the constant phonon mean free path at very low frequencies, was found to have values much less than the specimen diameter. This could be due to a precipitation of cadmium impurity in the alloy matrix.

Finally, in order that the reader might easily obtain a feel for the relative magnitudes of the effects discussed, we summarize the results by listing in Table III the various conductivities discussed for the most impure sample, No. 7. It is evident from the last two columns that the additional phonon scattering mechanism evident in our data is of quite comparable size to the electron-phonon interaction. It is seen that, to within experimental error, the columns add properly, indicating internal consistency of analysis.

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Transport Equation for a Fermi Liquid in Random Scattering Centers. I. A Quasiparticle Description in the Macroscopic and Low-Temperature Limit*

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The transport properties of an interacting fermion gas in the presence of randomly distributed scattering centers and a weak longitudinal force field are studied on the basis of a transport equation for the bare-particle distribution function. This equation, valid for arbitrary wavelength, frequency, and temperature, is derived by a generalization of a simple method due to Résibois. For the case of electrons, the transport equation is given in terms of the mean total electric field in the medium, thereby allowing a direct calculation of the transport coefficients of physical interest. The general theory is applied to the case of a slowly and smoothly varying driving field, low temperatures, and weak and dilute scattering centers. It is shown, up to second order in the interfermion interaction, that a transport equation for a quasiparticle distribution function can be derived. This equation has the form originally suggested by Landau with the interparticle and impurity scattering terms added. The connection between the bare-particle and the quasiparticle distribution functions is also obtained.

1. INTRODUCTION

THE linear electromagnetic properties of solids, such as metals and semiconductors, at low temperatures are determined to a large extent by the impurities in the crystal. Within the one-electron approximation, these properties are calculated on the basis of quantum transport equations, which have been derived by various authors.[1–5] All these theories neglect the electron-electron interaction and are, therefore, incapable of describing *a priori* such effects as the screening of the impurity potential, electron-electron scattering, and other more subtle many-body effects. Although the effects of the screening and the electron-electron scattering have at times been considered in various applications in a qualitative way, no convincing and self-contained theory has been given as yet even for these simple effects.

Recently progress has been made in the calculation of the transport properties of a degenerate, homogeneous, normal interacting electron gas in dilute random impurities. Langer[6] in a series of papers has been able

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to evaluate the current in this system induced by a static and uniform longitudinal electric field at low temperatures, taking into account all many-body effects. Betbeder-Matibet and Nozières[7] have derived a transport equation for the Landau quasiparticles for the same system. Their equation describes the linear response of the system to an electric field of long wavelength $1/q$ and small frequency ω, taking into account again the many-body effects to all orders in the electron-electron interaction. Since it is valid, however, only at zero temperature, their equation does not describe the effects due to electron-electron scattering. Actually, such transport equations for the Landau quasiparticles in the presence of random impurities and at zero temperature have been suggested by Silin,[8] Heine,[9] and Heine *et al.*[10] on the basis of the original phenomenological theory of Landau[11–15] for the pure Fermi liquid.

The investigations of Langer[6] and Betbeder-Matibet and Nozières[7] make use of the sophisticated mathematical techniques of Green's functions and many-body perturbation theory.[14,16] They parallel the derivations of the Landau-Silin transport equation for the Fermi liquid in the absence of impurities, as suggested by Landau[11] and developed by Luttinger and Nozières[14,17] for zero temperature and by Eliashberg[18] for low temperatures. All these works are very complicated mathematically, but have the redeeming feature that they take into account rigorously the electron-electron interaction to all orders, once the convergence of the expansion in powers of the strength of the electron-electron interaction is assumed. However, so far only the macroscopic limit (i.e., $q \ll k_F$ and $\omega \ll \omega_F$, where k_F and ω_F are the wave vector and frequency associated with the Fermi surface of the system) and the case of zero temperature have been treated successfully for the homogeneous Fermi liquid in the presence of dilute impurities. That is, no transport equation describing both the impurity and interparticle scattering has been derived.

In the present work we approach the same problem of the transport properties of a homogeneous system of interacting fermions in the presence of randomly distributed scattering centers from a different point of view. We wish to focus our attention on bare particles rather than on quasiparticles and thus derive a transport equation for the bare-particle distribution function, which is sufficient for the evaluation of the linear response of all interesting one-particle observables, such as the particle and current densities. As we shall see, such a transport equation for the bare-particle distribution function can be established with fairly simple mathematical techniques and for *arbitrary* wavelength, frequency, and temperature. We thus have a sufficiently broad basis from which we may attempt various approximations in different cases. Furthermore, whenever a quasiparticle description proves possible and useful, we can find the connection between the bare-particle and the appropriate quasiparticle distribution functions.

For the case of a homogeneous normal Fermi liquid in the absence of impurities, Résibois[19] has derived recently a transport equation for the bare-particle distribution function by simplifying and improving a method due to Konstantinov and Perel.[20] The method is based on a reclassification of an expansion of the distribution function in powers of the interparticle interaction. This is carried out with the help of a suitable diagrammatic representation of the terms of the series, once use is made of the usual contraction theorem.[21] The coefficients of the transport equation are given in a power series in the interparticle interaction, represented by appropriate diagrams. Résibois[22] applied this transport equation to a neutral Fermi liquid (i.e., with short-range interaction) in the macroscopic limit ($q \ll k_F$, $\omega \ll \omega_F$) and at zero temperature and proved, up to second order of the interparticle interaction, that the Landau[11] transport equation for a quasiparticle distribution function results. The connection between the bare-particle and quasiparticle distribution functions was also found. Watabe and Dagonnier[23] considered the collision term at low temperatures within the same framework and were able to prove that it can be arranged in the form of the interquasiparticle scattering term suggested by Landau.[11]

In this work we generalize the method of Résibois[19] so that it can be applied to the case of the inhomogeneous system of interest, namely the interacting fermions in the presence of impurities. The bare-particle distribution function is defined and its average over a random

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distribution of the impurities is formally expanded in powers of both the interparticle and impurity interactions in Sec. 2. In Sec. 3, the diagrammatic representation of the terms of the series is given with particular emphasis on the new elements that the presence of the impurities and the process of averaging over their distribution engender. Some important lemmas, that eliminate or simplify a large number of diagrams, are stated and their proofs indicated in the same section. The statements and/or proofs of these lemmas have been generalized to allow for the presence of the impurities. In Sec. 4, we make use of some of these lemmas to reclassify the diagrams representing the distribution function and thereby derive, quite simply, a general transport equation for the impurity-averaged bare-particle distribution function for arbitrary wave vector and frequency of the longitudinal driving field and all temperatures. The coefficients of the transport equation are given in a power series in both the interparticle and impurity interactions.

The important case of electrons, interacting with a long-range Coulomb force, is considered in detail in Sec. 5. In this case it is desirable, and at times necessary, to express the distribution function in terms of the average electric field in the medium, rather than the external electric field. We accomplish this within the framework of this theory by showing that, in terms of the mean total electric field, the distribution function is given by a subset of diagrams, all of which are regular as $q \to 0$. A transport equation for the distribution function is then obtained with coefficients which are regular as $q \to 0$.

In Sec. 6 we carry out the first application of this theory. We examine the transport equation in the macroscopic limit ($q \ll k_F$, $\omega \ll \omega_F$) for low temperatures and dilute and weak scattering centers. This case is of interest in the theory of the transport properties of metals. The scattering due to impurities is evaluated to first order in their concentration and the first Born approximation for the cross section of each scattering center. The effects of the interparticle interaction are considered only up to second order. Higher-order terms are forbiddingly complicated and, on the other hand, a second-order calculation provides most of the physical information of interest. We show that the transport equation for the bare-particle distribution function contains terms of obscure physical significance. However, a transformation is possible that introduces a quasiparticle distribution function which is shown to satisfy a transport equation of the Landau[11] form with interquasiparticle and quasiparticle-impurity scattering terms simply added. This quasiparticle distribution function, however, which is appropriate when the impurities are present, is different from that of the pure Fermi liquid.

We wish to report that we have applied the general theory developed here to the derivation of a transport equation for a system of dynamically independent fermions interacting with random impurities. No restrictions on the wave vector, the frequency, or the temperature have been made. The coefficients of the equation have been calculated to *all* orders in the potential of a single impurity and up to second order in the impurity density. This equation represents a generalization of the one derived by Luttinger and Kohn.[1] The details of this work will appear in a future publication.

We plan to apply the general theory developed here to other cases of more physical interest in subsequent publications.

2. BARE-PARTICLE DISTRIBUTION FUNCTION

In this section, we define and study formally the bare-particle distribution function for a normal Fermi liquid in the presence of random scattering centers. This function determines the linear response of some one-particle properties of the system to an external longitudinal field of force.

In the absence of any external field, the system under consideration consists of n_p interacting fermions in the presence of n_i impurities, all within a unit volume. Its Hamiltonian,

$$H=H_0+H'=H_0+V_p+V_i, \tag{2.1}$$

consists of the kinetic energy H_0, the interparticle interaction V_p, and the interaction of the particles with the impurities V_i. In terms of the creation and destruction operators $c_k^\dagger$, c_k for fermions, we have

$$H_0=\sum_k \epsilon_k c_k^\dagger c_k, \quad \epsilon_k=\mathbf{k}^2/2m \tag{2.2}$$

where k denotes the wave vector $\mathbf{k}$ of a plane-wave one-particle state and its spin σ, and $\hbar$ is taken as unity. The interparticle interaction energy is

$$V_p=\tfrac{1}{4}\sum_{kll'k'} V(kll'k')c_k^\dagger c_l^\dagger c_{l'}c_{k'}; \tag{2.3}$$

the use of the antisymmetrized interaction matrix element

$$V(kll'k')=[v(k-k')-v(k-l')]\delta_{k+l,k'+l'} \tag{2.4}$$

will bring about an important simplification later in the diagrammatic analysis. Here $v(k-k')=v(\mathbf{k}-\mathbf{k}')\delta_{\sigma\sigma'}$ is the Fourier transform of the interparticle potential $v_p(\mathbf{r}_i-\mathbf{r}_j)$, i.e.,

$$v(\mathbf{q})=\int d^3r\, e^{-i\mathbf{q}\cdot\mathbf{r}}v_p(\mathbf{r}), \tag{2.5}$$

with $v(\mathbf{q})$ a real and even function of $\mathbf{q}$. The interaction energy of the particles with the set of n_i identical impurities fixed at positions $\mathbf{R}_j$ is taken to be of the

form

$$V_i=\sum_i \sum_{j=1}^{n_i} u(\mathbf{r}_i-\mathbf{R}_j)$$

$$=\sum_{kk'}[u(k-k')\sum_{j=1}^{n_i} e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}_j}]c_k^\dagger c_{k'}. \quad (2.6)$$

Here $u(k-k')$ is the Fourier transform of the interaction between a particle and a single impurity, i.e.,

$$u(k-k')=\delta_{\sigma\sigma'}\int d^3r\, e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}u(\mathbf{r}). \quad (2.7)$$

Without loss of generality we take $u(\mathbf{q}=0)=0$.

