}_c{}^{\text{op}}(\mathbf{r}) = e \sum_{i=1}^{A} \tfrac{1}{2}[1+\tau_3{}^{(i)}]\delta(\mathbf{r}-\mathbf{r}_i)(\mathbf{p}_i/m) \quad (6b')$$

(m being the nucleon mass), and

$$\boldsymbol{\mu}_s{}^{\text{op}}(\mathbf{r}) = (e/2m)\{\mu_p \sum_{i=1}^{A} \tfrac{1}{2}[1+\tau_3{}^{(i)}]\delta(\mathbf{r}-\mathbf{r}_i)\boldsymbol{\sigma}^{(i)} + \mu_n \sum_{i=1}^{A} \tfrac{1}{2}[1-\tau_3{}^{(i)}]\delta(\mathbf{r}-\mathbf{r}_i)\boldsymbol{\sigma}^{(i)}\} \quad (6c')$$

(with the nucleon magnetic moments $\mu_p = 2.78$ and $\mu_n = -1.91$). The vector spherical harmonics contained in Eqs. (5) are

$$\mathbf{Y}_{LL'}{}^{M}(\hat{r}) = \sum_{mm'} (L'm, 1m' \mid LM)\, Y_{L'm}(\hat{r})\, \mathbf{e}_{m'}. \quad (7a')$$

Using the property[14]

$$(\nabla^2+q^2) j_L(qr)\mathbf{Y}_{LL}{}^{M}(\hat{r}) = 0, \quad (7b')$$

one may also write

$$eZ\tilde{T}_{LM}{}^{e}(q) = i^L q^{-1} \int d^3r\, \mathbf{J}^{\text{op}}(\mathbf{r})\cdot \nabla\times j_L(qr)\mathbf{Y}_{LL}{}^{M}(\hat{r}), \quad (8a)$$

$$eZ\tilde{T}_{LM}{}^{m}(q) = i^L \int d^3r\, \mathbf{J}^{\text{op}}(\mathbf{r})\cdot j_L(qr)\mathbf{Y}_{LL}{}^{M}(\hat{r}), \quad (8b)$$

with a total current-density operator

$$\mathbf{J}^{\text{op}}(\mathbf{r}) = \mathbf{j}_c{}^{\text{op}}(\mathbf{r}) + \nabla\times \boldsymbol{\mu}_s{}^{\text{op}}(\mathbf{r}). \quad (8c')$$

Reduced matrix elements are defined by the Wigner-Eckart theorem:

$$\langle J_f M_f \mid \tilde{O}_{LM} \mid J_i M_i\rangle = \hat{J}_f{}^{-1}(J_i M_i, LM \mid J_f M_f) \times \langle J_f \| \tilde{O}_L \| J_i\rangle. \quad (9')$$

The reduced matrix elements of the Born multipole operators which appear in Eq. (2) therefore contain the matrix elements of the (charge, current, and magnetization) density operators. Before the reduced matrix elements of the latter can be written down, however, the density operators must be expanded in terms of multipole density operators. This is necessary to incorporate all multipole aspects of the nuclear densities and transitions: for example, the moments of a permanently deformed nuclear ground state. Introducing the vector symbol $\mathbf{w}^{\text{op}}(\mathbf{r})$ to stand for $\mathbf{j}_c{}^{\text{op}}(\mathbf{r})$ or $\boldsymbol{\mu}_s{}^{\text{op}}(\mathbf{r})$, as the case may be, we expand the density operators:

$$\rho^{\text{op}}(\mathbf{r}) = \sum_{lm} \tilde{\rho}_{lm}(r)\, Y_{lm}{}^{*}(\hat{r}), \quad (10a')$$

$$\mathbf{w}^{\text{op}}(\mathbf{r}) = \sum_{ll'm} \tilde{w}_{ll'm}(r)\, \mathbf{Y}_{ll'}{}^{m*}(\hat{r}). \quad (10b')$$

The multipole density operators thus introduced are, of course, given by the inverse equations

$$\tilde{\rho}_{lm}(r) = \int Y_{lm}(\hat{r})\, \rho^{\text{op}}(\mathbf{r})\, d\hat{r}, \quad (11a')$$

$$\tilde{w}_{ll'm}(r) = \int \mathbf{Y}_{ll'}{}^{m}(\hat{r})\cdot \mathbf{w}^{\text{op}}(\mathbf{r})\, d\hat{r} = \sum_{\mu\nu} (l'\mu, 1\nu \mid lm) \int Y_{l'\mu}(\hat{r})\, w_\nu{}^{\text{op}}(\mathbf{r})\, d\hat{r}, \quad (11b')$$

where $w_\nu{}^{\text{op}}(\mathbf{r}) \equiv \mathbf{e}_\nu\cdot \mathbf{w}^{\text{op}}(\mathbf{r})$ is the νth spherical component of $\mathbf{w}^{\text{op}}(\mathbf{r})$. One may introduce reduced matrix elements of these multipole density operators,

$$\langle J_f \| \tilde{\rho}_l(r) \| J_i\rangle \equiv \rho_l{}^{if}(r), \quad (12a')$$

$$\langle J_f \| \tilde{w}_{ll'}(r) \| J_i\rangle \equiv w_{ll'}{}^{if}(r), \quad (12b')$$

and refer to these quantities as the "transition multipole densities"; they are now functions, not operators, and are the basic quantities of our theory. (The above formalism includes the ground-state densities; but for these, special definitions and notations will be introduced in Sec. III.) The transition matrix elements of the density operators needed in Eq. (2), via Eqs. (5), are thus given by[15]

$$\langle J_fM_f \mid \rho^{\text{op}}(\mathbf{r}) \mid J_iM_i\rangle = \hat{J}_f^{-1} \sum_{lm} (J_iM_i, lm \mid J_fM_f) \times \rho_l^{if}(r) Y_{lm}^*(\hat{r}), \quad (13a')$$

$$\langle J_fM_f \mid \mathbf{w}^{\text{op}}(\mathbf{r}) \mid J_iM_i\rangle = \hat{J}_f^{-1} \sum_{ll'm} (J_iM_i, lm \mid J_fM_f) \times w_{ll'}^{if}(r) \mathbf{Y}_{ll'}^{m*}(\hat{r}), \quad (13b')$$

using Eq. (9).

We may separate the convection-current-density operator into two parts:

$$\mathbf{j}_c^{\text{op}}(\mathbf{r}) = \mathbf{j}_0^{\text{op}}(\mathbf{r}) + \mathbf{j}^{\text{op}}(\mathbf{r}), \quad (14a')$$

which, by using Eq. (10b) are given by

$$\mathbf{j}_0^{\text{op}}(\mathbf{r}) = \sum_{lm} \tilde{j}_{llm}(r) \mathbf{Y}_{ll}^{m*}(\hat{r}), \quad (14b')$$

$$\mathbf{j}^{\text{op}}(\mathbf{r}) = \sum_{lm} \sum_{l'=l\pm1} \tilde{j}_{ll'm}(r) \mathbf{Y}_{ll'}^{m*}(\hat{r}), \quad (14c')$$

the former term having the property[16]

$$\nabla \cdot \mathbf{j}_0^{\text{op}}(\mathbf{r}) = 0. \quad (14d')$$

Therefore, we may set

$$\mathbf{j}_0^{\text{op}}(\mathbf{r}) = \nabla \times \boldsymbol{\mu}_0^{\text{op}}(\mathbf{r}), \quad (15a')$$

thereby introducing an "orbital" magnetization density operator $\boldsymbol{\mu}_0^{\text{op}}(\mathbf{r})$, which represents the contribution of closed-loop parts of the convection current to the total magnetization density. Adding this to the spin magnetization gives us the total magnetization density operator:

$$\boldsymbol{\mu}^{\text{op}}(\mathbf{r}) = \boldsymbol{\mu}_0^{\text{op}}(\mathbf{r}) + \boldsymbol{\mu}_s^{\text{op}}(\mathbf{r}). \quad (15b')$$

The total current operator, Eq. (8c), may then be written as

$$\mathbf{J}^{\text{op}}(\mathbf{r}) = \mathbf{j}^{\text{op}}(\mathbf{r}) + \nabla \times \boldsymbol{\mu}^{\text{op}}(\mathbf{r}); \quad (15c')$$

its new constituents $\mathbf{j}^{\text{op}}$ and $\boldsymbol{\mu}^{\text{op}}$ are then multipole-expanded according to Eq. (14c) for $\mathbf{j}^{\text{op}}$ (i.e., they contain no $l=l'$ multipole component) and in the form

$$\boldsymbol{\mu}^{\text{op}}(\mathbf{r}) = \sum_{ll'm} \tilde{\mu}_{ll'm}(r) \mathbf{Y}_{ll'}^{m*}(\hat{r}) \quad (16')$$

for $\boldsymbol{\mu}^{\text{op}}$. Equations (15c) and (8c) show that Eqs. (5b) and (5c) may be rewritten in the same form but with $\mathbf{j}_c^{\text{op}}$, $\boldsymbol{\mu}_s^{\text{op}}$ replaced by $\mathbf{j}^{\text{op}}$, $\boldsymbol{\mu}^{\text{op}}$, respectively.

While the divergence of $\mathbf{j}_0^{\text{op}}(\mathbf{r})$ vanishes from Eq. (14d), the divergence of $\mathbf{j}^{\text{op}}(\mathbf{r})$ may be related to $\rho^{\text{op}}(\mathbf{r})$ by the use of the continuity equation[14]; indeed, $\mathbf{j}^{\text{op}}(\mathbf{r})$ is the only part of $\mathbf{j}_c^{\text{op}}(\mathbf{r})$ which enters in the continuity equation

$$\nabla \cdot \mathbf{j}^{\text{op}}(\mathbf{r}) = -i[\tilde{H}, \rho^{\text{op}}(\mathbf{r})], \quad (17a')$$

in which $\tilde{H}$ is the nuclear Hamiltonian which operates on the nucleon coordinates only. If the multipole expansions are inserted, we find

$$l^{1/2} r^{l-1} (d/dr)[r^{1-l} \tilde{j}_{ll-1m}(r)] - (l+1)^{1/2} r^{-l-2} (d/dr) \times [r^{l+2} \tilde{j}_{ll+1m}(r)] = -i\hat{l}[\tilde{H}, \tilde{\rho}_{lm}(r)]. \quad (17b')$$

We also may insert now the multipole expansions, Eqs. (10a), (14c), and (16), into the form factors of Eqs. (5). Taking matrix elements, then, and using the definitions of the transition multipole densities, Eqs. (12), the reduced matrix elements (or "transition form factors") may be expressed in terms of the latter as follows:

$$eZ\langle J_f \| \tilde{M}_L(q) \| J_i\rangle = i^L \int r^2 j_L(qr) \rho_L^{if}(r) dr, \quad (18a)$$

$$eZ\langle J_f \| \tilde{T}_L^e(q) \| J_i\rangle = i^{L+1} \hat{L}^{-1} (L+1)^{1/2} \times \int r^2 j_{L-1}(qr) j_{LL-1}^{if}(r) dr - i^{L+1} \hat{L}^{-1} L^{1/2} \int r^2 j_{L+1}(qr) j_{LL+1}^{if}(r) dr + i^L q \int r^2 j_L(qr) \mu_{LL}^{if}(r) dr, \quad (18b)$$

$$eZ\langle J_f \| \tilde{T}_L^m(q) \| J_i\rangle = i^{L+1} q \hat{L}^{-1} (L+1)^{1/2} \times \int r^2 j_{L-1}(qr) \mu_{LL-1}^{if}(r) dr - i^{L+1} q \hat{L}^{-1} L^{1/2} \times \int r^2 j_{L+1}(qr) \mu_{LL+1}^{if}(r) dr. \quad (18c)$$

From parity and time-reversal invariance, these reduced matrix elements are real.[14] Note that there is no explicit contribution to T_L^m from the current $\mathbf{j}$; but this is only apparent since the appropriate j term has been incorporated in $\boldsymbol{\mu}$ by Eqs. (15a) and (15b). The corresponding contribution of $\boldsymbol{\mu}$ to T_L^e is present and causes the so-called electric spin-flip transitions.

In a similar way, the continuity equation (17b) is expressed as

$$L^{1/2} r^{L-1} (d/dr)[r^{1-L} j_{LL-1}^{if}(r)] - (L+1)^{1/2} r^{-L-2} (d/dr) \times [r^{L+2} j_{LL+1}^{if}(r)] = i\omega \hat{L} \rho_L^{if}(r), \quad (19')$$

i.e., in terms of the transition multipole densities. Using this equation and partial integration, it is easy to show that in the limit $q \to \omega$, where one may approximate

[15] D. S. Onley, J. T. Teynolds, and L. E. Wright, Phys. Rev. **134**, B945 (1964).

[16] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

$j_L(\omega r)\approx(\omega r)^L/(2L+1)!!$, the matrix elements are related by

$$\langle J_f \| \tilde{T}_L^e(\omega) \| J_i\rangle \approx [(L+1)/L]^{1/2}\langle J_f \| \tilde{M}_L(\omega) \| J_i\rangle, \tag{20}$$

as is well known[6] (Siegert theorem).