In the presence of an external longitudinal field of force of wave vector $\mathbf{q}$ and frequency ω,

$$\mathbf{F}_{q\omega}e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}=-i\mathbf{q}\phi_{q\omega}e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}, \quad (2.8)$$

the interaction Hamiltonian is

$$H''(t)=\phi_{q\omega}e^{-i\omega t}\sum_k \rho_{-q}{}^k. \quad (2.9)$$

Here we have introduced the operator

$$\rho_q{}^k=c_{k-q/2}{}^\dagger c_{k+q/2}, \quad (2.10)$$

and ω is understood to have a positive imaginary infinitesimal part to insure the adiabatic switching of the field.

We assume that in the remote past, i.e., before the external field has become significant, the system is in thermal equilibrium at temperature $T=1/k\beta$. Its density matrix is then

$$\rho_0=e^{-\beta(H-\mu N)}/\mathrm{Tr}e^{-\beta(H-\mu N)}, \quad (2.11)$$

where N is the fermion number operator and μ, the chemical potential, is determined from the condition $\mathrm{Tr}\rho_0 N=n_p$. The density matrix at time t, up to terms linear in the external field, is given by first-order perturbation theory as

$$\rho(t)=\rho_0-i\int_0^\infty d\tau\, e^{-iH\tau}[H''(t-\tau),\rho_0]e^{iH\tau}. \quad (2.12)$$

Using an identity given by Kubo[24] which expresses $[H'',\rho_0]$ in terms of the Fourier transform of the current density operator,

$$J_{-q}{}^\alpha=\sum_k \frac{k_\alpha}{m}\rho_{-q}{}^k=\sum_k \frac{k_\alpha}{m}c_{k+q/2}{}^\dagger c_{k-q/2}, \quad (2.13)$$

we find, with the help of (2.8), for the steady-state linear response

$$\rho(t)-\rho_0=e^{-i\omega t}F_{q\omega}{}^\alpha\int_0^\infty d\tau\, e^{i\omega\tau}\int_0^\beta d\gamma\, e^{-iH\tau}\rho_0$$

$$\times e^{\gamma H}J_{-q}{}^\alpha e^{-\gamma H}e^{iH\tau}. \quad (2.14)$$

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(We use the summation convention for repeated Cartesian indices.) Note that the simple form (2.13) of the current density is a consequence of the fact that all forces on the fermions can be described in terms of scalar potentials independent of the velocities of the particles.

We are specifically interested in the response of the system as it manifests itself on quantities like the induced current density, particle density, and, in general, all one-particle quantities. Let A be the sum of one-particle operators, i.e., of the form

$$A=\sum_i a(i)=\sum_{kq'}\langle k-\tfrac{1}{2}q'|a|k+\tfrac{1}{2}q'\rangle\rho_{q'}{}^k, \quad (2.15)$$

where $\rho_q{}^k$ is given by (2.10); then the change in the mean value of such an observable induced by the external field is given by

$$\mathrm{Tr}\{A[\rho(t)-\rho_0]\}=\sum_{kq'}\langle k-\tfrac{1}{2}q'|a|k+\tfrac{1}{2}q'\rangle f_{q'}(kt), \quad (2.16)$$

where

$$f_{q'}(kt)=\mathrm{Tr}\{\rho_{q'}{}^k[\rho(t)-\rho_0]\} \quad (2.17)$$

is the bare-particle distribution function for a particular arrangement of the impurities. The observed macroscopic quantities are determined by an ensemble average over an appropriate distribution of impurity centers. In the following we shall take this distribution to be completely random, i.e., we define the averaged bare-particle distribution function as

$$\langle f_{q'}(kt)\rangle_i\equiv\int d^3R_1\int d^3R_2\cdots f_{q'}(kt). \quad (2.18)$$

With this we can calculate, using (2.16), macroscopic quantities such as the induced current and particle densities, for which the corresponding operators a do not depend on the impurity distribution. The important feature of this averaging is that it restores "momentum conservation." We conclude that $\langle f_{q'}(kt)\rangle_i$ is different from zero only for $\mathbf{q}'=\mathbf{q}$, the wave vector of the driving field. Using (2.14) and the cyclic invariance of the trace, we have for the desired averaged bare-particle distribution function

$$\langle f_q(kt)\rangle_i=f_{q\omega}(k)e^{-i\omega t}, \quad (2.19)$$

where

$$f_{q\omega}(k)=F_{q\omega}{}^\alpha\int_0^\infty d\tau\, e^{i\omega\tau}$$

$$\times\int_0^\beta d\gamma\langle\mathrm{Tr}\{\rho_0 e^{\gamma H}J_{-q}{}^\alpha e^{-\gamma H}e^{iH\tau}\rho_q{}^k e^{-iH\tau}\}\rangle_i. \quad (2.20)$$

The primary purpose of this work is to derive a general transport equation for $f_{q\omega}(k)$ for neutral and charged normal Fermi liquids.

In order to obtain a transport equation for $f_{q\omega}(k)$, we first rewrite (2.20) in terms of the time-development

operators which are defined by the equations

$$e^{-iH\tau}=e^{-iH_0\tau}U(\tau,0), \qquad e^{iH\tau}=U(0,\tau)e^{iH_0\tau}, \tag{2.21}$$

$$e^{-\gamma H}=e^{-\gamma H_0}U(-i\gamma,0), \qquad e^{\gamma H}=U(0,-i\gamma)e^{\gamma H_0}. \tag{2.22}$$

We have from (2.20)

$$f_{q\omega}(k)=F_{q\omega}{}^{\alpha}\int_0^{\infty}d\tau\, e^{i\omega\tau}\int_0^{\beta}d\gamma\langle\langle U(-i\beta,-i\gamma)J_{-q}{}^{\alpha}(-i\gamma)$$

$$\times U(-i\gamma,0)U(0,\tau)\rho_q{}^{k}(\tau)U(\tau,0)\rangle\langle U(-i\beta,0)\rangle^{-1}\rangle_i, \tag{2.23}$$

where $J_{-q}{}^{\alpha}(\tau)$ and $\rho_q{}^{k}(\tau)$ are the corresponding operators in the interaction picture for either real or imaginary times, e.g.,

$$c_k(\tau)=e^{iH_0\tau}c_ke^{-iH_0\tau}=e^{-i\epsilon_k\tau}c_k, \tag{2.24}$$

and $\langle\cdots\rangle$ denotes the average over the free particle grand-canonical ensemble, i.e.,

$$\langle A\rangle\equiv \mathrm{Tr}\{e^{-\beta(H_0-\mu N)}A\}/\mathrm{Tr}e^{-\beta(H_0-\mu N)}. \tag{2.25}$$

A perturbation expansion of $f_{q\omega}(k)$ in powers of $H'=V_p+V_i$ is obtained by inserting in (2.23) the expansions for the quantities U. These are

$$U(\tau,0)=\sum_{r=0}^{\infty}(-i)^r\int_0^{\tau}dt_r\cdots\int_0^{t_2}dt_1H'(t_r)\cdots H'(t_1), \tag{2.26}$$

$$U(0,\tau)=\sum_{p=0}^{\infty}i^p\int_0^{\tau}dt_p'\cdots\int_0^{t_2'}dt_1'H'(t_1')\cdots H'(t_p'), \tag{2.27}$$

$$U(-i\lambda,-i\lambda_0)=\sum_{n=0}^{\infty}(-1)^n\int_{\lambda_0}^{\lambda}d\gamma_n\cdots\int_{\lambda_0}^{\gamma_2}d\gamma_1H'(-i\gamma_n)\cdots H'(-i\gamma_1), \tag{2.28}$$

where $H'(t)$ is the perturbation H' in the interaction picture, defined as in (2.24). The resulting imaginary time integrals can easily be rearranged so that all integration variables γ_i can be put in increasing order form 0 to β, i.e., $\beta\geq\cdots\geq\gamma_{i+1}\geq\gamma_i\geq\cdots\geq 0$. The real time integrations, on the other hand, consist of two sets of integrations, arising from $U(\tau,0)$ and $U(0,\tau)$, the limits in one set being independent of those in the other. As Résibois[19] pointed out, great advantage results if all integration variables are arranged in an increasing sequence. Such an ordering greatly facilitates the real-time integrations. This transformation is achieved with the help of the following identity[19] for arbitrary functions $\phi_i(t), \phi_i'(t)$:

$$\int_0^{\tau}dt_p'\cdots\int_0^{t_2'}dt_1'\,\phi_p'(t_1')\cdots\phi_1'(t_p')\int_0^{\tau}dt_r\cdots\int_0^{t_2}dt_1\,\phi_r(t_r)\cdots\phi_1(t_1)$$

$$=\int_0^{\tau}d\tau_{p+r}\cdots\int_0^{\tau_2}d\tau_1\sum_{\mathcal{P}}\mathcal{P}\phi_p'(\tau_\alpha)\cdots\phi_1'(\tau_\beta)\phi_r(\tau_\gamma)\cdots\phi_1(\tau_\delta), \tag{2.29}$$