The preceding formalism is valid for elastic electron scattering processes as well; in that special case, however, it may be developed further by the introduction of the static nuclear multipole moments. This is done in Sec. III.

III. ELASTIC SCATTERING FROM NUCLEAR MOMENTS

For elastic scattering one has $\omega=0$ and $J_f=J_i$, which we shall designate in common by J. We shall introduce as the basic quantities of elastic scattering the "(static) multipole densities" of the nuclear ground state analogously to Eqs. (12) but without the superscripts:

$$\langle J \| \tilde{\rho}_l(r) \| J\rangle=\rho_l(r), \tag{21a$'$}$$

$$\langle J \| \tilde{\mu}_{ll'}(r) \| J\rangle=\mu_{ll'}(r). \tag{21b$'$}$$

Only $\mu_{ll\pm1}(r)$ is needed here, not $j_{ll'}(r)$ or $\mu_{ll}(r)$, since the latter vanish; the only place where they appear is the matrix element of $\tilde{T}_L^e$, Eq. (18b). It has been shown, however,[17] that this matrix element vanishes in elastic scattering due to parity and time-reversal invariance of the interaction. The same argument also shows that only even or odd values of L occur in the matrix elements of $\tilde{M}_L$ or $\tilde{T}_L^m$, respectively, so that only even charge and odd magnetic multipole moments enter in the nuclear ground-state distribution.

Conventionally,[18] these static multipole moments are introduced by defining the operators

$$Q_{LM}{}^{\mathrm{op}}=(4\pi)^{1/2}\hat{L}^{-1}\int \rho^{\mathrm{op}}(\mathbf{r})r^L Y_{LM}(\hat{r})d^3r, \tag{22a$'$}$$

$$M_{LM}{}^{\mathrm{op}}=-(4\pi)^{1/2}\hat{L}^{-1}\int [\nabla\cdot\boldsymbol{\mu}^{\mathrm{op}}(\mathbf{r})]r^L Y_{LM}(\hat{r})d^3r, \tag{22b$'$}$$

the electric and magnetic multipole-moment operators. The latter contains $\boldsymbol{\mu}^{\mathrm{op}}=\boldsymbol{\mu}_s{}^{\mathrm{op}}+\boldsymbol{\mu}_0{}^{\mathrm{op}}$, i.e., the spin and the orbital magnetization. The nuclear charge and magnetic moments themselves are defined[19] as the expectation values in the stretched configuration:

$$eQ_L=\langle JJ | Q_{L0}{}^{\mathrm{op}} | JJ\rangle, \tag{23a$'$}$$

$$(e/2m)M_L=\langle JJ | M_{L0}{}^{\mathrm{op}} | JJ\rangle. \tag{23b$'$}$$

Parity immediately shows that Q_L is nonvanishing only for even L and that M_L is nonvanishing only for odd L.

In order to relate this to the usual lowest nuclear moments Ze (charge monopole), Q (charge quadrupole), or μ (magnetic dipole), given by[1]

$$\int \rho(\mathbf{r})d^3r=Ze, \tag{24a$'$}$$

$$\int \rho(\mathbf{r})r^2(3\cos^2\vartheta-1)d^3r=eQ, \tag{24b$'$}$$

$$\int \mu_Z(\mathbf{r})d^3r=(e/2m)\mu, \tag{24c$'$}$$

we have to clarify how the charge density function $\rho(\mathbf{r})$ and magnetization density function $\boldsymbol{\mu}(\mathbf{r})$ of Eqs. (24) have to be defined in relation to Sec. II. To be consistent with the conventions of Eqs. (23) and (24), we must choose the static charge and magnetization density functions as the expectation values of the operators in the stretched configuration (i.e., with the nuclear spin as close as possible to the z axis):

$$\rho(\mathbf{r})=\langle JJ | \rho^{\mathrm{op}}(\mathbf{r}) | JJ\rangle, \tag{25a$'$}$$

$$\boldsymbol{\mu}(\mathbf{r})=\langle JJ | \boldsymbol{\mu}^{\mathrm{op}}(\mathbf{r}) | JJ\rangle. \tag{25b$'$}$$

[Only the z component of $\boldsymbol{\mu}(\mathbf{r})$ exists.] Both ρ and $\boldsymbol{\mu}$ are even functions of $\mathbf{r}$, from parity. With this choice, it then follows from Eqs. (23) and (24) that

$$Q_0=Z, \tag{26a$'$}$$

$$Q_2=\tfrac{1}{2}Q, \tag{26b$'$}$$

$$M_1=\mu, \tag{26c$'$}$$

in agreement with the conventions.[1,19]

The multipole expansion of the static densities given by Eqs. (25) may be given in terms of the multipole densities of Eqs. (21). We simply use the expansion of the operators, Eqs. (10a) and (16), and the Wigner-Eckart theorem, and obtain

$$\rho(\mathbf{r})=\hat{J}^{-1}\sum_{L=0,\mathrm{even}}^{\leq 2J}(JJ,L0 | JJ)\rho_L(r)Y_{L0}{}^*(\hat{r}), \tag{27a$'$}$$

$$\boldsymbol{\mu}(\mathbf{r})=\hat{J}^{-1}\sum_{L=1,\mathrm{odd}}^{\leq 2J}\sum_{L'=L\pm1}(JJ,L0 | JJ)\mu_{LL'}(r)\mathbf{Y}_{LL'}{}^{0*}(\hat{r}). \tag{27b$'$}$$

The restrictions on the sums come about from the conditions that ρ, $\boldsymbol{\mu}$ be even functions of $\mathbf{r}$ and that Eq. (18b) vanish for elastic scattering (this rules out the terms μ_{LL}). The correspondingly defined current $\mathbf{j}(\mathbf{r})\equiv\langle JJ | \mathbf{j}^{\mathrm{op}}(r) | JJ\rangle$ vanishes, since it may be expressed, analogously to Eqs. (27), by $j_{LL'}(r)$, which are zero for elastic scattering as mentioned before. For this reason, we may use $\boldsymbol{\mu}^{\mathrm{op}}(\mathbf{r})$ in the definition, Eq. (22b), instead of [in terms of density functions *à la* Eq. (25b)] the customary $\boldsymbol{\mu}_{\mathrm{tot}}(\mathbf{r})\equiv\boldsymbol{\mu}_c(\mathbf{r})+\boldsymbol{\mu}_s(\mathbf{r})$, where $\boldsymbol{\mu}_c$ is given by the expression, appropriate to elastic scattering,

[17] R. H. Pratt, J. D. Walecka, and T. A. Griffy, Nucl. Phys. **64**, 677 (1965).

[18] See, e.g., S. De Benedetti, *Nuclear Interactions* (John Wiley & Sons, Inc., New York, 1964), pp. 31, 32.

[19] C. Schwartz, Phys. Rev. **97**, 380 (1955).

$\mathbf{j}_c(\mathbf{r}) = \nabla\times\boldsymbol{\mu}_c(\mathbf{r})$. One may show using[20]

$$\int \nabla\cdot\boldsymbol{\mu}_c(\mathbf{r})r^L Y_{LM}d^3r = (L+1)^{-1}\int r^L Y_{LM}\nabla\cdot(\mathbf{r}\times\mathbf{j}_c) \tag{28'}$$

that the use of either $\boldsymbol{\mu}(\mathbf{r})$ or $\boldsymbol{\mu}_{\text{tot}}(\mathbf{r})$ in Eq. (22b) gives the same result, since on the right-hand side of Eq. (28) one has (in the elastic case) $\mathbf{j}_c\equiv\mathbf{j}_0$.

For the special case of spherically symmetric distributions, where $\rho(\mathbf{r})\to\rho(r)$ or $\boldsymbol{\mu}(\mathbf{r})\to\boldsymbol{\mu}(r)$, the multipole densities reduce to

$$\rho_L(r) = (4\pi)^{1/2}\hat{J}\rho(r)\delta_{L0}, \tag{29a'}$$

$$\mu_{LL'}(r) = (4\pi)^{1/2}\hat{J}(JJ,10\,|\,JJ)^{-1}\mu_z(r)\delta_{L1}\delta_{L'0}. \tag{29b'}$$

Finally, from the defining equations (23), and the definition of $\rho(\mathbf{r})$, $\boldsymbol{\mu}(\mathbf{r})$ and their multipole expansions, the multipole moments Q_L and M_L may be expressed in terms of the multipole densities:

$$eQ_L = (4\pi)^{-1/2}\hat{L}^{-1}\hat{J}^{-1}(JJ,L0\,|\,JJ)\int r^L\rho_L(r)d^3r, \tag{30a'}$$

$$(e/2m)M_L = (L/4\pi)^{1/2}\hat{J}^{-1}(JJ,L0\,|\,JJ)\times\int r^{L-1}\mu_{LL-1}(r)d^3r. \tag{30b'}$$

The even character of $\rho(\mathbf{r})$, $\boldsymbol{\mu}(\mathbf{r})$ in Eqs. (27) shows again that only Q_L with even L and M_L with odd L exist. The derivation of Eq. (30a) is straightforward, whereas Eq. (30b) is a special case of the magnetic form factor, whose derivation will be described below.

Note the appearance of the projection factor

$$(JJ,L0\,|\,JJ) = \hat{J}(2J)!\left[(2J-L)!(2J+L+1)!\right]^{-1/2}, \tag{31a'}$$

which limits the multipolarities to $L\leq 2J$. For the case of an electric quadrupole, it becomes

$$(JJ,20\,|\,JJ) = \left(\frac{J}{J+1}\frac{2J-1}{2J+3}\right)^{1/2}, \tag{31b'}$$

giving the familiar result that nuclei of spin $J=0$ or $\frac{1}{2}$ may not possess an electric quadrupole moment. This illustrates the fact that we deal with the spectroscopic rather than with the intrinsic moments, since our formalism is developed in the laboratory system. Although these nuclei may have nonvanishing intrinsic quadrupole moments (or, more generally, intrinsic multipole moments, or expansion terms of the intrinsic densities, that are not limited by $L\leq 2J$), these do not show up in the laboratory, where the quantum-mechanical impossibility of lining up the body axis exactly with the space-fixed z axis, and the precession around the latter, wash them out, even though the spectroscopic moments and densities were defined as the expectation values in the stretched configuration which makes the lineup optimal.

[20] See Ref. 18, p. 32, footnote.

Next, we wish to express the elastic cross section, Eq. (2), in terms of the static multipole moments Q_L, M_L and the corresponding form factors $F_L^C(q)$, $F_L^M(q)$ normalized to unity for $q\to 0$; and finally, we wish to express the latter in terms of the multipole densities $\rho_L(r)$, $\mu_{LL'}(r)$. We note that for elastic scattering, $\Delta^2=q^2$ and

$$q=2k_1\sin(\vartheta/2). \tag{32'}$$

First, introducing the quantity

$$\tilde{T}_{LM}^C(q) = i^L q^{-L}(2L+1)!!\int j_L(qr)Y_{LM}(\hat{r})\rho^{\text{op}}(\mathbf{r})d^3r = q^{-L}(2L+1)!!eZ\tilde{M}_{LM}(q), \tag{33a}$$

we find that

$$\tilde{T}_{LM}^C(0) = i^L(4\pi)^{-1/2}\hat{L}Q_{LM}^{\text{op}} \tag{33b'}$$

for the limit of zero momentum transfer. If further, we define an elastic Coulomb form factor by

$$F_L^C(q) = \langle J\,||\,\tilde{T}_L^C(q)\,||\,J\rangle/\langle J\,||\,\tilde{T}_L^C(0)\,||\,J\rangle, \tag{34a}$$

which has the property

$$F_L^C(0)=1, \tag{34b}$$

then we find, using Eq. (23a) and the Wigner-Eckart theorem,

$$eQ_LF_L^C(q) = (-1)^{L/2}(4\pi)^{1/2}\hat{L}^{-1}\hat{J}^{-1}(JJ,L0\,|\,JJ)\times\langle J\,||\,\tilde{T}_L^C(q)\,||\,J\rangle. \tag{34c}$$

A similar procedure will be used for the magnetic form factor. We introduce the quantity

$$\tilde{T}_{LM}^M(q) = q^{-L}[L(L+1)]^{1/2}(2L+1)!!eZ\tilde{T}_{LM}^m, \tag{35a}$$

and find the limit, using Eq. (8b),

$$\tilde{T}_{LM}^M(0) = (-1)^L\int d^3r\mathbf{J}^{\text{op}}\cdot\mathbf{L}[r^LY_{LM}(\hat{r})], \tag{35b'}$$

since

$$\mathbf{L}Y_{LM}(\hat{r}) = [L(L+1)]^{1/2}\mathbf{Y}_{LL}^M(\hat{r}). \tag{35c'}$$