where $\mathcal{P}$ represents all possible permutations of the $(p+r)$ arguments such that $\tau_\alpha<\cdots<\tau_\beta$ and $\tau_\gamma>\cdots>\tau_\delta$. This identity expresses two series of integrals, each running independently from 0 to τ, as a sum of integrals ordered along a single time scale 0 to τ. Thus, with the use of (2.29), we find for the averaged bare-particle distribution function the perturbation expansion

$$f_{q\omega}(k)=F_{q\omega}{}^{\alpha}\sum_{n,m=0}^{\infty}\sum_{p,r=0}^{\infty}(-1)^{n+m}i^p(-i)^r\int_0^{\beta}d\gamma_{n+m+1}\cdots\int_0^{\gamma_2}d\gamma_1\int_0^{\infty}d\tau e^{i\omega\tau}\int_0^{\tau}d\tau_{p+r}\cdots\int_0^{\tau_2}d\tau_1$$

$$\times\sum_{\mathcal{P}}\mathcal{P}\langle\langle H'(-i\gamma_{n+m+1})\cdots J_{-q}{}^{\alpha}(-i\gamma_{n+1})\cdots H'(-i\gamma_1)H'(\tau_\alpha)\cdots H'(\tau_\beta)$$

$$\times\rho_q{}^{k}(\tau)H'(\tau_\gamma)\cdots H'(\tau_\delta)\rangle\langle U(-i\beta,0)\rangle^{-1}\rangle, \tag{2.30}$$

where the permutation operator $\mathcal{P}$ has been defined in (2.29). The factor $\langle U(-i\beta,0)\rangle^{-1}$ has not been expanded in powers of H' for reasons to become clear in the following.

By decomposing the expression (2.30) in a way which is most conveniently represented by a certain set of diagrams, we shall be able to derive a transport equation for $f_{q\omega}(k)$. A systematic expansion of the coefficients of this transport equation will also be represented in terms of diagrams.

3. DIAGRAMMATIC REPRESENTATION

In this section, we develop the connection, previously mentioned, between the expansion (2.30) and a certain set of diagrams. These diagrams, which represent the various terms in the expansion, are quite similar to those of Résibois.[19] Differences occur, however, because of the presence of the impurities and because of our use of the antisymmetrized matrix element of the interparticle interaction (2.3). The use of the matrix element (2.4) results in the advantage that both direct and exchange interactions are handled together, so that fewer diagrams need be considered. At the end of this section, we shall consider the method by which the average over the random distribution of impurities is incorporated in the diagram technique.

From (2.3), (2.6), and (2.13) we note that $H'(\tau)$ and $J_{-q}{}^{\alpha}(\tau)$ are just sums of products of destruction and creation operators in the interaction picture, as given by (2.24). Thus, for the evaluation of (2.30), we need the average of a product of fermion operators over the free-particle grand-canonical ensemble as in (2.25). According to the theorem of Bloch and DeDominicis,[21] such an average is given by the sum of all possible complete systems of contractions of pairs of fermion operators. The only nonvanishing contractions are

$$\langle c_k{}^{\dagger}(\tau)c_k(\tau')\rangle = e^{i\epsilon_k(\tau-\tau')}F_k\,,$$
$$\langle c_k(\tau')c_k{}^{\dagger}(\tau)\rangle = e^{i\epsilon_k(\tau-\tau')}(1-F_k)\,, \qquad (3.1)$$

where

$$F_k=[1+e^{\beta(\epsilon_k-\mu)}]^{-1} \qquad (3.2)$$

is the Fermi distribution function for the unperturbed energy ϵ_k, and μ the chemical potential. Note that (3.1) holds also for imaginary τ, τ'.

We shall represent a complete system of contractions by an appropriate diagram. To begin with, in order to represent the time sequence of the various operators as they appear in (2.30), (a) we draw r dots at points $\tau_\delta, \cdots, \tau_\gamma$ just above a horizontal axis running from 0 on the right to τ on the left; these dots correspond, respectively, to $H'(\tau_\delta), \cdots, H'(\tau_\gamma)$ [see Fig. 1(a)]. The end of the axis at τ represents $\rho_q{}^k(\tau)$. (b) We draw p dots at points $\tau_\alpha, \cdots, \tau_\beta$ corresponding to $H'(\tau_\alpha), \cdots, H'(\tau_\beta)$ just below the horizontal axis. Note that a particular permutation $\mathcal{P}$ in (2.30) corresponds to a particular ordering of the p dots below the axis in relation to the r dots above the axis and that all permutations $\mathcal{P}$ are taken into account by considering all such possible relative orderings of the dots below the axis with respect to the dots above the axis. (c) Along a vertical axis connected to the horizontal axis at 0 and running down from 0 to $-i\beta$, we draw $(n+m)$ dots at the points $-i\gamma_1, \cdots, -i\gamma_n, -i\gamma_{n+2}, \cdots, -i\gamma_{n+m+1}$ corresponding to $H'(-i\gamma_{n+m+1}), \cdots, H'(-i\gamma_1)$ and a dot at $-i\gamma_{n+1}$ corresponding to $J_{-q}{}^{\alpha}(-i\gamma_{n+1})$ [see Fig. 1(a)]. Thus, for a given arrangement of the dots the ordering of two fermion operators in a contraction is uniquely determined. Since each fermion operator is associated with a dot, the operator associated with the dot farther along the oriented contour C shown in Fig. 1(b) is always placed to the left of the other operator in the contraction. Of course, it follows from the form of the Hamiltonian that in a contraction of two operators associated with the same dot the creation operator is always placed to the left of the destruction operator.

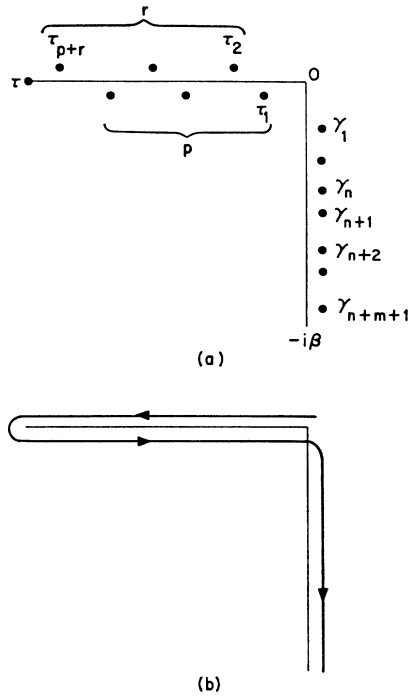

FIG. 1. (a) The time sequence of the operators in the expansion (2.30). (b) The contour C determining the order of operators in contractions.

We can now draw a diagram corresponding to a complete system of contractions in the usual way: A contraction of two operators $c_k{}^{\dagger}$, $c_{k'}$ is drawn as a solid line directed from the vertex where $c_k{}^{\dagger}$ is found to the vertex at which $c_{k'}$ appears (this is a fermion line of momentum k). Since $H'=V_p+V_i$, we have two types of vertices for H'. The interparticle vertex V_p is drawn simply as the point of intersection of two fermion lines.[25] The impurity vertex V_i is indicated by a point where two fermion lines meet a dashed line of (incoming) momentum $k-k'$ (k being the momentum of outgoing fermion line and k' the momentum of the incoming line). The current vertex $J_{-q}{}^{\alpha}$ is indicated by a point at $-i\gamma_{n+1}$ where two fermion lines (of incoming and outgoing momenta $k'-\frac{1}{2}q$ and $k'+\frac{1}{2}q$, respectively) meet a wavy line of momentum $\mathbf{q}$. Finally, the vertex $\rho_q{}^k$ is simply denoted by the end of the horizontal axis at τ, where one fermion line of momentum $k-\frac{1}{2}q$ leaves and another of momentum $k+\frac{1}{2}q$ enters. Because of the

[25] For more details, see Ref. 14.

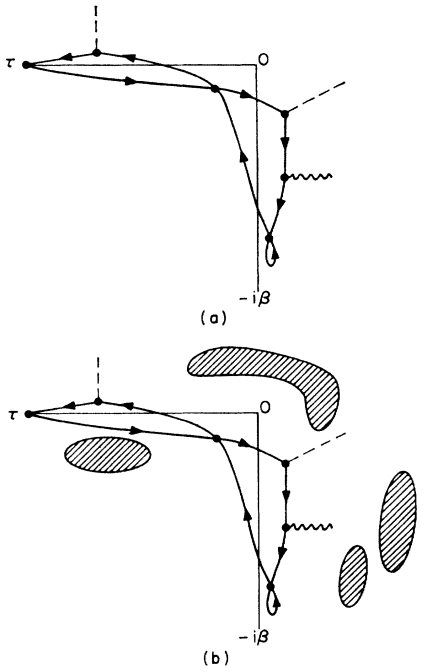

FIG. 2. (a) G_c: a typical connected diagram for the distribution function before impurity averaging. (b) A disconnected diagram with a connected part G_c.

summation over the permutations $\mathcal{P}$, diagrams with different relative orderings between the vertices above and below the horizontal axis are distinct, i.e., the contributions of all such diagrams should be added. A typical diagram contributing to $f_{q\omega}(k)$ is found in Fig. 2(a).

The contribution of a diagram to $f_{q\omega}(k)$ can easily be written down with the help of (3.1), the Bloch-DeDominicis theorem[21] and (2.30). Before we do this, however, we shall note and prove a few lemmas which considerably simplify (2.30) and make possible the derivation of a transport equation for $f_{q\omega}(k)$.

Lemma 1. Any diagram with a real time vertex to the left of all fermion lines directly connected to it may be ignored.

To see this, we note that for each diagram with such a vertex below the real time axis, there is another diagram that differs from it only in that the vertex in question is above the axis. The condition that the vertex be to the left of the lines connected to it implies that the orientation of the lines with respect to the contour C in Fig. 1(b) is the same in both diagrams and thus their contributions to (2.30) are equal except for a difference in sign due to the factors $i^p(-i)^r$. It follows that their sum vanishes.

Lemma 2. In (2.30) we can keep only the connected diagrams for the numerator and ignore the denominator.