Equation (35b) is to be related to the magnetic multipole-moment operator M_{LM}^{op} of Eq. (22b). In the vector identity of Eq. (28) we replace $\mathbf{j}_c$ by $\mathbf{J}^{\text{op}}$ on the right-hand side, and on the left-hand side we may replace $\boldsymbol{\mu}_c$ by $\boldsymbol{\mu}^{\text{op}}$, since, as was stated before, the difference vanishes if expectation values are taken. The left-hand side then gives $-i^L(4\pi)^{-1/2}\hat{L}(L+1)M_{LM}^{\text{op}}$, whereas the right-hand side may be transformed by partial integrations into $i\tilde{T}_{LM}^M(0)$, so that finally,

$$\tilde{T}_{LM}^M(0) = i^{L+1}(L+1)(4\pi)^{-1/2}\hat{L}M_{LM}^{\text{op}}. \tag{36d'}$$

Now we define the magnetic form factor by

$$F_L{}^M(q)=\langle J\,\|\,\tilde{T}_L{}^M(q)\,\|\,J\rangle/\langle J\,\|\,\tilde{T}_L{}^M(0)\,\|\,J\rangle, \tag{37a}$$

with the property

$$F_L{}^M(0)=1, \tag{37b}$$

and from Eq. (23b) we get

$$(e/2m)M_LF_L{}^M(q)=(-1)^{(L+1)/2}(4\pi)^{1/2}(L+1)^{-1}\hat{L}^{-1}\hat{J}^{-1}\times(JJ,L0\,|\,JJ)\langle J\,\|\,\tilde{T}_L{}^M(q)\,\|\,J\rangle. \tag{37c}$$

The differential cross section, Eq. (2), specialized to elastic scattering and after use of Eqs. (33a), (35a), (34c), (37c), and (31a), becomes

$$\frac{d\sigma}{\Omega d}=\sigma_M\Bigg\{\sum_{L=0,\mathrm{even}}^{\leq 2J}q^{2L}Q_L{}^2\,|\,F_L{}^C(q)\,|^2\frac{2L+1}{[(2L+1)!!]^2}\times\frac{(2J-L)!(2J+L+1)!}{(2J)!(2J+1)!}+(\tfrac{1}{2}+\tan^2\tfrac{1}{2}\vartheta)\sum_{L=1,\mathrm{odd}}^{\leq 2J}q^{2L}\times\frac{M_L{}^2}{4m^2}\,|\,F_L{}^M(q)\,|^2\frac{L+1}{L}\frac{2L+1}{[(2L+1)!!]^2}\times\frac{(2J-L)!(2J+L+1)!}{(2J)!(2J+1)!}\Bigg\}. \tag{38a}$$

This is the generalization of the differential cross section given by Griffy and Yu,[21] and we agree with their Eqs. (3) as compared to our Eqs. (34c) and (37c) except for our factor $i^{L+1}\hat{J}^{-1}$, which comes from a different definition of the reduced matrix elements, and for our factor $(L+1)^{-1}$ in Eq. (37c).

We would finally like to express the quantities $eQ_LF_L{}^C(q)$ and $(e/2m)M_LF_L{}^M(q)$ in terms of the static multipole densities $\rho_L(r)$ and $\mu_{LL\pm1}(r)$, Eqs. (21). We use the definitions, Eqs. (34c) and (37c), and the expression given by Eq. (8b) for $\tilde{T}_{LM}{}^m(q)$, with Eq. (15c) for $\mathbf{J}^{\mathrm{op}}(\mathbf{r})$. The matrix elements of the operators in the stretched configuration $|\,JJ\rangle$ are expanded as in Eqs. (13), and the vector identities of Edmonds [Ref. 16, p. 84] are employed. We find:

$$eQ_LF_L{}^C(q)=(4\pi)^{1/2}q^{-L}(2L+1)!!\hat{L}^{-1}\hat{J}^{-1}\times(JJ,L0\,|\,JJ)\int r^2j_L(qr)\rho_L(r)dr, \tag{38b}$$

$$(e/2m)M_LF_L{}^M(q)=[L/(L+1)]^{1/2}(4\pi)^{1/2}q^{1-L}\times(2L+1)!!\hat{L}^{-1}\hat{J}^{-1}(JJ,L0\,|\,JJ)\Bigg\{(L+1)^{1/2}\hat{L}^{-1}\times\int r^2j_{L-1}(qr)\mu_{LL-1}(r)dr-L^{1/2}\hat{L}^{-1}\times\int r^2j_{L+1}(qr)\mu_{LL+1}(r)dr\Bigg\}. \tag{38c}$$

[21] T. A. Griffy and D. U. L. Yu, Phys. Rev. **139**, B880 (1965).

Equations (38) are the elastic special case of the general equations (2) and (18). Equations (30) expressing the static nuclear multipole moments in terms of the multipole densities are seen to be just the limit $q\to0$ of Eqs. (38b) and (38c).

Equations (38b) and (38c) are the most general case of the nuclear ground-state form factors [of which Eq. (1) is the simplest case, for a spherically symmetric nucleus and considering the charge distribution only], expressed by the static multipole densities $\rho_L(r)$, $\mu_{LL\pm1}(r)$ which describe the nuclear ground state. The conventional procedure, which can still be used, has been to choose reasonable analytic expressions for the multipole densities [especially the monopole charge density $\rho_0(r)$, see Eq. (29a)], insert the form factors calculated with these expressions into $d\sigma/d\Omega$, and try to obtain a fit with the experimental differential cross section as a function of q. For the higher multipole contributions, the procedure becomes necessarily successively more uncertain. We shall discuss examples of this method.

In principle, inelastic scattering may be treated in exactly the same way. In other words, the inelastic cross section to an excited nuclear level, Eq. (2), may be measured as a function of q, analytic forms may be chosen for the transition densities $\rho_L{}^{if}(r)$, $j_{LL\pm1}{}^{if}(r)$, and $\mu_{LL'}{}^{if}(r)$, and the form factors may be calculated from Eqs. (18) and fitted to experiment. The method that has most generally been chosen, however, for an analysis of the inelastic scattering data has been to construct a model for the nuclear transition and calculate the transition densities from such a model rather than just fit them phenomenologically to the data. In the following sections we will develop a hydrodynamical model, a generalization of Tassie's model,[8] in which we assume that the transition densities may be described by quantized incompressible and irrotational collective vibrations of the ground-state densities (treated as a liquid drop), the latter being assumed to have been obtained from a phenomenological fit to the elastic scattering data.

In the remainder of this section we will show examples of ground-state fits for ^{10}B. Figure 1 presents data of the Coulomb form factors, measured by Stovall, Goldemberg, and Isabelle.[22] They are plotted versus the squared "effective momentum transfer" $q_{\mathrm{eff}}{}^2$, where

$$q_{\mathrm{eff}}=q[1+\tfrac{3}{2}(\alpha Z/k_1R)], \tag{39}$$

in which R is the radius of the (assumed uniform spherical) charge distribution. The factor in parentheses roughly corrects[23] for the fact that the Born approximation overestimates the nuclear-charge radius as compared to the more exact phase-shift analysis. The

[22] T. Stovall, J. Goldemberg, and D. Isabelle, Nucl. Phys. **86**, 225 (1966).

[23] L. R. B. Elton, *Nuclear Sizes* (Oxford University Press, Oxford, England, 1961).

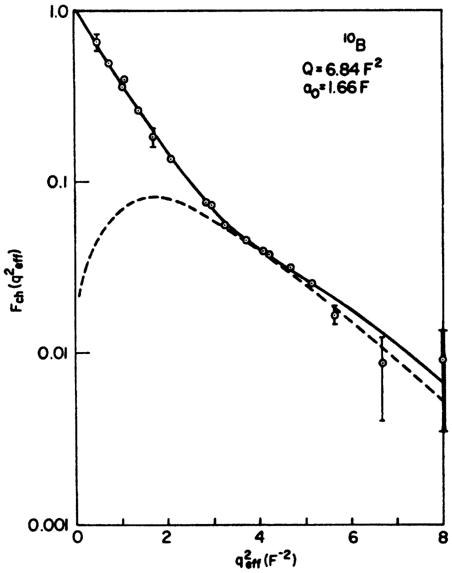

FIG. 1. Charge form factor F_{ch} of the deformed ground-state charge distribution of ^{10}B, measured by Stovall, Goldemberg, and Isabelle (figure taken from Ref. 22, including the fit), and fitted with a p-shell harmonic-oscillator charge distribution (solid curve). The broken curve gives the contribution of the deformed part. The value $a_0=1.66$ F corresponds to a root-mean-square charge radius (Ref. 22) of $a=2.45$ F.

data were taken at angles mostly less than $\vartheta=90°$, so that the magnetic terms in Eq. (38a) gave only small corrections. After the usual experimental (Schwinger, bremsstrahlung, and ionization) and theoretical corrections (center-of-mass motion, unfolding of proton charge distribution), Stovall *et al.*[22] found that the form factors could best be fitted, assuming only monopole and quadrupole contributions, with the following densities. For the $L=0$ part of $\rho(\mathbf{r})$ of Eq. (27a) [see also Eq. (29a)],

$$\rho_0(r)=(4\pi)^{1/2}\hat{J}\,\frac{eZ}{\pi^{3/2}a_0^3}\,\frac{2}{2+3\alpha}\left(1+\alpha\frac{r^2}{a_0^2}\right)\exp(-r^2/a_0^2), \quad (40a)$$

with $\alpha=(Z-2)/3$. This is the harmonic-oscillator charge distribution (a_0= oscillator potential-well radius) which was used by Hofstadter[24] to fit the $1p$-shell nuclear form factors, especially those of ^{12}C and ^{16}O. For the $L=2$ part of $\rho(\mathbf{r})$, the best fit was achieved[22] by

$$\rho_2(r)=\frac{\hat{J}}{(JJ,20\,|\,JJ)}\,\frac{4}{3\pi\cdot 5^{1/2}}\,\frac{eQ}{a_0^7}\,r^2\exp(-r^2/a_0^2), \quad (40b)$$

which is the density given by the spherically symmetric oscillator shell model with incomplete p shells.[25] The figure plots the expressions (for the ^{10}B ground state, $J=3^+$)

$$F_{ch}=\left[\,|F_0^C(q)|^2+\frac{q^4}{75}\frac{Q^2}{Z^2}|F_2^C(q)|^2\right]^{1/2} \quad (40c)$$

(solid curve), and plots the quadrupole contribution alone,

$$\frac{q^2}{(75)^{1/2}}\frac{Q}{Z}|F_2^C(q)| \quad (40d)$$

(broken curve), calculated from Eq. (38b), using the charge distributions of Eqs. (40a) and (40b). The quantities a_0 and Q are taken as parameters, and the best fit determines their values as $a_0=1.66$ F and $Q=6.84$ F^2. It is also possible to consider the deformed charge distribution $\rho_2(r)$ as arising from a hydrodynamic vibration of the undeformed shell-model distribution, $\rho(r)=(4\pi)^{-1/2}\hat{J}^{-1}\rho_0(r)$, which is frozen into a permanently deformed state[25] (see Appendix B). This procedure, however, does not lead to a good fit[22] for the ^{10}B ground-state form factor (see Fig. 2).

The charge distributions $\rho_0(r)$ and $\rho_2(r)$ which gave the best fit, Eqs. (40a) and (40b), are plotted in Fig. 3, normalized in such a way that $\rho_0(0)=1$. This figure also shows the ground-state magnetization density of ^{10}B obtained from the elastic scattering experiments at $\vartheta=180°$ of Rand, Frosch, and Yearian.[26] Their data are presented in Fig. 4. At 180°, the Coulomb terms in Eqs. (38a) drop out, and the rest is plotted versus q^2,

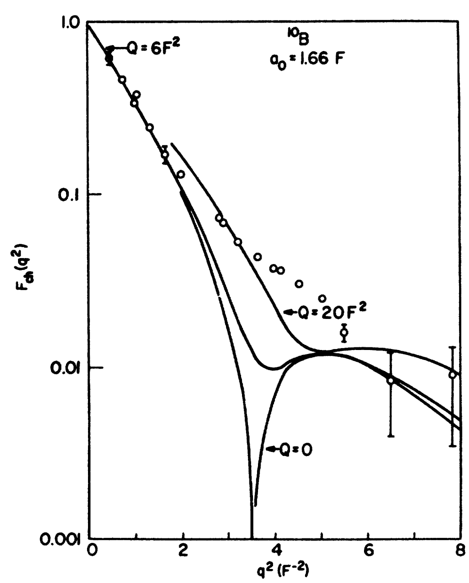

FIG. 2. Attempts to fit the charge form factor F_{ch} of the ^{10}B ground state with a deformed oscillator charge distribution, for various values of the quadrupole moment Q. (Figure taken from Ref. 22, including the fit.)

[24] R. Hofstadter, Ann. Rev. Nucl. Sci. **7**, 231 (1957).

[25] U. Meyer-Berkhout, K. W. Ford, and A. E. S. Green, Ann. Phys. (N.Y.) **8**, 119 (1959).

[26] R. F. Rand, R. Frosch, and M. R. Yearian, Phys. Rev. **144**, 859 (1966).

normalized to unity for $q \to 0$; i.e., the quantity plotted is

$$F_{mag}^2 = |F_1^M(q)|^2 + (q^4/175)(\Omega^2/\mu^2)|F_3^M(q)|^2, \quad (40e)$$

with the notation for the octupole moment being $M_3 \equiv \Omega$. The data are consistent with a vanishing octupole contribution, which is compatible with a spherically symmetric distribution of magnetization; but they also admit a maximum possible value $\Omega/\mu = 0.9$ F^2, and the corresponding upper limit of the magnetic octupole contribution [second term in Eq. (40e)] is shown as a dashed curve. The data, after unfolding the form factor of the nucleon, and after center-of-mass correction, have been fitted by Rand *et al.*[26] with a spherically symmetric distribution of the *p*-shell harmonic-oscillator type, Eq. (40a), but with different values of a_0 and α, namely, $a_0 = 1.42$ F and $\alpha = 2.0$. The quantity plotted in Fig. 3 (broken curve) for this distribution is essentially the function $\mu_{10}(r)$ of Eq. (27b) [see also Eq. (29b)], normalized in such a way that

$$\int \mu_{10}(r) r^2 dr = \int \rho_0(r) r^2 dr. \quad (40f)$$

One notices that the magnetization tends to be located more on the edge of the nucleus, since the inner shell couples to angular momentum zero and thus does not contribute to the orbital magnetization. However, the fit shown here need ***not be unique***.

The dashed curve of Fig. 4 gives the octupole magnetic form factor evaluated[26] with a corresponding magnetization function $\mu_{32}(r)$ of the form of Eq. (40b). We shall neglect it for our purposes and shall assume the spherically symmetric magnetization density that is compatible with the data. In the following sections, the preceding densities for the ground state of ^{10}B shall be employed as the input of a hydrodynamic model which is used to describe the electroexcitation of electric and magnetic levels, both in general and applied to ^{10}B as an example.

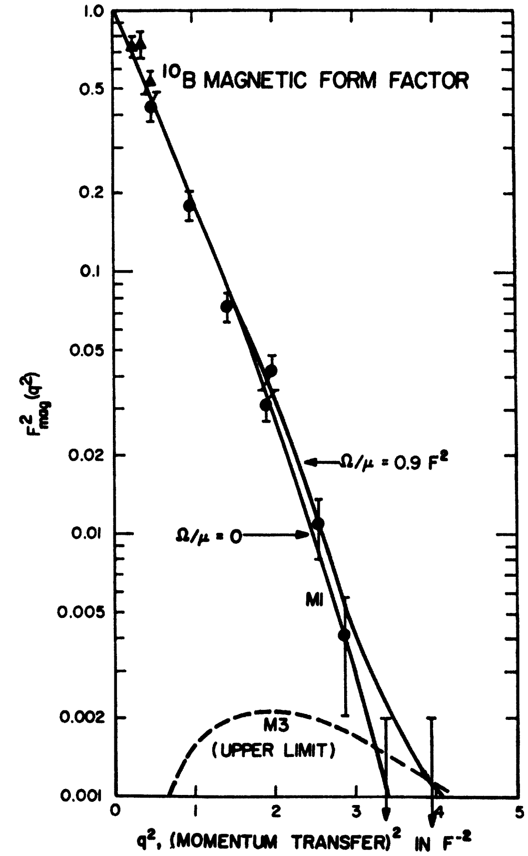

FIG. 4. Magnetic form factor F_{mag}^2 of the ground-state magnetization distribution of ^{10}B, measured by Rand, Frosch, and Yearian (figure taken from Ref. 26, including the fit), and fit by a *p*-shell harmonic-oscillator charge distribution, for values $\Omega/\mu = 0$ or 0.9 F^2. The broken curve gives the maximal contribution of a magnetic octupole term.

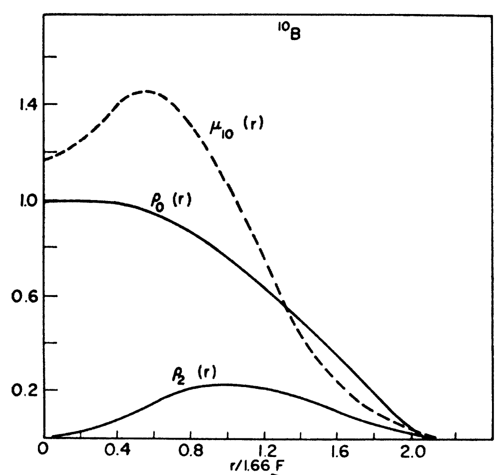

FIG. 3. Plot of the monopole part $\rho_0(r)$ and the quadrupole part $\rho_2(r)$ of the ^{10}B ground-state charge distribution, and of its spherically symmetric magnetization density $\mu_{10}(r)$ (broken curve).

IV. INELASTIC SCATTERING AND CHARGE VIBRATIONS OF A DEFORMED NUCLEUS

At the end of Sec. III, it was shown how the static nuclear densities may be obtained by phenomenologically fitting the form factors, calculated using assumed distributions, to the experimental data. We have stated that in principle a similar procedure may be used for obtaining the transition densities, which are contained in Eqs. (18), by fitting the cross section, Eq. (2), which contains the form factors of Eqs. (18), to the electroexcitation data. This procedure appears, however, to be less unique here than in the elastic case, and it has been customary to postulate models for the transition densities which are based on the ground-state densities. In this spirit, we shall develop a hydrodynamic model which describes the transitions of a deformed nucleus by the incompressible, irrotational vibrations of its ground state. For a spherical nucleus, such a model was introduced by Tassie.[3,4] In our general case, we deal with the nonspherical ground-state charge density $\rho(\mathbf{r})$ of Eq. (27a) (only charge vibrations will be considered in this section). Together with the convection current

$$\mathbf{j}_c(\mathbf{r})=\rho(\mathbf{r})\mathbf{v}(\mathbf{r}), \qquad (41a')$$

$\mathbf{v}(\mathbf{r})$ being the velocity field, $\rho(\mathbf{r})$ satisfies the continuity equation

$$\nabla\cdot\mathbf{j}_c(\mathbf{r})+[\partial\rho(\mathbf{r})/\partial t]=0. \qquad (41b')$$

The motion of nuclear matter will now be taken as incompressible and irrotational. Incompressibility implies

$$d\rho(\mathbf{r})/dt\equiv\mathbf{v}(\mathbf{r})\cdot\nabla\rho(\mathbf{r})+[\partial\rho(\mathbf{r})/\partial t]=0. \qquad (42a')$$

Combined with Eqs. (41), Eq. (42a) leads to

$$\nabla\cdot\mathbf{v}(\mathbf{r})=0. \qquad (42b')$$

The assumption of irrotational flow,

$$\nabla\times\mathbf{v}(\mathbf{r})=0, \qquad (43a')$$

permits the introduction of a velocity potential $\Phi(\mathbf{r})$,

$$\mathbf{v}(\mathbf{r})=\nabla\Phi(\mathbf{r}), \qquad (43b')$$

which, due to Eq. (42b), satisfies the Laplace equation,

$$\nabla^2\Phi(\mathbf{r})=0, \qquad (44a')$$

with the regular solution

$$\Phi(\mathbf{r})=\sum_{lm}\beta_{lm}r^lY_{lm}^*(\hat{r}). \qquad (44b')$$

Monopole vibrations ($l=0$) are excluded by the assumption of incompressibility, but by relaxing this condition for the monopole case, we shall incorporate the monopole case below. With harmonically vibrating densities $\propto\exp(-i\omega t)$, the continuity equation becomes

$$\nabla\cdot[\rho(\mathbf{r})\mathbf{v}(\mathbf{r})]=i\omega\rho(\mathbf{r}), \qquad (45a')$$

and using Eq. (42b), we have

$$\rho(\mathbf{r})=(i\omega)^{-1}\mathbf{v}(\mathbf{r})\cdot\nabla\rho(\mathbf{r}). \qquad (45b')$$

Collective vibrations of this kind will lead to a harmonically varying deformation of an originally spherical reference surface $r=r_0$ imbedded in our nuclear matter, given by

$$r=r_0[1+\sum_{lm}\alpha_{lm}(r_0/R)^{l-2}Y_{lm}^*(\hat{r})], \qquad (46a')$$

with harmonically varying expansion parameters α_{lm}, which are rendered dimensionless by the use of a suitable reference radius R. Calculating $(\partial r/\partial t)_\rho\equiv\partial\Phi/\partial r$ from Eq. (44b) and Eq. (46a) and equating, we see that the expansion in Eq. (46a) is justified and that

$$\beta_{lm}=\dot{\alpha}_{lm}/lR^{l-2}=-i\omega\alpha_{lm}/lR^{l-2}. \qquad (46b')$$

From this and Eqs. (45b) and (44b), we have

$$\rho(\mathbf{r})=-\sum_{lm}(\alpha_{lm}/lR^{l-2})\nabla[r^lY_{lm}^*(\hat{r})]\cdot\nabla\rho(\mathbf{r}). \qquad (47a')$$

Quantizing this thus-far classical treatment, α_{lm} is changed from a harmonically varying function of time into the operator

$$\alpha_{lm}=C_l[a_{lm}^\dagger+(-1)^m a_{l-m}], \qquad (47b')$$

with $a_{lm}^\dagger$ the creation operator of a multipole phonon (l,m). The numerical factor C_l in the Bohr-Mottelson theory[27] is a combination of inertial and restoring-force parameters, but we shall here simply consider it an adjustable parameter determining the amplitude of the collective vibration. The quantized hydrodynamic model now takes $\rho(\mathbf{r})$ on the right-hand side of Eq. (47a) as the ground-state charge density, and the entire expression (47a) then represents the transition density, in view of the appearance of the expansion coefficients α_{lm} which create or absorb phonons.

At this point, one may generalize Eq. (47a) to include monopoles, $l=0$. In this case, the form of $\Phi(\mathbf{r})$ will be changed so that it is no longer a solution of the Laplace equation, implying that the condition of incompressibility is relaxed. The new expression is

$$\rho^{\text{tr}}(\mathbf{r})=-\sum_{lm}\frac{\alpha_{lm}}{(l+k_l)R^{l+k_l-2}}\nabla[r^{l+k_l}Y_{lm}^*(\hat{r})]\cdot\nabla^{(l)}\rho(\mathbf{r}), \qquad (47c')$$

where $k_l=2\delta_{l0}$, and where we define

$$\nabla^{(l)}\rho(\mathbf{r})=r^{-3}\nabla[r^3\rho(\mathbf{r})], \quad l=0$$
$$=\nabla\rho(\mathbf{r}), \quad l\geq1. \qquad (47d')$$

In this form, we find

$$\int d^3r\rho^{\text{tr}}(\mathbf{r})=0, \qquad (47e')$$

with all terms in the sum on the right-hand side of Eq. (47c) integrating to zero, including the monopole. This implies (classically) that the deformation is matter-conserving[28] (which for $l\geq1$ simply means volume-conserving). Quantum-mechanically, it serves to satisfy the Schucan condition,[29] as will be pointed out below.

Finally, the form of Eq. (47c), which is the basic equation of our hydrodynamic model, will now be taken over for the operators of the densities in the sense of Sec. II, and the multipole charge-density operator will

[27] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-Fys. Medd. 27, No. 16 (1953).
[28] C. Werntz and H. Überall, Phys. Rev. 149, 762 (1966).
[29] T. Schucan, Nucl. Phys. 61, 417 (1965).

be found from Eq. (11a), i.e.,

$$\tilde{\rho}_{LM}{}^{\mathrm{tr}}(r)=-\int Y_{LM}(\hat{r})d\hat{r}\sum_{lm}\frac{\alpha_{lm}}{(l+k_l)R^{l+k_l-2}}\times\nabla[r^{l+k_l}Y_{lm}{}^*(\hat{r})]\cdot\nabla^{(l)}\rho^{\mathrm{op}}(\mathbf{r}).\quad(48')$$

It operates on a hybrid representation described both by the nuclear coordinates (to give the classical densities) and by the collective variables (to describe their quantized collective vibrations, i.e., phonons). Using the equations and techniques of Edmonds,[16] the differentiations in Eq. (48) may be performed explicitly.