This is the "linked cluster theorem" for this problem and it is of the essence in any perturbation theory of a many-body problem. To prove this lemma, we note first that we can expand the denominator $\langle U(-i\beta,0)\rangle$ in a power series in H', as in (2.28), and represent it by diagrams. These diagrams are similar to the ones we have described above, except that their vertices lie only along the imaginary axis and do not include the vertex $J_{-q}{}^{\alpha}(-i\gamma_{n+1})$. Thus, the graphs G_U for $\langle U(-i\beta,0)\rangle$ can be components of a disconnected graph for $f_{q\omega}(k)$. Let us now consider a particular connected graph G_c of $f_{q\omega}(k)$ [which, therefore, includes the $\rho_q{}^k$ and $J_{-q}{}^{\alpha}$ vertices; see Fig. 2(a)] as well as all possible disconnected graphs with G_c as a component [see Fig. 2(b), for example] and sum their contributions to $f_{q\omega}(k)$. First note that the sum of the disconnected graphs with real time vertices (other than the vertices of G_c) is zero; this results from lemma 1. Thus, only disconnected graphs with vertices (other than those of G_c) exclusively on the imaginary time axis need be considered. That is, in the set under consideration we need keep only the graph G_c along with any of the graphs G_U for $\langle U(-i\beta,0)\rangle$. We can now sum the contributions of all disconnected graphs for the numerator of $f_{q\omega}(k)$ consisting of G_c and a particular G_U with its vertices in any possible relative ordering with respect to the vertices of G_c. Since none of the factors in the contributions of G_c and G_U is altered upon change of the relative ordering of the vertices of one part with respect to the other, the sum is given by the product of the contributions of G_c by itself and G_U by itself. We therefore conclude that the contribution to $f_{q\omega}(k)$ of all graphs with the same connected part G_c is just the contribution of G_c alone, ignoring the factor $\langle U(-i\beta,0)\rangle^{-1}$ in (2.30).

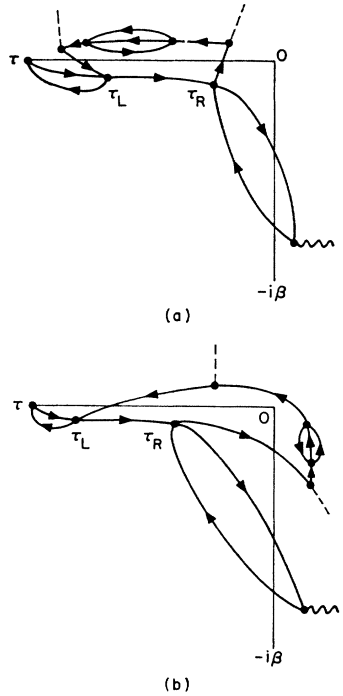

FIG. 3. (a) A diagram for $f_{q\omega}(k)$ with a "left insertion." (b) A diagram for $f_{q\omega}(k)$ with a "right insertion." (The bunching of impurity lines is not indicated.)

In order to complete the proof, we must show that a diagram in which the part connected to the $\rho_q{}^k$ vertex is disconnected from the part connected to the $J_{-q}{}^\alpha$ vertex may be ignored. For this purpose, we observe that in the case of the homogeneous system with no impurities such graphs are trivially vanishing, because neither part conserves momentum. When impurities are present, however, this argument does not hold and a different proof is necessary. We first note, as before, that summing over all relative time orderings of the vertices of the two different parts of the diagram, we can make the time (real and imaginary) integrals of the two parts completely independent of each other, and thus conclude that, leaving aside the integration

$$\int_0^\infty d\tau\, e^{i\omega\tau},$$

we can multiply the contributions of each part as if the other part were not present. Next, let us consider the sum of all parts connected to the $J_{-q}{}^\alpha$ vertex. If any of these parts has a vertex on the real time axis, then it has one to the left of all lines connected to it; by lemma 1, we can ignore all such parts. The sum of all other parts with vertices confined to the imaginary time axis is proportional to

$$Q=\int_0^\beta d\gamma\langle U(-i\beta,-i\gamma)J_{-q}{}^\alpha(-i\gamma)U(-i\gamma,0)\rangle \times\langle U(-i\beta,0)\rangle^{-1}, \quad (3.3)$$

as can easily be seen by a perturbation expansion of (3.3). But Q can also be written as

$$Q=\int_0^\beta d\gamma\ \mathrm{Tr}\{\rho_0 e^{\gamma H}J_{-q}{}^\alpha e^{-\gamma H}\}=\beta\ \mathrm{Tr}\{\rho_0 J_{-q}{}^\alpha\} \quad (3.4)$$

due to the invariance of the trace under cyclic permutation. The last expression of (3.4), however, vanishes, as it is the average current density in thermal equilibrium. Thus, the sum of such disconnected graphs for $f_{q\omega}(k)$ vanishes.

The lemma having been established, we can now rewrite (2.30) as

$$f_{q\omega}(k)=F_{q\omega}{}^\alpha \sum_{n,m=0}^{\infty}\sum_{p,r=0}^{\infty}(-1)^{n+m}i^p(-i)^r\int_0^\beta d\gamma_{n+m+1}\cdots\int_0^{\gamma_2}d\gamma_1\int_0^\infty d\tau\, e^{i\omega\tau}\int_0^\tau d\tau_{p+r}\cdots\int_0^{\tau_2}d\tau_1$$

$$\times\sum_{\mathcal{P}}\mathcal{P}\langle\langle H'(-i\gamma_{n+m+1})\cdots J_{-q}{}^\alpha(-i\gamma_{n+1})\cdots H'(-i\gamma_1)H'(\tau_\alpha)\cdots H'(\tau_\beta)\rho_q{}^k(\tau)H'(\tau_\gamma)\cdots H'(\tau_\delta)\rangle_c\rangle_i, \quad (3.5)$$

where now the subscript c prescribes that only connected diagrams are to be summed.

In order to discuss the next lemma, we first define an "insertion": this is a part of a graph for $f_{q\omega}(k)$ which includes neither the $J_{-q}{}^\alpha$ nor the $\rho_q{}^k$ vertices and is connected to the rest of the graph by only one entering and one exiting fermion line. Furthermore, a "left insertion" is an insertion in a fermion line with *any* of its vertices lying to the left of both ends of that line [see Fig. 3(a)]. Clearly, a graph with a left insertion can be ignored, according to lemma 1. A "right insertion" is defined as an insertion in a fermion line with *all* its vertices to the right of both ends of the line [see Fig. 3(b)].

Lemma 3. Consider the graphs of $f_{q\omega}(k)$ which have right insertions on a line leaving [entering] a real time vertex τ_R with momentum k and entering [leaving] a real time vertex $\tau_L(\tau_L>\tau_R)$ with momentum l. In calculating the sum of these graphs, we may ignore all right insertions with *real* time vertices if we replace the factor $e^{i\epsilon_k\tau_R}[e^{-i\epsilon_k\tau_R}]$, which occurs because the line leaves [enters] the vertex τ_R, by $e^{i\epsilon_l\tau_R}[e^{-i\epsilon_l\tau_R}]$. In other words, the contribution of the fermion line between vertices at τ_R and τ_L plus the sum of the contributions of this line with all possible right insertions may be written

$$\mathrm{Tr}\{\rho_0 c_l c_k{}^\dagger\}e^{i\epsilon_l(\tau_R-\tau_L)} \quad \text{or} \quad (-1)\mathrm{Tr}\{\rho_0 c_k{}^\dagger c_l\}e^{i\epsilon_l(\tau_R-\tau_L)}$$
$$[(-1)\ \mathrm{Tr}\{\rho_0 c_l{}^\dagger c_k\}e^{-i\epsilon_l(\tau_R-\tau_L)} \quad \text{or} \quad \mathrm{Tr}\{\rho_0 c_k c_l{}^\dagger\}e^{-i\epsilon_l(\tau_R-\tau_L)}], \quad (3.6)$$

respectively, for the case in which τ_L lies farther along the oriented contour C in Fig. 1(b) than τ_R or the inverse.

To prove these assertions, first note that in considering a right insertion, we can sum all diagrams which differ only in the relative orderings of the vertices in the right insertion with respect to the vertices in the rest of the diagram. This makes the integrations involving the time arguments of the vertices in the right insertion independent of all the other times in the diagram (except, of course, that τ_R provides an upper limit of these integrations). Thus, the sum of the contributions of all right insertions in the line between τ_R and τ_L is [for the case: momentum k leaving and τ_L beyond τ_R in the oriented contour C of Fig. 1(b)]

$$S=e^{i(\epsilon_k\tau_R-\epsilon_l\tau_L)}\langle U(-i\beta,0)U(0,\tau_R)c_l c_k{}^\dagger U(\tau_R,0)\rangle_c$$
$$=e^{i\epsilon_l(\tau_R-\tau_L)}\langle U(-i\beta,0)U(0,\tau_R)c_l(\tau_R)c_k{}^\dagger(\tau_R)U(\tau_R,0)\rangle \times\langle U(-i\beta,0)\rangle^{-1}. \quad (3.7)$$

The second form of (3.7) can be rewritten with the use

of (2.21) and (2.22) as

$$S=e^{i\epsilon l(\tau_R-\tau_L)}\,\mathrm{Tr}\{\rho_0 e^{iH\tau_R}c_l e^{-iH_0\tau_R}e^{iH_0\tau_R}c_k^{\dagger}e^{-iH\tau_R}\}$$
$$=e^{i\epsilon l(\tau_R-\tau_L)}\,\mathrm{Tr}\{\rho_0 c_l c_k^{\dagger}\}$$
$$=e^{i\epsilon l(\tau_R-\tau_L)}\langle U(-i\beta,0)c_l c_k^{\dagger}\rangle_c, \quad (3.8)$$

the second equality following from the cyclic invariance of the trace. This proves the lemma.

The importance of the lemma is seen as follows: Since all right insertions with real time vertices can be ignored, the time dependence of the sum of all contributing diagrams can be thought of as the time dependence of a line going directly from τ_R to τ_L with momentum l, i.e., $e^{\pm i\epsilon l(\tau_L-\tau_R)}$. This feature is crucial, as we shall see, for the derivation of a transport equation.

We wish to mention here that lemma 3, proven above, is a generalization to the case of inhomogeneous systems of a lemma used by Résibois[19] for homogeneous systems.

The real time integrations in (3.5) can be carried out explicitly, in view of the fact that every contraction has the simple time dependence shown in (3.1). The following lemma gives the result of these integrations in a convenient form.

Lemma 4. For a diagram of (3.5), the real time integrations yield the result

$$\prod_{j=1}^{p+r+1}\frac{1}{i(\Delta\epsilon_j-\omega)}, \quad (3.9)$$

where

$\Delta\epsilon_j=$[sum of the energies of all fermion lines going from right to left between τ_{j-1} and τ_j minus the sum of the energies of all fermion lines going from left to right in the same time interval; the two fermion lines connecting a right insertion to the rest of the diagram are to be ignored for all $\tau_j\leq\tau_R$, where τ_R is the time associated with the vertex to which the right-hand end of the right insertion is connected]. (3.10)

Here we have put $\tau_0=0$ and $\tau_{p+r+1}=\tau$.

The proof is easy, once we note that the real time integrals for a connected graph, like the ones in (3.5), can be expressed as convolutions over the time intervals $\tau_j-\tau_{j-i}$ $(j=1,\cdots,p+r+1)$ of the function

$$\exp[-i(\Delta\epsilon_j-\omega)(\tau_j-\tau_{j-1})],$$

i.e.,

$$\int_0^{\infty}d\tau_{p+r+1}\cdots\int_0^{\tau_2}d\tau_1\prod_{j=0}^{p+r+1}e^{-i(\Delta\epsilon_j-\omega)(\tau_j-\tau_{j-1})}, \quad (3.11)$$

where $\Delta\epsilon_j$ is given by (3.10). The multiple integral in (3.11) can be carried out by elementary means and is found to give (3.9). The provisions for the fermion lines connecting to a right insertion (with vertices on the imaginary time axis) are easily seen to be the result of lemma 3.

The integrations over the imaginary times can be carried out in a similar way. However, since they are not essential in the derivation of the transport equation for $f_{q\omega}(k)$, we shall not do this.

The last point to consider in connection with the evaluation of (3.5) is the averaging over the random distribution of the impurities. This point has been discussed by Edwards[26] and has been applied to a system with interparticle interactions by Langer.[6] From (2.6) we see that in any diagram of (3.5) with r impurity (dashed) lines we have the factor

$$\prod_{s=1}^{r}(u(q_s)\sum_{j=1}^{n_i}e^{-i\mathbf{q}_s\cdot\mathbf{R}_j}), \quad (3.12)$$

where $q_s=k_s-k_s'$ (k_s, k_s' being the momenta of outgoing and incoming fermion lines at the sth impurity vertex, respectively). Thus, impurity averaging amounts to evaluating, according to (2.18),

$$\langle\prod_{s=1}^{r}(\sum_{j=1}^{n_i}e^{-i\mathbf{q}_s\cdot\mathbf{R}_j})\rangle_i$$
$$=\int d^3R_1\cdots\prod_{s=1}^{r}(\sum_{j=1}^{n_i}e^{-i\mathbf{q}_s\cdot\mathbf{R}_j}). \quad (3.13)$$

Separating the integrand into terms that involve one $\mathbf{R}$, two $\mathbf{R}$'s, etc., it is easily found[6] that in the thermodynamic limit with an impurity density n_i, (3.13) can be written as a series in powers of n_i, namely,

$$\langle\prod_{s=0}^{r}(\sum_{j=1}^{n_i}e^{-i\mathbf{q}_s\cdot\mathbf{R}_j})\rangle_i=n_i\delta(\sum_{s=1}^{r}\mathbf{q}_s,\mathbf{0})$$
$$+n_i^2\sum_{\{\alpha\},\{\beta\}}{}'\,\delta(\sum_{a\in\{\alpha\}}\mathbf{q}_a,\mathbf{0})\delta(\sum_{b\in\{\beta\}}\mathbf{q}_b,\mathbf{0})+\cdots. \quad (3.14)$$

Here $\{\alpha\}$, $\{\beta\}$ are any pair of nonempty sets into which the set $\{1,\cdots,r\}$ can be partitioned. The sum $\sum'$ is a sum over all possible distinguishable pairs of sets $\{\alpha\}$, $\{\beta\}$, and $\delta(a,0)$ is the usual Kronecker delta $\delta_{a,0}$. The result (3.14) expresses the connection between the average over impurity positions and expansions in powers of the impurity density n_i; note that in this formula the term proportional to n_i^l has l Kronecker deltas.

We can incorporate the result (3.14) in the diagram matic representation of (3.5). Impurity lines, the momenta of which are constrained to sum to zero by a Kronecker delta, are drawn so as to form a bunch originating from a common point. Such a point diagrammatically represents the factor $n_i\delta(\sum\mathbf{q},\mathbf{0})$. Thus, the impurity average of the contribution of a diagram is calculated by bunching the dashed impurity lines together at points in all possible distinct ways and adding their separate contributions. A diagram of order n_i^l

[26] S. F. Edwards, Phil. Mag. 3, 1020 (1958).

has l bunches of coalescing impurity lines. This discussion is summarized in the following lemma.

Lemma 5. For each impurity vertex with momenta k (outgoing) and k' (incoming) we insert a factor $u(k-k')$. For each point at which a bunch of impurity lines meet we insert a factor $n_i\delta(\sum_s \mathbf{q}_s,\mathbf{0})$, where $\mathbf{q}_s=\mathbf{k}_s-\mathbf{k}_s'$.

Note that we need apply these rules only to diagrams that survive the previous lemmas. Also note that any diagram with a bunch consisting of only one dashed line can be ignored, since its contribution involves the factor $u(\mathbf{q})\delta(\mathbf{q},\mathbf{0})=u(\mathbf{0})\delta(\mathbf{q},\mathbf{0})=0$, according to the statement following (2.7).

We can now complete the diagrammatic representation by giving the rules for the contribution of each diagram that has to be considered in the calculation of $f_{q\omega}(k)$. These are easily verified by referring to (3.5), (3.1), and the previous lemmas. Clearly, only connected diagrams are to be considered, and only those with no left insertions or right insertions with real time vertices. The numerical value of such a diagram is found according to the following rules:

(1) For each fermion line of momentum k leaving an imaginary time vertex at $-i\gamma_n$ and entering another at $-i\gamma_m$, insert a factor $(1-F_k)e^{\epsilon k(\gamma_n-\gamma_m)}$, or $(-F_k)e^{\epsilon k(\gamma_n-\gamma_m)}$, depending on whether the line runs along the contour C in Fig. 1(b), or against it. If any of the vertices is on the real time axis, simply put its corresponding γ equal to zero. In addition, for each closed loop of fermion lines insert a factor of -1.

(2) For each interparticle vertex, insert a factor $V(kll'k')$ given by (2.4). $k,l(l',k')$ are the momenta of the outgoing (incoming) fermion lines. For the correct sign, the incoming line of momentum $k'(l')$ must always cross over to the outgoing line of momentum $k(l)$. For the whole diagram, we insert a symmetry factor $(\frac{1}{2})^m$; $m=$number of pairs of equivalent lines. Here "equivalent lines" are any two fermion lines which begin at one interparticle vertex and meet again at another, even if they go through other (noninterparticle) vertices in between. (For more details, see Ref. 27.)

(3) For each impurity vertex, insert a factor $u(k-k')$ given by (2.7), where $k(k')$ is the momentum of the outgoing (incoming) fermion line. For each point at which a bunch of impurity lines meet, insert a factor $n_i\delta[\sum_s (k_s-k_s'),0]$.

(4) For the $J_{-q}{}^{\alpha}(-i\gamma_{n+1})$ vertex, insert a factor $F_{q\omega}{}^{\alpha}k_{\alpha}'/m$, where $k'+\frac{1}{2}q(k'-\frac{1}{2}q)$ is the momentum of the outgoing (incoming) fermion line. For the $\rho_{-q}{}^{k}$ vertex, insert a factor 1.

(5) Insert a factor $[i(\Delta\epsilon_j-\omega)]^{-1}$ for each intermediate state $j(j=1,\cdots,p+r+1)$ on the real time axis, where $\Delta\epsilon_j$ is given by (3.10) for the fermion lines running between the vertices at τ_j and τ_{j-1}.

(6) Integrate over all imaginary time variables as indicated by $\int_0^{\beta} d\gamma_{n+m+1}\cdots\int_0^{\gamma_2} d\gamma_1$.

(7) Sum over the momenta of all fermion lines, except the momenta $k+\frac{1}{2}q$, and $k-\frac{1}{2}q$ associated with the two lines connected to the $\rho_q{}^{k}$ vertex.

(8) For each interaction vertex above the real time axis, insert a factor of $-i$, and for each interaction below, a factor i. For each interaction vertex on the imaginary time axis, a factor of -1.

Care must be taken to include all distinct diagrams, including the ones that arise from different orderings of the vertices above and below the real time axis.

4. DERIVATION OF A TRANSPORT EQUATION FOR THE DISTRIBUTION FUNCTION

In this section, we use some of the features of the diagrammatic description we developed in the previous section to derive an inhomogeneous integral equation for $f_{q\omega}(k)$; this is the desired transport equation. The method of derivation parallels that of Résibois.[19] The impurities necessitate only trivial modifications in the derivation of the transport equation, once the averaging over their random distribution is effected.