In the intermediate steps, the following relations were found useful:

$$\mathbf{Y}_{Jl}{}^M(\hat{r})\cdot\mathbf{Y}_{J'l'}{}^{M'}(\hat{r})=(4\pi)^{-1/2}(-1)^{J-J'+1}\hat{J}\hat{J}'\hat{l}\hat{l}'\times\sum_{LM_L}(-1)^L L^{-1}(l0,l'0\mid L0)(JM,J'M'\mid LM_L)\times W(J1Ll';lJ')Y_{LM_L}(\hat{r}),\quad(49a')$$

$$Y_{Jl}{}^{M*}(\hat{r})=(-1)^{l+1-J+M}\mathbf{Y}_{Jl}{}^{-M}(\hat{r}).\quad(49b')$$

We obtain the transition multipole charge-density operator:

$$\tilde{\rho}_{LM}{}^{\mathrm{tr}}(r)=-\frac{\alpha_{00}}{(4\pi)^{1/2}r^2}\frac{d}{dr}[r^3\tilde{\rho}_{LM}(r)]-\sum_{l\geq1,m}\frac{\alpha_{lm}}{(4\pi)^{1/2}\hat{L}R^{l-2}}\left(\frac{2l-1}{l}\right)^{1/2}\hat{l}^2r^{l-1}\sum_{l'm'}(-1)^{L+l+l'}(lm,l'm'\mid LM)$$
$$\times\{[(l'+1)(2l'+3)]^{1/2}(l-10,l'+10\mid L0)W(l1Ll'+1;l-1l')r^{l'}(d/dr)\times[r^{-l'}\tilde{\rho}_{l'm'}(r)]-[l'(2l'-1)]^{1/2}(l-10,l'-10\mid L0)W(l1Ll'-1;l-1l')r^{-l'-1}(d/dr)[r^{l'+1}\tilde{\rho}_{l'm'}(r)]\}.\quad(50')$$

Note that the expansion (10a) has been used for $\rho^{\mathrm{op}}(\mathbf{r})$, so that the multipole charge-density operators $\tilde{\rho}_{l'm'}(r)$ appear in Eq. (50). The transition multipole density function $\rho_L{}^{if}(r)$ is obtained from Eq. (50) by the definition (12a). Acting on the ground state $|J_iM_i\rangle$, the operator α_{lm} creates one phonon of the collective vibration,

$$\alpha_{lm}\mid J_iM_i\rangle=C_l\mid J_iM_i\rangle\mid lm\rangle,\quad(51a')$$

whereas the final state $|J_fM_f\rangle$ has one phonon λ coupled to the ground state,

$$\mid J_fM_f\rangle=\sum_{\mu M_i'}(J_iM_i',\lambda\mu\mid J_fM_f)\mid J_iM_i'\rangle\mid\lambda\mu\rangle.\quad(51b')$$

The transition matrix element of the product $\alpha_{lm}\tilde{\rho}_{l'm'}(r)$ is therefore

$$\langle J_fM_f\mid\alpha_{lm}\tilde{\rho}_{l'm'}(r)\mid J_iM_i\rangle=C_l\hat{J}_i{}^{-1}\sum_{M_i'}(J_iM_i',lm\mid J_fM_f)(J_iM_i,l'm'\mid J_iM_i')\rho_{l'}(r),\quad(51c')$$

which by use of Eq. (21a) contains the multipole densities of the ground state, $\rho_{l'}(r)$. This gives the transition multipole density

$$\rho_L{}^{if}(r)=-(4\pi)^{-1/2}C_0\delta_{J_iJ_f}r^{-2}\frac{d}{dr}[r^3\rho_L(r)]-(4\pi)^{-1/2}\hat{J}_f\sum_{l\geq1}C_l\left(\frac{2l-1}{l}\right)^{1/2}\hat{l}^2r\left(\frac{r}{R}\right)^{l-2}\sum_{l'=0,\mathrm{even}}^{\leq2J_i}W(J_il'J_fl;J_iL)$$
$$\times\{[(l'+1)(2l'+3)]^{1/2}(l-10,l'+10\mid L0)W(l1Ll'+1;l-1l')r^{l'}(d/dr)[r^{-l'}\rho_{l'}(r)]-[l'(2l'-1)]^{1/2}(l-10,l'-10\mid L0)W(l1Ll'-1;l-1l')r^{-l'-1}(d/dr)[r^{l'+1}\rho_{l'}(r)]\},\quad(52')$$

expressed in terms of the ground-state multipole densities $\rho_{l'}(r)$ by virtue of our hydrodynamical model. It is to be used in Eq. (18a) for a calculation of the Coulomb transition matrix element.

Because of the orthogonality of initial and final states, the condition

$$\int\rho_0{}^{if}(r)d^3r=0\quad(53a')$$

must hold for the monopole transitions $L=0$, as pointed out by Schucan.[29] This is satisfied by our Eq. (52); for the C_0 term this follows by direct integration and shows that our monopole contribution in Eq. (47c) was correctly chosen. For the C_l terms, one has $l'=l$ if $L=0$; the first of the two terms in braces vanishes because $(l-10,l+10\mid 00)=0$ and the second term integrates to zero.

For a spherically symmetric ground state where Eq. (29a) holds, one finds the multipole transition densities

$$\rho_L{}^{if}(r)=-C_L\hat{J}_fr(r/R)^{L+k_L-2}\hat{r}\cdot\nabla^{(L)}\rho(r),\quad(53b')$$

together with the triangle condition

$$\Delta(J_iLJ_f).\quad(53c')$$

This is in essence the result of Tassie,[3] which, however, did not include the monopole case $L=0$. With $\rho_L{}^{if}(r)$

given by Eq. (52), it is now straightforward to evaluate the Coulomb matrix element of Eq. (18a). We find

$$eZ\langle J_f \| \tilde{M}_L(q) \| J_i\rangle = -i^L(4\pi)^{-1/2}C_0\delta_{J_iJ_f}I_{L2L}{}^{-3}(q) - i^L(4\pi)^{-1/2}\hat{J}_f \sum_{l\geq 1} C_l[(2l-1)/l]^{1/2}\hat{l}^2R^{2-l} \sum_{l'=0,\text{even}}^{\leq 2J_i} W(J_il'J_fl;J_iL)$$
$$\times\{[(l'+1)(2l'+3)]^{1/2}(l-10,l'+10\,|\,L0)W(l1Ll'+1;l-1l')I_{Ll l'}{}^{l'}(q)$$
$$-[l'(2l'-1)]^{1/2}(l-10,l'-10\,|\,L0)W(l1Ll'-1;l-1l')I_{Ll l'}{}^{-l'-1}(q)\}, \quad (54a)$$

in which the integrals $I_{Lll'}{}^n(q)$, obtained by partial integration, are defined by

$$I_{Lll'}{}^n(q) = q\int_0^\infty dr\rho_{l'}(r)r^{l+1}$$
$$\times[j_{L+1}(qr)-(L+l+n+1)(qr)^{-1}j_L(qr)]. \quad (54b)$$

The coefficients C_l may be chosen so that Eq. (54a) is real. Note that in these equations three different kinds of multipolarities are involved which should not be confused, namely, (a) L, the multipolarity of the transition caused by the electroexcitation, (b) l, the multipolarity of the collective nuclear vibration whose amplitude is C_l, and (c) l', the (even) multipolarity of the expansion of a deformed ground-state density according to Eq. (27a). Since the corresponding moments $\rho_{l'}(r)$ are here assumed to be known from a fit to the elastic scattering cross section, the only unknown quantities in our model are the amplitudes C_l of the collective vibrations.

As an example of inelastic charge scattering, we shall here consider the transition of ^{10}B from the $J_i=3^+$ ground state to the $J_f=4^+$ level at $\omega=6.02$ MeV. The transition is electric quadrupole, $L=2$, but as we shall see, the deformation of the ground state causes an additional $L=4$ component to enter which is proportional to the quadrupole moment Q. The form factor of this level was measured by Fricke, Bishop, and Isabelle[30] at angles between $\vartheta=50°$ and 135° where the Coulomb form factor is dominant over the transverse electric one. We used the $l'=0$ and 2 components of the ground-state charge distribution given by Eqs. (40a) and (40b) and the quadrupole moment $Q=6.84$ F^2. Only a quadrupole nuclear collective vibration, $l=2$, was assumed, so that the only unknown parameter contained in the model was C_2. The Coulomb matrix elements which contribute to the example considered are given by

$$eZ\langle J_f \| \tilde{M}_2(q) \| J_i\rangle = (4\pi)^{-1/2}C_2(3/7^{1/2})$$
$$\times\{I_{220}{}^0+(3^{1/2}/4)[(3/7)I_{222}{}^2+I_{222}{}^{-3}]\}, \quad (55a)$$
$$eZ\langle J_f \| \tilde{M}_4(q) \| J_i\rangle = -(4\pi)^{-1/2}C_2(330/7^3)^{1/2}I_{422}{}^2. \quad (55b)$$

The leading term in $\langle\tilde{M}_2\rangle$ is $I_{220}{}^0$ which depends on $\rho_0(r)$ of the ground state. The three other I's depend on $\rho_2(r)$ and are proportional to Q. Using ρ_0, ρ_2 of Eqs. (40), the integrals may be performed analytically. The quantity $F_{\text{in}}{}^2(q)$ given in Ref. 30 is in our notation given by

$$F_{\text{in}}{}^2(q) = 4\pi\hat{J}_i{}^{-2}(\Delta^2/q^2)^2 \sum_{L=2,4} |\langle J_f \| \tilde{M}_L(q) \| J_i\rangle|^2. \quad (55c)$$

Figure 5 shows our results, compared with the experimental data. For $Q=0$, only the $L=2$ term is present, and adjustment of our curve to the data at $q=1.0$ F^{-1} determines $C_2=0.41$. Using the experimental $Q=6.84$ F^2 requires a slight readjustment to $C_2=0.36$ due to the small additional terms $I_{222}{}^2$ and $I_{222}{}^{-3}$ in $\langle\tilde{M}_2\rangle$. In addition, a large $L=4$ term $\langle\tilde{M}_4\rangle$ appears which completely fills in the dip to zero in $\langle\tilde{M}_2\rangle$ at $q\approx 2.2$ F^{-1}. This exact zero is due to our use of the Born approximation, but its filling-in due to a use of phase-shift analysis would only be very slight[22] for a light nucleus such as ^{10}B. Although the data do not extend to large enough values of q to completely confirm our predictions, the highest data points do go in the right direction; and if we compare with the elastic scattering situation, Fig. 1, the effect of the ground-state deformation on the inelastic form factor seems to be quite analogous.[31] It is

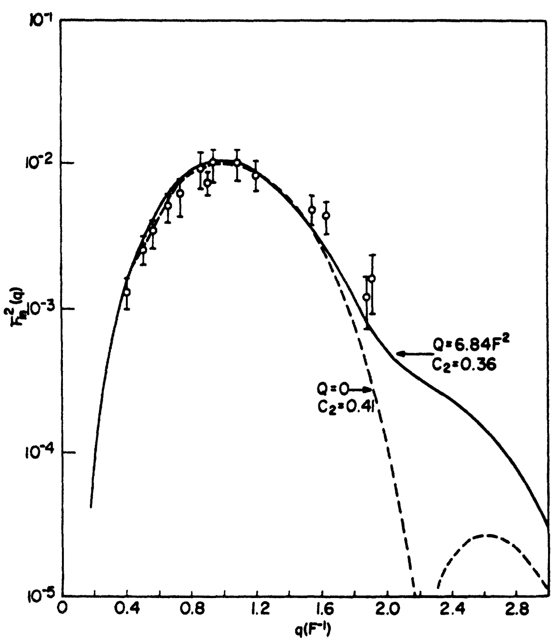

FIG. 5. Inelastic squared charge form factor $F_{\text{in}}{}^2$ of the $J_f=4^+$, $\omega=6.02$-MeV level of ^{10}B, measured by Fricke, Bishop, and Isabelle (Ref. 30), and fitted with the hydrodynamic model of Sec. IV. Broken curve: $Q=0$, and amplitude of quadrupole collective vibration $C_2=0.41$. Solid curve: $Q=6.84$ F^2 and $C_2=0.36$.

[30] G. Fricke, G. R. Bishop, and D. B. Isabelle, Nucl. Phys. 67, 187 (1965).

[31] A similar filling-in of the Born minimum of inelastic form factors due to the effects of a ground-state deformation was recently obtained also on the basis of the Helm model [R. Raphael and M. Rosen (to be published)].

desirable that inelastic form factors of deformed light nuclei[32] should be investigated experimentally at large enough momentum transfers to confirm the filling-in of the diffraction minima by deformation effects and to verify the predictions of the hydrodynamic model.

V. EXISTENCE OF COLLECTIVE MAGNETIC TRANSITIONS

In this section, we will investigate the properties of any collective magnetic transitions, similar to the collective charge transitions of Sec. IV, provided they exist. It will turn out that the observed $M1$ transitions at low momentum transfers can probably not be explained by collective vibrations of the magnetization density (of the quadrupole type, $l=2$) but should be single-particle excitations. However, collective quadrupole-type vibrations may show up as $M1$ transitions at high momentum transfer, and one may look for them there. (To the author's knowledge, no study of $M1$ transitions at $\vartheta=180°$ and $q>100$ MeV/c has been undertaken as yet.) If the vibrating ground-state magnetization is deformed, the deformation may sometimes contribute some transition strength even at lower values of q, however.