The key to this derivation is the realization that in a collision process the system goes from an initial state to a final state each with $\Delta\epsilon\to 0$ in the limit $\mathbf{q}\to 0$, while for all intermediate states $\Delta\epsilon_j\neq 0$ (in the same limit). We thus separate contributions to any diagram for $f_{q\omega}(k)$ by looking for denominators $(\Delta\epsilon_j-\omega)$ [see rule (5)] such that in the limit $\mathbf{q}\to 0$, $\Delta\epsilon_j\to 0$, where $\Delta\epsilon_j$ is given by (3.10). The simplest example of such a denominator occurs when only two fermion lines are involved in $\Delta\epsilon_j$ and the whole diagram is cut into two unconnected pieces by cutting these two fermion lines. (This implies that all impurity lines belonging to a single bunch are part of one or the other of the two pieces of the diagram so cut.) This assertion follows because the momenta of the two severed lines are constrained by the over-all momentum conservation for the random impurity distribution to differ by the external momentum $\mathbf{q}$; thus for these lines, the energy difference $\Delta\epsilon_j=\epsilon_{k'+q/2}-\epsilon_{k'-q/2}$ vanishes in the limit $\mathbf{q}\to 0$. In fact, not only is this the simplest way for $\lim\Delta\epsilon_j\to 0$, but it is the only way a $\Delta\epsilon_j$ can vanish in any diagram which gives a nonzero contribution to $f_{q\omega}(k)$. To see this, we recall that, according to (3.10), the fermion lines of right or left insertions must not be considered in calculating $\Delta\epsilon_j$; therefore, the only way that a denominator $\Delta\epsilon_j$ can vanish is to have one or more pairs of lines, the contribution of which to $\Delta\epsilon_j$ vanishes separately as $\mathbf{q}\to 0$. Now if there are two or more pairs of lines involved in this $\Delta\epsilon_j$, then there will be at least one vertex to the left (other than the $\rho_q{}^{k}$ vertex) which is to the left of all lines connected to it; by lemma 1, however, we may ignore such a diagram in the evaluation of $f_{q\omega}(k)$. We conclude that in looking

[27] J. L. Sigel, thesis, M. I. T., 1968 (unpublished).

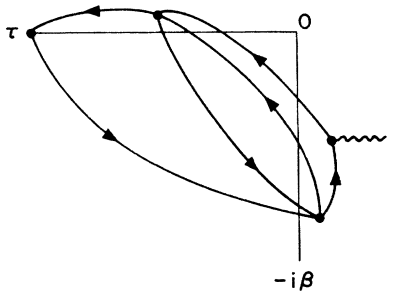

FIG. 4. A destruction diagram.

for denominators with $\Delta\epsilon_j \to 0$ as $\mathbf{q} \to 0$ we have to consider only denominators of the form

$$(\epsilon_{k'+q/2} - \epsilon_{k'-q/2} - \omega).$$

We wish to emphasize that without the reestablishment of momentum conservation by the process of averaging over the random impurity distribution there would be no denominators of the type discussed above; the averaging makes the existence of such denominators possible and they are found in the same way for both pure and impure systems. As we shall see below, the derivation of the transport equation also depends crucially on this point.

If the fermion lines are cut for all j with $\Delta\epsilon_j \to 0$ as $q \to 0$, the most general diagram for $f_{q\omega}(k)$ is split into several unconnected pieces. These consist of the "destruction region," a sequence of "diagonal fragments," both of which we define and evaluate below, and the two cut fermion lines connected to $\rho_q{}^k$ vertex.

The "destruction region" is defined as that part of the diagram which is connected to the $J_{-q}{}^{\alpha}$ vertex on the imaginary time axis and which includes all of the diagram up to the first pair of cut fermion lines encountered as one goes from right to left on the diagram. Note that right insertions in lines belonging to other parts of the diagram are not included in the "destruction region." To calculate the contribution of a destruction diagram, one includes all imaginary time integrals, summations, denominators, and factors of all vertices and unbroken fermion lines (as stated in the rules of Sec. 3) which are part of the "desctruction region" of the diagram. For the two broken fermion lines to the left of the "destruction region" one does *not* include the corresponding denominator $i(\Delta\epsilon_j - \omega)$ or the summation over k, but one *does* include the statistical factors $(1-F_{k\pm q/2})$ or $(-F_{k\pm q/2})$ associated with these lines. (It is easy to check from the earlier definitions that the statistical factors of the fermion lines are completely determined by the associated interaction vertices to their right.) Thus, the contribution of a destruction diagram depends, in addition to $\mathbf{q}$ and ω, on the mean momentum k of the broken fermion lines. The over-all sign is chosen so that if we draw a destruction diagram with the two cut lines brought together at the end of the real axis, we can calculate its contribution by use of the rules for the evaluation of diagrams for $f_{q\omega}(k)$ [except that we delete the denominator $i(\Delta\epsilon_j - \omega)$]. An example of such a diagram is shown in Fig. 4.

We now define the driving function $D_{q\omega}(k)$ as the sum of the contributions of all destruction diagrams with a fermion line of momentum $k+\frac{1}{2}q$ leaving and another of momentum $k-\frac{1}{2}q$ entering at the left end of the diagrams.

A "diagonal fragment" consists of the part of a diagram for $f_{q\omega}(k)$ that is found between two pairs of cut fermion lines. Note that any right insertion in one of the lines of the "diagonal fragment" is taken to be part of the fragment. To calculate the contribution of a fragment diagram, one includes all imaginary time integrals (associated with right insertions) summations, denominators, and factors of all vertices and unbroken fermion lines (as stated in the rules of the previous section) which belong to the "diagonal fragment." For the broken fermion lines to the left (of momenta $k\pm\frac{1}{2}q$) and to the right of the fragment (of momenta $k'\pm\frac{1}{2}q$) one does *not* include the corresponding denominators nor the summations over k and k'; however, one *does* include the statistical factors $(1-F_{k\pm q/2})$ or $(-F_{k\pm q/2})$ associated with the lines to the *left* of the fragment diagram. (As noted before, these factors are completely determined; the factors associated with the broken lines to the right are *not* included, as they are attached to the following fragment diagram or to the destruction diagram.) Thus the contribution of a fragment diagram depends, in addition to $\mathbf{q}$ and ω, on the mean momenta k and k' of the left and right broken fermion lines, respectively. In drawing a fragment diagram in the following, we shall bring together the broken fermion lines to the left and right at τ and 0 on the real axis, respectively. Its contribution can then be calculated according to the rules for the diagrams for $f_{q\omega}(k)$; we need only omit the statistical factors of the two fermion lines at the right brought together at the origin, and multiply by -1 to account for the extra closed loop introduced by the joining of these two lines. Examples of such diagrams are exhibited in Sec. 6.

We now define the collision function $W_{q\omega}(kk')$ as the sum of the contributions of all fragment diagrams with lines of momenta $k\pm\frac{1}{2}q$ to the left and of momenta $k'\pm\frac{1}{2}q$ to the right.

With the help of the functions $D_{q\omega}(k)$ and $W_{q\omega}(kk')$, we can derive a transport equation for the distribution function $f_{q\omega}(k)$ as follows. We classify the diagrams for $f_{q\omega}(k)$ according to the number of denominators with $\Delta\epsilon_j \to 0$ as $\mathbf{q} \to 0$. We then add together the ones that have the same number of such denominators. For example, the contribution to $f_{q\omega}(k)$ of all diagrams with one such denominator is simply

$$[i(\epsilon_{k+q/2} - \epsilon_{k-q/2} - \omega)]^{-1} D_{q\omega}(k),$$

as follows from the definition of $D_{q\omega}(k)$. The contribution to $f_{q\omega}(k)$ of all diagrams with two such denomina-

tors is

$$\sum_{k_1}[i(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega)]^{-1}W_{q\omega}(kk_1)\times[i(\epsilon_{k_1+q/2}-\epsilon_{k_1-q/2}-\omega)]^{-1}D_{q\omega}(k_1).$$

Summing all the contributions, we get

$$f_{q\omega}(k)=[i(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega)]^{-1}\times\{D_{q\omega}(k)+\sum_{n=1}^{\infty}\sum_{k_1}\cdots\sum_{k_n}W_{q\omega}(kk_1)\times[i(\epsilon_{k_1+q/2}-\epsilon_{k_1-q/2}-\omega)]^{-1}\cdots W_{q\omega}(k_{n-1}k_n)\times[i(\epsilon_{k_n+q/2}-\epsilon_{k_n-q/2}-\omega)]^{-1}D_{q\omega}(k_n)\}. \quad (4.1)$$

Expression (4.1) is clearly the iteration of the integral equation

$$f_{q\omega}(k)=[i(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega)]^{-1}\times\{D_{q\omega}(k)+\sum_{k'}W_{q\omega}(kk')f_{q\omega}(k')\}, \quad (4.2)$$

or equivalently

$$i(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega)f_{q\omega}(k)=D_{q\omega}(k)+\sum_{k'}W_{q\omega}(kk')f_{q\omega}(k'). \quad (4.3)$$

Equation (4.3) is the desired transport equation for the (impurity-averaged) bare-particle distribution function $f_{q\omega}(k)$. The inhomogeneous term in (4.3), the driving function $D_{q\omega}(k)$, is naturally proportional to the external driving field and may be conveniently written

$$D_{q\omega}(k)=F_{q\omega}{}^{\alpha}G_{q\omega}{}^{\alpha}(k). \quad (4.4)$$

It gives the effects of the acceleration of the particles due to the external field in the presence of the particle-particle and particle-impurity interactions. In particular it describes the effect of the external field of force on the collision processes of the particles. The kernel of this integral equation is given by the collision function $W_{q\omega}(kk')$, which describes the effects of all interactions of the particles in their full complexity (i.e., renormalization, scattering, etc.) in the absence of the driving field.

As described above, the coefficients $D_{q\omega}(k)$, $W_{q\omega}(kk')$ are given as the contributions of certain diagrams. Determining the coefficients in this way amounts to giving their values in terms of power series in the interparticle and particle-impurity interaction strengths; the convergence of these series is assumed here once and for all. Up to this point, no restrictions have been made on the magnitudes of the wave vector $\mathbf{q}$, the frequency ω, or the temperature T. Furthermore, both D and W can be expressed in a power series of the impurity density n_i, assuming such an expansion converges.

This transport equation is too general and too complicated to be capable of any direct application. It can be used, however, as a starting point for various approximations in different directions that, hopefully, will throw some light on the correct description of the effect of interparticle interactions on transport phenomena.