One may derive a continuity equation for the magnetization density $\boldsymbol{\mu}(\mathbf{r})$ also, since

$$\frac{\partial}{\partial t}\int \boldsymbol{\mu}(\mathbf{r})d^3r+\int \mathfrak{F}(\mathbf{r})\cdot d\mathbf{A}=0, \qquad (56a')$$

where the flow of magnetization is

$$\mathfrak{F}(\mathbf{r})=\boldsymbol{\mu}(\mathbf{r})\mathbf{v}(\mathbf{r}), \qquad (56b')$$

which is a dyadic. The desired continuity equation follows from a dyadic Gauss theorem[33]:

$$\mathfrak{F}\cdot\boldsymbol{\nabla}+(\partial\boldsymbol{\mu}/\partial t)=0. \qquad (56c')$$

If $\mathfrak{F}$ is written as the column matrix

$$\mathfrak{F}=\begin{pmatrix}\mathbf{F}_1\\ \mathbf{F}_2\\ \mathbf{F}_3\end{pmatrix}, \qquad (\mathfrak{F}\cdot\boldsymbol{\nabla})_i\equiv\boldsymbol{\nabla}\cdot\mathbf{F}_i, \qquad (56d')$$

then Eq. (56c) reads

$$\boldsymbol{\nabla}\cdot\mathbf{F}_i+(\partial\mu_i/\partial t)=0; \qquad (56e')$$

i.e., the usual continuity equation holds separately for each (Cartesian) component of $\boldsymbol{\mu}$, since

$$\mathbf{F}_i=\mu_i\mathbf{v}. \qquad (56f')$$

Hence, the derivations for $\rho(\mathbf{r})$, Eqs. (41) through (47), hold equally well for $\boldsymbol{\mu}(\mathbf{r})$, and we have

$$\boldsymbol{\mu}^{\mathrm{tr}}(\mathbf{r})=-\sum_{lm}\frac{\alpha_{lm}}{(l+k_l)R^{l+k_l-2}} \times\boldsymbol{\nabla}[r^{l+k_l}Y_{lm}^*(\hat{r})]\cdot\boldsymbol{\nabla}^{(l)}\boldsymbol{\mu}(\mathbf{r}). \qquad (56g')$$

By the same argument as before, and using Eq. (11b), we have

$$\tilde{\mu}_{LL'M}{}^{\mathrm{tr}}(r)=-\sum_{\mu\nu}(L'\mu,1\nu\mid LM)\int Y_{L'\mu}(\hat{r})d\hat{r} \times\sum_{lm}\frac{\alpha_{lm}}{(l+k_l)R^{l+k_l-2}}\boldsymbol{\nabla}[r^{l+k_l}Y_{lm}^*(\hat{r})]\cdot\boldsymbol{\nabla}^{(l)}\mu_\nu^{\mathrm{op}}(\mathbf{r}). \qquad (57')$$

Again, the transition multipole density function is obtained as before:

$$\mu_{LL'}{}^{if}(r)=-(4\pi)^{-1/2}C_0\delta_{J_iJ_f}r^{-2}\frac{d}{dr}[r^3\mu_{LL'}(r)]+(4\pi)^{-1/2}(-1)^{L+L'}\hat{J}_f\hat{L}\sum_{l\geq 1}C_l\left(\frac{2l-1}{l}\right)^{1/2}\hat{l}^2r\left(\frac{r}{R}\right)^{l-2}$$
$$\times\sum_{l'=1,\mathrm{odd}}^{\leq 2J_i}\sum_{l''=l'\pm1}\hat{l}'W(J_il'J_fl;J_iL)W(ll''L1;L'l')\{[(l''+1)(2l''+3)]^{1/2}(l-10,l''+10\mid L'0)$$
$$\times W(l1L'l''+1;l-1l'')r^{l''}(d/dr)[r^{-l''}\mu_{l'l''}(r)]-[l''(2l''-1)]^{1/2}(l-10,l''-10\mid L'0)$$
$$\times W(l1L'l''-1;l-1l'')r^{-l''-1}(d/dr)[r^{l''+1}\mu_{l'l''}(r)]\}; \qquad (58')$$

this function is expressed in terms of the ground-state multipole magnetization densities $\mu_{l'l''}(r)$ by virtue of our hydrodynamical model. It is to be used in Eq. (18c) for a calculation of the transverse magnetic transition matrix element, or in Eq. (18b) for a calculation of the transverse electric spin-flip transition matrix element.

The analog to the Schucan condition[29] has to be satisfied for the $L=1$, $L'=0$ transitions:

$$\int\mu_{10}{}^{if}(r)d^3r=0, \qquad (59a')$$

which again follows from the orthogonality of the initial and final states. One may easily show that the condition is satisfied by our transition density, Eq. (58).

For a spherically symmetric ground state where Eq.

[32] For medium-heavy and heavy nuclei, a substantial filling-in of the diffraction minimum occurs due to the distortion of the electron wave function (as shown by phase-shift analysis results) which may mask or overwhelm the additional filling from deformation effects.

[33] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Co., Inc., New York, 1953), Vol. 1, p. 66, Eq. (1.6.18).

(29b) holds, one finds the magnetic multipole transition densities:

$$\mu_{LL'}{}^{if}(r)=(-1)^{L+L'}\hat{J}_i\hat{J}_f\hat{L}(J_iJ_i,10\,|\,J_iJ_i)^{-1}C_{L'}\times r(r/R)^{L'+k_{L'}-2}W(J_i1J_fL';J_iL)\hat{r}\cdot\nabla^{(L')}\mu_z(r). \tag{59b$'$}$$

Introducing the notation

$$\mathcal{G}_{LL'}(q)=\int_0^\infty r^2 j_{L'}(qr)\mu_{LL'}{}^{if}(r)\,dr, \tag{60a}$$

we may write the transverse magnetic matrix element of Eq. (18c) as

$$eZ\langle J_f\,||\,\tilde{T}_L{}^m(q)\,||\,J_i\rangle=i^{L+1}q(L+1)^{1/2}\hat{L}^{-1}\times\mathcal{G}_{LL-1}(q)-i^{L+1}qL^{1/2}\hat{L}^{-1}\mathcal{G}_{LL+1}(q) \tag{60b}$$

and the electric spin-flip part of the transverse electric matrix element of Eq. (18b) as

$$eZ\langle J_f\,||\,\tilde{T}_L{}^{e\mu}\,||\,J_i\rangle=i^Lq\mathcal{G}_{LL}(q). \tag{60c}$$

For Eq. (60a) we obtain

$$\mathcal{G}_{LL'}(q)=-(4\pi)^{-1/2}C_0\delta_{J_iJ_f}K_{L'2LL'}{}^{-3}(q)+(-1)^{L+L'}(4\pi)^{-1/2}\hat{J}_f\hat{L}\sum_{l\geq1}C_l[(2l-1)/l]^{1/2}\hat{l}^2R^{2-l}$$
$$\times\sum_{l'=1,\text{odd}}^{\leq 2J_i}\sum_{l''=l'\pm1}\hat{l}'W(J_il'J_fl;J_iL)W(ll''L1;L'l')\{[(l''+1)(2l''+3)]^{1/2}(l-10,l''+10\,|\,L'0)$$
$$\times W(l1L'l''+1;l-1l'')K_{L'll'l''}{}^{l''}(q)-[l''(2l''-1)]^{1/2}(l-10,l''-10\,|\,L'0)W(l1L'l''-1;l-1l'')K_{L'll'l''}{}^{-l''-1}(q)\}, \tag{60d}$$

in which the integrals $K_{L'll'l''}{}^n(q)$, obtained by partial integration, are defined by

$$K_{L'll'l''}{}^n(q)=q\int_0^\infty dr\mu_{l'l''}(r)r^{l+1}\times[j_{L'+1}(qr)-(L'+l+n+1)(qr)^{-1}j_{L'}(qr)]. \tag{60e}$$

As in the case of the charge transitions, one has the transition multipolarity L, the nuclear collective vibration multipolarity l (as defined by the amplitude parameter C_l), and the multipolarities of the (deformed) ground-state magnetization $(l'l'')$; for practical purposes L' is just an index. Again, the moments $\mu_{l'l''}(r)$ are assumed to be known from a fit to the elastic scattering cross section at 180°, and the only unknown quantities in our model are the collective vibration amplitudes C_l.

It is instructive to consider the limit for small values of q for the transition matrix elements given by our model of vibrating magnetization. From Eqs. (18b) and (18c) one sees that normally

$$\langle J_f\,||\,\tilde{T}_L{}^{e\mu}(q)\,||\,J_i\rangle\propto q^{L+1} \tag{61a}$$

for the electric spin-flip transitions and

$$\langle J_f\,||\,\tilde{T}_L{}^m(q)\,||\,J_i\rangle\propto q^L \tag{61b}$$

for the magnetic transitions. This limit need not be realized in the vibrating model. From Eqs. (60) we find the limits for $q\to0$ given in Table I, for the lowest multipolarities of magnetization vibrations l, and the lowest transition multipolarities ($L\leq2$, except for monopole transitions $l=0$ where $L=3$ is also considered).

Columns 1 and 2 list the multipolarity of matter vibration l. Columns 3 and 4 list the magnetic or spin-flip electric matrix elements that were found nonvanishing in our model and their transition multipolarities L, and column 5 gives their lowest power of q in the limit $q\to0$, from Eqs. (61). Column 6 shows the actual limiting power of q given by our model, Eqs. (60), and one sees that the lowest possible power of q is not reached by the $M1$ transition $T_1{}^m$ for $l=0$ monopole vibrations as well as for $l=2$ quadrupole vibrations. Column 7 presents the ground-state multipole densities contributing to the low-q limit of the $T_L{}^m$ or $T_L{}^{e\mu}$ in our model, and column 8 gives the corresponding ground-state moments $M_1=\mu$ or $M_3=\Omega$. Column 9 lists the accompanying Racah coefficient that involves the initial and final spins of the transition.

Dipole vibrations of magnetization, $l=1$, are of course possible only if part of the nuclear matter oscillates 180° out of phase against the other part. These involve the familiar isospin, spin-isospin, and spin wave modes, shown schematically in Fig. 6, which have been extensively discussed elsewhere.[34,35] We only wish to remark here that, owing to Morpurgo's selection rule,[36] spin waves do not appreciably get excited in electron scattering, and the other two modes involve only $\Delta T=1$ transitions. For the magnetic vibrations considered here, it is mainly the $\Delta T=1$ spin-isospin wave that contributes.

The conventionally termed "surface oscillations" are the quadrupole vibrations, $l=2$, which are possible in phase as well as in the three modes[37] of Fig. 6. Many experimentally observed $M1$ transitions, e.g., that[9] to the $\omega=7.477$-MeV level of ^{10}B, exhibit the q^1 dependence of $T_1{}^m$ allowed by Eq. (18c), but Table I shows that our vibrating model only allows a q^3 dependence. (To the

[34] H. Überall, Nuovo Cimento **41B**, 25 (1966).
[35] H. Überall, Nuovo Cimento Suppl. **4**, 781 (1966).
[36] G. Morpurgo, Phys. Rev. **110**, 721 (1958).
[37] R. Raphael, H. Überall, and C. Werntz, Phys. Rev. **152**, 899 (1966).

TABLE I. The lowest nonvanishing matter-vibration and magnetic or spin-flip electric multipoles, the limiting q dependence for $q \to 0$, and the contributing ground-state densities and moments.

Matter vibration	Multi-polarity l	Matrix element	Transition multi-polarity L	Lowest possible power of q	Lowest power in model	Ground-state densities contributing	Corres-ponding ground-state moment	Dependence on J_i, J_f
Monopole	0	T_1^m	1	q^1	q^3	μ_{10}, μ_{12}	μ	$\delta_{J_iJ_f}$
		T_3^m	3	q^3	q^3	μ_{32}	Ω	$\delta_{J_iJ_f}$
Dipole	1	$T_1^{e\mu}$	1	q^2	q^2	μ_{10}	μ	$W(J_i1J_f1;J_i1)$
		T_2^m	2	q^2	q^2	μ_{10}	μ	$W(J_i1J_f1;J_i2)$
Quadrupole	2	T_1^m	1	q^1	q^3	μ_{10}, μ_{12}	μ	$W(J_i1J_f2;J_i1)$
						μ_{32}	Ω	$W(J_i3J_f2;J_i1)$
		$T_2^{e\mu}$	2	q^3	q^3	μ_{10}, μ_{12}	μ	$W(J_i1J_f2;J_i2)$
						μ_{32}	Ω	$W(J_i3J_f2;J_i2)$

same order q^3, the T_3^m transition will also contribute, as shown in the following example.) A q^3 dependence only is allowed also for T_1^m furnished by monopole matter vibrations $l=0$, which, however, are expected to lie at higher energies owing to the small nuclear compressibility. We must conclude, therefore, that the $M1$ transitions observed at small momentum transfers cannot be explained by collective vibrations of nuclear magnetization but should be interpreted exclusively as single-particle transitions. The electric spin-flip transitions, however, do assume the minimal power of q in our model. They possess an intrinsically higher power of q than the transverse electric charge transitions, which behave as

$$\langle J_f \| \tilde{T}_L^{ej}(q) \| J_i \rangle \propto q^{L-1} \tag{61c}$$

according to Eq. (18b) and have not yet been conclusively identified. (They are characterized, besides their q dependence at low values of q, by the absence of a longitudinal matrix element.)

The matrix elements considered so far receive their contribution to the low-q limit mainly from the dominant spherically symmetric ground-state magnetization density $\mu_{10}(r)$, i.e., from the static dipole moment μ. For the case of the $M3$ monopole transition, however, we find that its existence at low-momentum transfers is based exclusively on the higher moment $\mu_{32}(r)$, i.e., on the presence of an octupole magnetic moment Ω.

The absence of collective $l=2$ $M1$ transitions at $q \to 0$ in our vibrating model is analogous to the absence of collective $M1$ photon transitions $(q=\omega)$ which was shown by Lipas[12] for even nuclei. The situation there is different from our case, however, since no spectroscopic magnetization densities are involved (although the orbital magnetic-moment operator considered by Lipas is incorporated in our theory).

It is interesting to point out, however, that although $l=2$ $M1$ transitions furnished by our model of a vibrating ground-state magnetization density will not be very conspicuous in experiments at low momentum transfers, they could appear at large values of momentum transfer and should be looked for in 180° scattering experiments at high energy. The q^3 dependence of T_1^m however (see Table I) will make them appear as if they were $M3$ transitions—unless the latter possibility is explicitly prohibited by the selection rule $\Delta(J_iLJ_f)$. For example, a transition $\frac{1}{2}^+ \to \frac{1}{2}^+$ that appears at 180° with a low-q dependence $T \propto q^3$ should only be an ($l=2$ or $l=0$) $M1$ transition as described by our model of a vibrating magnetization. A $1^+ \to 1^+$ transition with $T \propto q^3$ may be an ($l=2$ or $l=0$) $M1$ transition caused by a vibrating magnetization, or an ($l=2$) $E2$ spin-flip transition possibly, but not necessarily, caused by a vibrating magnetization. If the selection rule permits several values of L, then, for example, an $M1$ and an $M3$ transition may (incoherently) add together with the same low-q dependence T_1^m, $T_3^m \propto q^3$ on the vibrational model. This is the case in the following example.

Here we consider a $3^+ \to 2^+$ transition in ^{10}B and calculate the cross section at 180° assuming the model of a vibrating magnetization density which was described after Eq. (40e) for ^{10}B [$a_0=1.42$ F, $\alpha=2.0$; only $\mu_{10}(r)$ is assumed to be present]. We assume a dipole vibration

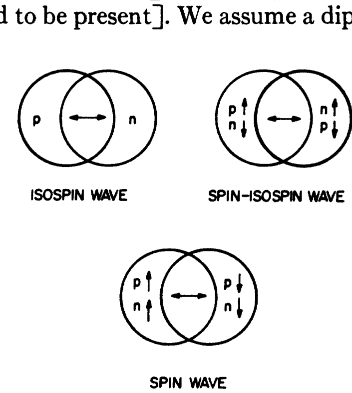

FIG. 6. Three possible modes of collective dipole vibrations (Refs. 34 and 35) of nucleon fluids $p\uparrow$, $p\downarrow$, $n\uparrow$, $n\downarrow$ (arrow indicates spin projection). Only the spin-isospin mode ($\Delta T=1$) contributes appreciably to the magnetization vibrations considered in the text.

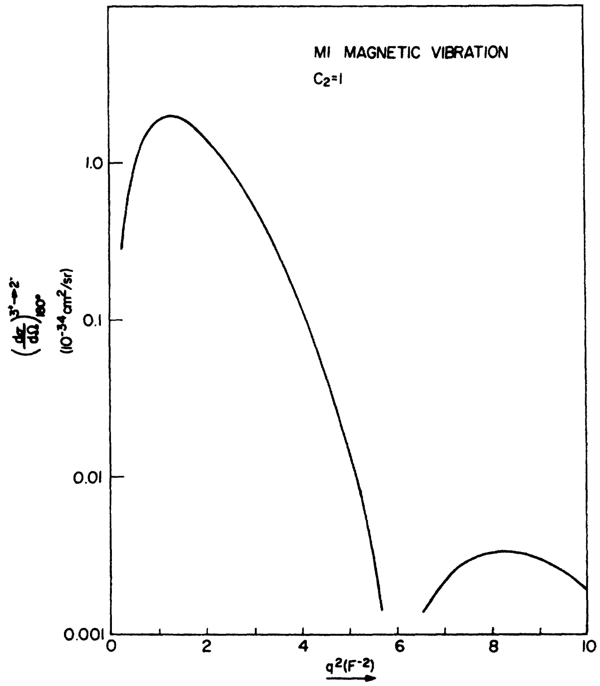

FIG. 7. Differential cross section at $\vartheta=180°$ of a hypothetical $3^+\rightarrow2^+$ $M1$ transition in ^{10}B caused by a quadrupole ($l=2$) collective vibration of a spherically symmetric ground-state magnetization distribution.

of the magnetization, $l=2$, and calculate $d\sigma/d\Omega$ at $\vartheta=180°$ from Eq. (2), using the matrix element Eq. (60b) of our model. From Eqs. (60) it turns out that both $L=1$ and $L=3$ transitions will take place with the same q dependence T_1^m, $T_3^m \propto q^3$. After the corrections for the nucleonic form factor and the center-of-mass motion were made following Ref. 26, the results

$$eZ\langle J_f \| \tilde{T}_1^m(q) \| J_i\rangle = -(4\pi)^{-1/2}C_2\hat{J}_f \times W(J_i1J_f2;J_i1)qK_{2210}^0(q), \quad (62a)$$

$$eZ\langle J_f \| \tilde{T}_3^m(q) \| J_i\rangle = -2i(4\pi)^{-1/2}C_2\hat{J}_f \times W(J_i1J_f2;J_i3)qK_{2210}^0(q) \quad (62b)$$

led to a differential cross section $(d\sigma/d\Omega)_{180°}$ plotted versus q^2 (taking the vibration amplitude $C_2=1$) in Fig. 7. Its maximum is about one order of magnitude smaller than that of the $M1$ cross sections that are found[9] at low momentum transfers with q dependence $T_1^m \propto q^1$, and the form is probably similar to conventional $M3$ transitions, so that it may be distinguished from the latter only by the action of selection rules, as pointed out above. It is practically certain that the diffraction minimum would again have been filled in as in the case of charge vibrations, had a deformed ground-state magnetization density been assumed.

VI. SUMMARY

On the basis of the Born approximation, we have given the most general expressions for the cross sections for the elastic and inelastic scattering of electrons by deformed nuclei of arbitrary spin. The latter were described by multipole expansions of their static charge and magnetization densities, and the elastic cross section was given in terms of the static moments, multiplied by the corresponding electric and magnetic form factors. Examples are quoted for ground-state charge and magnetization densities of ^{10}B obtained by fits to the experimental form factors. Inelastic cross sections are likewise expressed in the most general fashion by transition multipole densities of charge, current, and magnetization. Since these may not be determined from experiment as unambiguously as the static densities, we developed a model of collective vibrations of the static charge and magnetization densities for describing the transitions. For charge transitions this is a generalization of Tassie's model to deformed nuclei (and to include monopole vibrations and transitions), and it is shown to lead to a filling in of the diffraction minima of inelastic form factors due to the ground-state deformations, similar to the familiar filling in of elastic diffraction minima by the deformation. Collective vibrations of the magnetization density are often found to behave for low momentum transfers according to a higher power of q than conventional magnetic transitions (so that the conventional magnetic transitions must be explained by single-particle rather than collective effects), but these collective vibrations will show up at larger momentum transfers, where they may be distinguished from higher-order magnetic transitions if the latter are forbidden by selection rules.

In the following Appendices, we shall consider some individual problems related to the main topics of the paper: A: connections between spectroscopic and intrinsic nuclear quantities, B: ground-state densities described by a collective deformation, C: relation of our theory to Helm-type models,[6] and D: definitions of root-mean-square charge and magnetic radii of the nucleus.

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APPENDIX A: INTRINSIC VIBRATIONS

The theory of this paper has dealt exclusively with spectroscopic quantities such as moments and densities, so that, for example, Q_L will not exist for nuclei with $J_i<\frac{1}{2}L$, due to the projection factors $(J_iJ_i, L0 \mid J_iJ_i)$. Likewise, in our model of collective vibrations of a deformed nucleus, C_l in Eq. (47b) is the vibration amplitude of a spectroscopic density. For even-even nuclei which are intrinsically deformed, we know that there are, for example, two types of quadrupole ($l=2$) vibrations, β and γ type, upon either of which rotational bands may be based. By a rotation through Euler angles Θ, one transforms the spectroscopic vibration amplitudes $\alpha_{lm}(\hat{r})$ to the intrinsic ones $a_{lm}(\hat{r})$ according to

$$\alpha_{lm}(\hat{r})=\sum_{m''}D_{mm''}{}^{l}(\Theta)a_{lm''}(\hat{r}'), \qquad (A1')$$

where $\hat{r}'$ denotes the intrinsic coordinates. For quadrupole vibrations, $l=2$, the operators $a_{22}\equiv a_{2-2}$ create the γ vibrations, and the operator $a_{20}\equiv\beta_0+a_{20}'$ creates the β vibrations, where β_0 gives the constant equilibrium deformation. If, in Eq. (51c), $\alpha_{lm}(\hat{r})$ is replaced by the expression of Eq. (A1) and the rotational wave functions

$$|J_iM_i\rangle=(8\pi^2)^{-1/2}\hat{J}_iD_{M_iK_i}{}^{J_i}(\Theta)\chi_{K_i}{}^{i}(\hat{r}'), \quad (A2a')$$

$$|J_fM_f\rangle=(8\pi^2)^{-1/2}\hat{J}_fD_{M_fK_f}{}^{J_f}(\Theta)\chi_{K_f}{}^{f}(\hat{r}'), \quad (A2b')$$

are used [where $\chi_K(\hat{r}')$ are the intrinsic wave functions], then the functions $D_{MK}{}^{J}(\Theta)$ may be integrated out, and we regain precisely the right-hand side of Eq. (51c), but this time with the explicit expression for the vibration amplitude C_l:

$$C_2=\hat{J}_i\hat{J}_f{}^{-1}(J_iK_i,2m''\,|\,J_fK_f)\langle\chi_{K_f}{}^{f}\,|\,a_{2m''}\,|\,\chi_{K_i}{}^{i}\rangle. \qquad (A3')$$

For β vibrations, $m''=0$, the factor $(J_iK_i,20\,|\,J_fK_f)$ therefore introduces the familiar selection rule $K_f=K_i$, and for γ vibrations, $(J_iK_i,22\,|\,J_fK_f)$ introduces the selection rule $K_f=K_i+2$. This example shows that our spectroscopic amplitude factor C_l is not an arbitrary or artificial quantity, but if considered in greater detail may be expressed by the properties of the familiar intrinsic vibrations if so desired.

APPENDIX B: STATIC DEFORMED NUCLEAR MODEL

We may assume that a static deformed charge distribution is described by exactly the kind of incompressible, irrotational collective vibration treated in Sec. IV of an originally spherically symmetric charge distribution $\rho(r)$ with the vibration then "frozen" into its static position. We may then simply use Eq. (53b) for the static multipole densities $\rho_L(r)$ and exclude monopole vibrations, since they do not cause a deformation. The $L=0$ multipole density is simply given by the original spherical distribution. For the deformation amplitudes, we introduce appropriate constant parameters β_L, so that we may write for the densities $\rho_L(r)$ of Eq. (27a):

$$\rho_0(r)=(4\pi)^{1/2}\hat{J}\rho(r), \qquad (B1a')$$

$$\rho_L(r)=-\beta_L\hat{J}(JJ,L0\,|\,JJ)^{-1}r(r/R)^{L-2}\rho'(r), \qquad L\geq 2. \qquad (B1b')$$

This leads to expressions for the electric multipole moments and form factors of Eq. (38b) as follows:

$$eQ_0F_0{}^{C}(q)=4\pi\int r^2j_0(qr)\rho(r)dr, \qquad (B2a)$$

$$eQ_LF_L{}^{C}(q)=(4\pi)^{1/2}\beta_L(2L+1)!!\hat{L}^{-1}q^{1-L}R^{2-L}\times\int r^{L+1}j_{L-1}(qr)\rho(r)dr, \qquad L\geq 2. \qquad (B2b)$$

In the limit $q\to 0$, we recover the static moments:

$$eQ_0=\int\rho(r)d^3r=Ze, \qquad (B3a')$$

$$eQ_L=(4\pi)^{-1/2}\beta_L\hat{L}R^{2-L}\int r^{2L-2}\rho(r)d^3r, \qquad L\geq 2, \qquad (B3b')$$

which shows that the deformation amplitudes β_L are uniquely related to the static moments Q_L for a given spherical density $\rho(r)$. As was shown in Fig. 2, this "frozen deformed" static density, with $\rho(r)$ obtained from a spherical harmonic oscillator, could not fit very well the experimental charge form factor of ^{10}B. Nevertheless, it represents a possible model which may work better elsewhere.