5. CASE OF COULOMB INTERACTIONS

In this section we consider the physically important case of electrons, which interact among themselves via a long-range Coulomb force. In this case one is not so much interested in the actual value of the induced current density as in the conductivity, since the latter, in conjunction with Maxwell's equations, determines all electromagnetic properties of the system. Since the conductivity is defined in reference to the electric field in the medium and not just the external electric field, it is desirable to have a transport equation for the distribution function in which the driving term is given in terms of the electric field in the medium. We shall derive such a transport equation for $f_{q\omega}(k)$ in this section.

It is convenient to note that the distribution function $f_{q\omega}(k)$ can be written in a form different from (2.20). If we use for $\rho(t)$ the form given by (2.12), rather than (2.14), in Eqs. (2.17) and (2.19), we find

$$f_{q\omega}(k)=-i\chi_{q\omega}(k)\phi_{q\omega}. \quad (5.1)$$

Here,

$$\chi_{q\omega}(k)=\int_0^{\infty}d\tau\, e^{i\omega\tau}\sum_{k'}\Delta\chi_q(k'k|\tau), \quad (5.2)$$

$$\Delta\chi_q(k'k|\tau)=\chi_q(k'k|0^+,\tau)-\chi_q(k'k|0^-,\tau), \quad (5.3)$$

and the latter functions are the correlation functions

$$\chi_q(k'k|0^+,\tau)=\langle \mathrm{Tr}\{\rho_0 e^{iH\tau}\rho_q{}^k e^{-iH\tau}\rho_{-q}{}^{k'}\}\rangle_i, \quad (5.4)$$

$$\chi_q(k'k|0^-,\tau)=\langle \mathrm{Tr}\{\rho_0\rho_{-q}{}^{k'}e^{iH\tau}\rho_q{}^k e^{-iH\tau}\}\rangle_i. \quad (5.5)$$

Recall that $\phi_{q\omega}$ is the Fourier transform of the potential of the *external* longitudinal field of force.

The potential of the *total* field of force in the medium $\bar{\phi}_{q\omega}$ is

$$\bar{\phi}_{q\omega}=\phi_{q\omega}/\mathcal{E}(q\omega), \quad (5.6)$$

where $\mathcal{E}(q\omega)$ is the wave-vector and frequency-dependent longitudinal dielectric constant of the medium, i.e.,

$$\mathcal{E}^{-1}(q\omega)=1-iv(q)\sum_k\chi_{q\omega}(k). \quad (5.7)$$

Here $v(q)=4\pi e^2/q^2$ is the Fourier transform of the Coulomb interaction between the electrons, according to (2.5). Thus, combining (5.1), (5.6), and (5.7), we can write

$$f_{q\omega}(k)=\frac{-i\chi_{q\omega}(k)}{1-iv(q)\sum_{k'}\chi_{q\omega}(k')}\bar{\phi}_{q\omega}, \quad (5.8)$$

which gives the distribution function in terms of the mean total potential for the field of force in the medium.

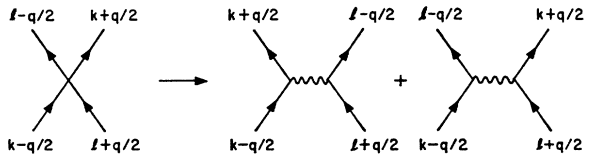

FIG. 5. The decomposition of the antisymmetrized interparticle vertex.

We shall now show that the coefficient of $\tilde{\phi}_{q\omega}$ in (5.8) can be represented by a particular class of diagrams contributing to $f_{q\omega}(k)/\phi_{q\omega}$. The method of proof is similar to that used by Izuyama[28] for the longitudinal conductivity of the electron-phonon system.

A perturbation expansion of $\chi(k'k|0^{\pm},\tau)$ in powers of H' is obtained by use of (2.21) and (2.22), i.e.,

$$\chi_q(k'k|0^+,\tau)=\langle\langle U(-i\beta,0)U(0,\tau)\rho_q{}^k(\tau)U(\tau,0)\rho_{-q}{}^{k'}\rangle \times\langle U(-i\beta,0)\rangle^{-1}\rangle_i, \quad (5.9)$$

$$\chi_q(k'k|0^-,\tau)=\langle\langle U(-i\beta,0)\rho_{-q}{}^{k'}U(0,\tau)\rho_q{}^k(\tau)U(\tau,0)\rangle \times\langle U(-i\beta,0)\rangle^{-1}\rangle_i, \quad (5.10)$$

where the evolution operators U are given by (2.26)–(2.28). These can be represented by diagrams quite similar to those for $f_{q\omega}(k)$, and a linked-cluster theorem can similarly be proved for them. If we now consider the points on the contour C in Fig. 1(b) and denote by t^+ the point $t(0\leqslant t<\tau)$ on the upper section of C between 0 and τ, by t^- the corresponding point on the lower section of C, and by $-i\gamma(0\leqslant\gamma\leqslant\beta)$ a point on the section of C along the imaginary axis, we can write symbolically

$$\chi_q(k'k|0^{\pm},\tau) =\langle\langle P\{U_C(-i\beta,0^+)\rho_{-q}{}^{k'}(0^{\pm})\rho_q{}^k(\tau)\}\rangle_c\rangle_i, \quad (5.11)$$

where $U_C(-i\beta,0^+)$ is the evolution operator along the contour C and P an ordering operator on the contour C. For any two points z_1, z_2 on the contour C, we can define more generally

$$\chi_q(k'k|z_1,z_2)=\langle\langle P\{U_C(-i\beta,0^+)\rho_{-q}{}^{k'}(z_1)\rho_q{}^k(z_2)\}\rangle \times\langle U(-i\beta,0)\rangle^{-1}\rangle_i =\langle\langle P\{U_C(-i\beta,0^+) \times\rho_{-q}{}^{k'}(z_1)\rho_q{}^k(z_2)\}\rangle_c\rangle_i. \quad (5.12)$$

We note a property of this function, namely,

$$\chi_q(k'k|t^{\pm},\tau)=\chi_q(k'k|0^{\pm},\tau-t). \quad (5.13)$$

This is easily shown to follow from the definition (5.12) and the invariance of the trace under cyclic permutations. It will prove useful below, where we demonstrate that $\chi_{q\omega}(k)$ can be expressed in terms of its *proper* diagrams.

In order to define these, we first observe that in the case of interacting electrons in a uniform neutralizing background, the interaction V_p can be written in the form

$$V_p=\tfrac{1}{4}\sum_{kl}\sum_q[v(q)-v(k-l)] \times c_{k+q/2}{}^{\dagger}c_{l-q/2}{}^{\dagger}c_{l+q/2}c_{k-q/2}, \quad (5.14)$$

where in the case under consideration

$$v(q)=4\pi e^2/q^2, \quad q\neq 0 \qquad =0, \quad q=0. \quad (5.15)$$

The vanishing of $v(0)$ results from the presence of the uniform neutralizing background charge. It proves convenient to separate in (5.14) the direct and exchange terms. For this purpose, we depict each interparticle vertex by the two elements shown in Fig. 5. The first element in Fig. 5 represents the direct term $v(q)$ while the second represents the exchange term $-v(k-l)$ in (5.14). The wavy line in the first element is seen to carry momentum $\mathbf{q}$ and gives a contribution $v(q)$ according to rule 2 of Sec. 3; we shall refer to such wavy interaction lines as Coulomb lines. We now define a connected diagram of $\chi_q(k'k|z_1,z_2)$ as improper if eliminating a single Coulomb line splits it into two unconnected graphs. All diagrams that cannot be split into two separate parts by eliminating a single Coulomb line are called proper.

It is clear that in an improper diagram of $\chi_q(k'k|z_1,z_2)$ a bunch of impurity lines belongs entirely to one part or the other into which such a diagram can be split by eliminating a Coulomb line. Since we have an over-all conservation of momentum for every bunch of impurity lines, it follows that the Coulomb line of an improper diagram of $\chi_q(k'k|z_1,z_2)$, the elimination of which would completely split the diagram, must have momentum equal to the wave vector $\mathbf{q}$ of the external field. Also, one of the vertices $\rho_{-q}{}^{k'}(z_1)$, $\rho_q{}^k(z_2)$ must belong to one of the separated parts of the diagram, while the other must belong to the second part. The exclusion of $\mathbf{q}=0$ in (5.15) eliminates the possibility of both vertices $\rho_{-q}{}^k$, $\rho_q{}^k$ belonging to the same separated part of an improper diagram.

Let us now denote the sum of the contributions of all proper diagrams of $\chi_q(k'k|z_1,z_2)$ by $\tilde{\chi}_q(k'k|z_1z_2)$, i.e., we define, in a symbolic notation,

$$\tilde{\chi}_q(k'k|z_1,z_2)=\langle\langle P\{U_C(-i\beta,0^+)\rho_{-q}{}^{k'}(z_1) \times\rho_q{}^k(z_2)\}\rangle_c\rangle_i{}^{\text{proper}}. \quad (5.16)$$

This function has the properties

$$\tilde{\chi}_q(k'k|0^+,t^+)=\tilde{\chi}_q(k'k|0^+,t^-)=\tilde{\chi}_q(k'k|0^+,t), \quad (5.17)$$

$$\tilde{\chi}_q(k'k|0^-,t^+)=\tilde{\chi}_q(k'k|0^-,t^-)=\tilde{\chi}_q(k'k|0^-,t), \quad (5.18)$$

$$\tilde{\chi}_q(k'k|0^+,-i\gamma)=\tilde{\chi}_q(k'k|0^-,-i\gamma), \quad (5.19)$$

which will prove useful later. To prove the first, we note that in

$$\tilde{\chi}_q(k'k|0^+,t^+)=\langle\langle U(-i\beta,0)U(0,\tau)U(\tau,t) \times\rho_q{}^k(t)U(t,0)\rho_{-q}{}^{k'}\rangle_c\rangle_i{}^{\text{proper}}$$

[28] T. Izuyama, Progr. Theoret. Phys. (Kyoto) **25**, 964 (1961).

we can put $U(0,\tau)U(\tau,t)=U(0,t)$ even if only proper diagrams are considered, since the contribution to $U_C(t^-,t^+)=U(t,\tau)U(\tau,t)$ of each diagram that has one or more vertices on the section (t^+,t^-) of C vanishes. This replacement proves that

$$\tilde{\chi}_q(k'k|0^+,t^+)=\tilde{\chi}_q(k'k|0^+,t).$$

Similar arguments prove the remaining relations in (5.17)–(5.19).