One may equally well assume a collective deformation of an originally spherically symmetric magnetization density $\mu_s(r)$ frozen into a static deformed state. Starting from Eq. (59b) and introducing appropriate constant deformation parameters $\gamma_{L'}$, where $L'=L\pm 1$, one may write [see also Eq. (29b)]

$$\mu_{10}(r)=(4\pi)^{1/2}\hat{J}(JJ,10\,|\,JJ)^{-1}\mu_s(r), \qquad (B4a')$$

$$\mu_{LL'}(r)=-\gamma_{L'}\hat{J}(L'0,10\,|\,L0)(JJ,L0\,|\,JJ)^{-1}\times r(r/R)^{L'-2}\mu_s'(r), \qquad (LL')\neq(10). \qquad (B4b')$$

Using this, we obtain the form factors

$$(e/2m)M_1F_1{}^{M}(q)=\int j_0(qr)\mu_s(r)d^3r+(4\pi/5)^{1/2}\gamma_2q\int r^3j_1(qr)\mu_s(r)dr, \qquad (B5a)$$

$$(e/2m)M_LF_L{}^{M}(q)=(4\pi)^{1/2}L(2L-1)!!q^{2-L}R^{3-L}\times\left[\gamma_{L-1}(2L-1)^{-1/2}\int r^Lj_{L-2}(qr)\mu_s(r)dr+\gamma_{L+1}(2L+3)^{-1/2}R^{-2}\int r^{L+2}j_L(qr)\mu_s(r)dr\right], \qquad L\geq 3 \qquad (B5b)$$

and the static moments ($q\to 0$):

$$(e/2m)M_1=\int \mu_z(r)\,d^3r=(e/2m)\mu, \qquad \text{(B6a}')$$

$$(e/2m)M_L=(4\pi)^{1/2}\gamma_{L-1}L(2L-1)^{1/2}R^{3-L}\times\int r^{2L-2}\mu_z(r)\,dr, \qquad L\geq 3 \quad \text{(B6b}')$$

with the $\gamma_{L'}$ related to the static moments M_L.

APPENDIX C: HELM-TYPE MODEL

The original Helm model[6,38] combines a simple assumed (static or transition) distribution with a folded-in surface smearing. The latter is often assumed to be of Gaussian form,

$$\rho_g(\mathbf{r})=(2\pi g^2)^{-3/2}\exp(-r^2/2g^2), \qquad \text{(C1a}')$$

g being a surface thickness parameter. This assumption has the consequence that all form factors get multiplied with the Fourier transform of $\rho_g(\mathbf{r})$, i.e.,

$$f_g(q)=\exp(-g^2q^2/2); \qquad \text{(C1b}')$$

this is true also for the transverse form factors, as has been shown in Ref. 6. The original distributions which $\rho_g(\mathbf{r})$ gets folded into we shall designate by a superscript (0). In the spirit of the Helm model, one may choose for the spherically symmetric part of the charge density, $\rho_0^{(0)}(r)$, a step function $\theta(r-R)$ which cuts off at a radius R; we define

$$\theta(x)=1, \quad x<0$$
$$=0, \quad x>0. \qquad \text{(C2}')$$

The higher moments which, for example, in the frozen deformed model are given by derivatives of $\rho_0^{(0)}(r)$ one may take proportional to $\delta(r-R)$. With the proper normalization obtained from Eq. (30a) we have

$$\rho_0^{(0)}(r)=3(4\pi)^{-1/2}\hat{J}eZR^{-3}\theta(r-R), \qquad \text{(C3a}')$$

$$\rho_L^{(0)}(r)=(4\pi)^{-1/2}(JJ,L0\,|\,JJ)^{-1}\hat{L}\hat{J}R^{-L-2}\times eQ_L\delta(r-R), \qquad L\geq 2 \quad \text{(C3b}')$$

which immediately leads to the static charge form factors

$$F_0^C(q)=(3/qR)f_g(q)j_1(qR), \qquad \text{(C4a)}$$

$$F_L^C(q)=(2L+1)!!(qR)^{-L}f_g(q)j_L(qR), \qquad L\geq 2. \qquad \text{(C4b)}$$

Note the assumption is implied that the surface smearing $\rho_g(\mathbf{r})$ remains undeformed. The same procedure may be used for the magnetization density, which is likewise convoluted with $\rho_{\bar{g}}(\mathbf{r})$ (one assumes $\bar{g}\neq g$): using Eqs. (30b) for normalization, one has

$$\mu_{10}^{(0)}(r)=3(4\pi)^{-1/2}\hat{J}(JJ,10\,|\,JJ)^{-1}(e\mu/2m)\bar{R}^{-3}\theta(r-\bar{R}), \qquad \text{(C5a}')$$

$$\mu_{LL-1}^{(0)}(r)=(4\pi L)^{-1/2}\hat{J}(JJ,L0\,|\,JJ)^{-1}(eM_L/2m)\times\bar{R}^{-L-1}\delta(r-\bar{R}), \qquad L\geq 3 \quad \text{(C5b}')$$

also taking a different radius parameter $\bar{R}$. We may consider Eq. (C5b) as a direct consequence of the "frozen deformed" model of Appendix B, Eq. (B4b), which by comparison gives the deformation parameter

$$\gamma_{L-1}=\tfrac{1}{3}(4\pi/L)^{1/2}(L-10,10\,|\,L0)^{-1}R^{1-L}(M_L/\mu). \qquad \text{(C5c}')$$

Since only one type of parameter $\gamma_{L'}$ is involved, Eq. (C5c) immediately gives an expression for γ_{L+1}, which from Eqs. (B4b) permits the calculation of $\mu_{LL+1}^{(0)}(r)$ with the result

$$\mu_{LL+1}^{(0)}(r)=-(4\pi)^{-1/2}(L+1)^{1/2}(L+2)^{-1}\times\hat{J}(JJ,L0\,|\,JJ)^{-1}(eM_{L+2}/2m)\bar{R}^{-L-3}\delta(r-R), \qquad L\geq 1. \quad \text{(C5d}')$$

This leads to the static magnetic form factors

$$F_1^M(q)=f_{\bar{g}}(q)\{(3/q\bar{R})j_1(q\bar{R})+\tfrac{1}{3}(M_3/\mu)\bar{R}^{-2}j_2(q\bar{R})\}, \qquad \text{(C6a)}$$

$$F_L^M(q)=(2L-1)!!(q\bar{R})^{1-L}f_{\bar{g}}(q)\times\{j_{L-1}(q\bar{R})+[L/(L+2)](M_{L+2}/M_L)\times\bar{R}^{-2}j_{L+1}(q\bar{R})\}, \qquad L\geq 3. \quad \text{(C6b)}$$

The expressions for $\rho_L(r)$ and $\mu_{LL'}(r)$ given by the frozen deformed model, Eqs. (B1) and (B4), or by the Helm model, Eqs. (C3) and (C5), may also be used to find expressions for the integrals $I_{Ll l'}{}^n(q)$, Eqs. (54b), and $K^n{}_{L'l l' l''}(q)$, Eq. (60e), of the collective vibration model.

It is noteworthy that our previous results[6] of the Helm model for inelastic electron scattering from spherically symmetric nuclei can quite easily be derived from our general Eqs. (18) for the inelastic form factors. For example, by assuming

$$\rho_L^{if}(r)=eZi^{-L}\beta_L R^{-2}\delta(r-R) \qquad \text{(C7a)}$$

(and folding in the surface smearing), one has

$$\langle J_f\,||\,\tilde{M}_L(q)\,||\,J_i\rangle=\beta_L f_g(q)j_L(qR), \qquad \text{(C7b)}$$

our previous result. The quantity β_L is here simply understood as an amplitude parameter, to be determined by experiment, and we thus avoided defining it in the somewhat vague manner of our previous derivation.[6] Furthermore, setting

$$\mu_{LL'}{}^{if}(r)=eZi^{-L'}(\gamma_{LL'}/2m)\bar{R}^{-2}\delta(r-\bar{R}), \qquad \text{(C7c)}$$

[38] H. Crannell, R. Helm, H. Kendall, J. Oeser, and M. Yearian, Phys. Rev., 123, 923 (1961).

one obtains from Eqs. (18) our previous results[6] for the transverse magnetic transition matrix elements,

$$\langle J_f \| \tilde{T}_L{}^m(q) \| J_i\rangle = -(q/2m)f_{\bar{g}}(q)\hat{L}^{-1} \times\{L^{1/2}\gamma_{LL+1}j_{L+1}(q\bar{R})+(L+1)^{1/2}\gamma_{LL-1}j_{L-1}(q\bar{R})\}, \quad \text{(C7d)}$$

again with $\gamma_{LL'}$ a simple constant-amplitude parameter. For the electric spin-flip matrix element, one finds

$$\langle J_f \| \tilde{T}_L{}^{e\mu}(q) \| J_i\rangle = (q/2m)\gamma_{LL}f_{\bar{g}}(q)j_L(q\bar{R}), \quad \text{(C7e)}$$

as before. As to the transverse electric matrix element due to the convection current, we note that in Ref. 6 it was shown that the continuity equation of curl-free flow could be satisfied only with $j_{LL+1}{}^{if}(r)\equiv 0$ and $j_{LL-1}{}^{if}(r)$ proportional to the ground-state density $\rho_0{}^{(0)}(r)$ in the Helm model, i.e., $\theta(r-R)$. From Eqs. (18), we thus obtain the previous result

$$\langle J_f \| \tilde{T}_L{}^{ej}(q) \| J_i\rangle = \beta_L[(L+1)/L]^{1/2}(\omega/q)f_g(q)j_L(qR), \quad \text{(C7f)}$$

the factors (especially the appearance of β_L) being determined by the condition that for $q\rightarrow\omega$, M_L and $T_L{}^{ej}$ should satisfy the Siegert theorem, Eq. (20).

APPENDIX D: CHARGE AND MAGNETIZATION RADII

For the ground-state densities at small momentum transfer, one cannot determine the entire shape but only one parameter characterizing the spatial extent of the density, i.e., a root-mean-square radius. Expanding the Bessel function

$$j_0(qr)\approx 1-q^2r^2/6+\cdots, \quad \text{(D1')}$$

one finds from Eq. (38b)

$$F_0{}^C(q)\approx 1-(q^2/6)\langle r^2\rangle_c, \quad \text{(D2a)}$$

with a mean-square charge radius

$$\langle r^2\rangle_c = (4\pi)^{1/2}(Ze\hat{J})^{-1}\int r^4\rho_0(r)\,dr. \quad \text{(D2b)}$$

Since higher moments enter only to order q^4, the electric part of the elastic cross section, Eq. (38a), may be expanded as

$$(d\sigma/d\Omega)_e\approx Z^2\sigma_M[1-(q^2/3)\langle r^2\rangle_c+O(q^4)], \quad \text{(D2c)}$$

valid to order q^2. For the model of Appendix B, Eq. (D2b) becomes

$$\langle r^2\rangle_c = (Ze)^{-1}\int r^2\rho(r)\,d^3r, \quad \text{(D2d)}$$

and for the Helm model,

$$\langle r^2\rangle_c = (3/5)R^2+3g^2. \quad \text{(D2e)}$$

This equation gives a connection between the radius parameter R, usually taken as $R=1.10A^{1/3}$F, and the surface thickness g after the charge radius has been determined from experiment.

If we define a mean-square radius $\langle r^2\rangle_M$ for the magnetization density by

$$F_1{}^M(q)\approx 1-(q^2/6)\langle r^2\rangle_M, \quad \text{(D3a)}$$

then Eq. (38c) gives (through an expansion of the Bessel functions):

$$\langle r^2\rangle_M = (4\pi)^{1/2}[(e/2m)\mu_*\hat{J}]^{-1}(JJ,10\,|\,JJ) \times\left[\int r^4\mu_{10}(r)\,dr+(2^{1/2}/5)\int r^4\mu_{12}(r)\,dr\right]. \quad \text{(D3b)}$$

Higher moments contribute to order q^4 only, so the magnetic part of the elastic cross section, Eq. (38a), may be expanded as

$$(d\sigma/d\Omega)_m\approx\sigma_M(1+2\tan^2\tfrac{1}{2}\vartheta)(q^2/4m^2)(\mu^2/3) \times[(J+1)/J][1-(q^2/3)\langle r^2\rangle_M+\sigma(q^4)]. \quad \text{(D3c)}$$

Since at $\vartheta=180°$ the magnetic cross section alone survives, this formula may be used for determining the magnetization radii of light nuclei.

For the model of Appendix B the mean-square radius is given by

$$\langle r^2\rangle_M = [(e/2m)\mu]^{-1}[1-(5\pi)^{-1/2}\gamma_2]\int r^2\mu_z(r)\,d^3r, \quad \text{(D3d)}$$

and for the Helm model, by

$$\langle r^2\rangle_M = (3/5)\bar{R}^2-(2/15)(M_3/\mu)+3\bar{g}^2. \quad \text{(D3e)}$$

For the magnetization, one may choose $\bar{R}=1.25A^{1/3}$F. From the experimentally determined $\langle r^2\rangle_M$, one then gets a relation for the surface thickness $\bar{g}$, which, however, involves the ratio of the octupole moment M_3 to the dipole moment μ.