Considering a general improper diagram for

$$\chi_q(k'k|0^\pm,\tau)$$

as made up of a proper diagram of $\tilde{\chi}_q(k'l'|0^\pm,z)$ connected with the $\rho_{-q}{}^{k'}(0^\pm)$ vertex and the rest of the diagram of $\chi_q(lk|z,\tau)$ connected with the $\rho_q{}^k(\tau)$ vertex along with the connecting Coulomb line at z, we find, in direct analogy with Izuyama,[28] that the functions $\chi_q(k'k|0^\pm,\tau)$ satisfy the integral equation

$$\chi_q(k'k|0^\pm,\tau)=\tilde{\chi}_q(k'k|0^\pm,\tau)$$
$$-iv(q)\int_C dz\sum_{ll'}\tilde{\chi}_q(k'l'|0^\pm,z)\chi_q(lk|z,\tau), \quad (5.20)$$

where the integral is taken over the contour C. By subtraction we have

$$\Delta\chi_q(k'k|\tau)=\Delta\tilde{\chi}_q(k'k|\tau)$$
$$-iv(q)\int_C dz\sum_{ll'}\Delta\tilde{\chi}_q(k'l'|z)\chi_q(lk|z,\tau), \quad (5.21)$$

where

$$\Delta\tilde{\chi}_q(k'k|z)=\tilde{\chi}_q(k'k|0^+,z)-\tilde{\chi}_q(k'k|0^-,z). \quad (5.22)$$

Now $\Delta\tilde{\chi}_q(k'k|z)$ on the imaginary part of the contour C vanishes due to (5.19), whereas on either side of the real axis it has the same value $\Delta\tilde{\chi}_q(k'k|t)$, according to (5.17) and (5.18). We thus find, if we use (5.13), that

$$\Delta\chi_q(k'k|\tau)=\Delta\tilde{\chi}_q(k'k|\tau)$$
$$-iv(q)\sum_{ll'}\int_0^\tau dt\,\Delta\tilde{\chi}_q(k'l'|t)\Delta\chi_q(lk|\tau-t). \quad (5.23)$$

Defining

$$\tilde{\chi}_{q\omega}(k)=\int_0^\infty d\tau\, e^{i\omega\tau}\sum_{k'}\Delta\tilde{\chi}_q(k'k|\tau), \quad (5.24)$$

in direct analogy with $\chi_{q\omega}(k)$ [see (5.2)], and performing the indicated operations on (5.23), we find with the help of the convolution theorem

$$\chi_{q\omega}(k)=\tilde{\chi}_{q\omega}(k)-iv(q)\chi_{q\omega}(k)\sum_{k'}\tilde{\chi}_{q\omega}(k') \quad (5.25)$$

or

$$\chi_{q\omega}(k)=\frac{\tilde{\chi}_{q\omega}(k)}{1+iv(q)\sum_{k'}\tilde{\chi}_{q\omega}(k')}. \quad (5.26)$$

Substituting (5.26) into (5.8), we finally find

$$f_{q\omega}(k)=-i\tilde{\chi}_{q\omega}(k)\tilde{\phi}_{q\omega}. \quad (5.27)$$

Comparing this with (5.1), we see that the distribution function can be obtained by keeping only the proper diagrams if at the same time we replace the external field by the mean total field in the medium.

Proceeding as in the previous section, we can now construct a transport equation for $f_{q\omega}(k)$ which involves the mean total field of force in the medium $\bar{F}_{q\omega}(k)$. Noting that the concept of a proper diagram is independent of the technique used for handling time integrations, we find in analogy with (4.3) and (4.4)

$$i(\epsilon_{k+q/2}-\epsilon_{k-q/2}-\omega)f_{q\omega}(k)$$
$$=\bar{F}_{q\omega}{}^\alpha\tilde{G}_{q\omega}{}^{\alpha(k)}+\sum_{k'}\tilde{W}_{q\omega}(kk')f_{q\omega}(k'), \quad (5.28)$$

where now $\tilde{W}_{q\omega}(kk')$ is just the sum of all proper diagrams for the collision function $W_{q\omega}(kk')$, and $\tilde{G}_{q\omega}(k)$ is the sum of all proper diagrams for the driving function $G_{q\omega}(k)$.

We shall see in Sec. 6 that this form of the transport equation for the electron gas is particularly useful in the case of slowly varying disturbances.

6. TRANSPORT EQUATION FOR SLOWLY VARYING FIELD, WEAK IMPURITY SCATTERING, AND LOW TEMPERATURES—A QUASIPARTICLE DESCRIPTION

With this section we begin an application of the general transport equation (4.3), or (5.28), to the case of long-wavelength, low-frequency, weak impurity scattering and low temperatures. As we shall see, this case is of practical importance for a degenerate Fermi gas of sufficiently high Fermi energy, as for example, the conduction electrons in an impure metal at low temperatures.

For short-range interparticle and impurity interactions the various functions involved in the theory are regular as $\mathbf{q}\to 0$, and thus the coefficients $G_{q\omega}{}^\alpha(k)$, $W_{q\omega}(kk')$ in the transport equation (4.3) can be expanded about $\mathbf{q}=0$ without difficulty. For the case of Coulomb interactions, the coefficients $\tilde{G}_{q\omega}{}^\alpha(k)$, $\tilde{W}_{q\omega}(kk')$ of the appropriate transport equation (5.28) are also nonsingular at $\mathbf{q}=0$, due to the fact that the improper diagrams, which involve the singular function $v(q)=4\pi e^2/q^2$, have been identified with the average induced electric field in the medium. Thus, we can obtain a transport equation for long wavelength and low frequency in both cases by Taylor expansions of the coefficients of the transport equation around $\mathbf{q}=\omega=0$.

Thus in the limit of small $\mathbf{q}$ and ω, we have

$$i\left(q^\alpha\frac{k_\alpha}{m}-\omega\right)f_{q\omega}(k)$$
$$-\sum_{k'}\left(q^\alpha\left.\frac{\partial W_{q\omega}(kk')}{\partial q^\alpha}\right|_{00}+\omega\left.\frac{\partial W_{q\omega}(kk')}{\partial\omega}\right|_{00}\right)f_{q\omega}(k')$$
$$-F_{q\omega}{}^\alpha G_{00}{}^\alpha(k)=\sum_{k'}W_{00}(kk')f_{q\omega}(k') \quad (6.1)$$

for the short-range interparticle case and similarly for the Coulomb case. The third and fourth terms on the left-hand side of (6.1), involving the derivatives of $W_{q\omega}$ evaluated at $\mathbf{q}=\omega=0$, describe the nondissipative effects of the interactions on the free motion of the particles as given by the first two terms. The term on the right-hand side of (6.1) involving $W_{00}(kk')$ describes the collision processes of the particles and is the same as in the static, homogeneous case. This term is made up of two contributions: (1) an impurity scattering part (i.e., one that involves a δ function of two energy arguments) of order Γ_i, where Γ_i is the width of the state due to impurity scattering, which to the lowest order is proportional to n_i, the impurity density; (2) an interparticle scattering part (i.e., terms that involve δ functions of four energy arguments) of order Γ_p the width due to interparticle collisions, which for low temperatures is $\Gamma_p \propto T^2$, as we shall see. In the inhomogeneous term of (6.1) we have kept only the lowest-order term $G_{00}{}^{\alpha}(k)$; it describes the acceleration of the particles by the external field in the presence of the other interactions affecting the particles. It will turn out, as we shall see, that the inhomogeneous term essentially forces k to be on the Fermi surface and thus by small ω and $\mathbf{q}$ we mean $\omega/\mu \ll 1$ and $q/k_F \sim qv_F/\mu \ll 1$, where v_F is the Fermi velocity.

Foregoing a discussion of internal consistency, we shall further restrict ourselves to low temperatures and weak impurity scattering. Effectively, this means that we shall formally treat Γ_p/μ and Γ_i/μ as small quantities of the same order of magnitude as qv_F/μ and ω/μ; for convenience, we characterize all these by a parameter λ. We shall calculate the coefficients of the transport equation (6.1) to lowest nonvanishing order in λ, making, however, no assumption about the relative size of the four parameters listed above. Doing this, we shall be able to evaluate the distribution function to the lowest order, namely, λ^{-1}. This limitation to a determination of $f_{q\omega}(k)$ to order λ^{-1} allows us to ignore the effects of the particle-impurity interaction on the driving term $G_{00}{}^{\alpha}(k)$, and on the renormalization terms associated with the derivatives of $W_{q\omega}(kk')$. Thus, for all terms of the left-hand side of (6.1) no impurity effects have to be calculated; only in evaluating $W_{00}(kk')$ do we have to include the effects of the impurities. These will be considered up to second order in V_i. This approximation makes the impurity-scattering terms proportional to the concentration n_i of the impurity centers and to the scattering cross section of each center evaluated in the lowest Born approximation.

A. Results for the Case of No Impurities

We now proceed to calculate the coefficients of the left-hand side of the approximate transport equation (6.1) in the absence of the impurities. Actually this calculation is the same as for a pure system and the results have been stated by Résibois.[22] Since the details of the evaluation of all relevant diagrams have been considered in the thesis of one of us,[27] we shall give here only the necessary results.

These calculations are performed up to second order in the interparticle potential V_p. We limit ourselves to this order, not because we expect higher-order terms to be negligible, but because we are unable to sum the diagrams to all orders. Higher-order terms in this method are very tedious to calculate and, on the other hand, the second-order calculation demonstrates all the new features of the quasiparticle description of the problem first introduced by Landau.[11]

We find up to second order in V_p for the left-hand side of (6.1)

$$iq^{\alpha}\frac{k_{\alpha}}{m}f_{q\omega}(k)-q^{\alpha}\sum_{k'}\left.\frac{\partial W_{q\omega}(kk')}{\partial q^{\alpha}}\right|_{00}f_{q\omega}(k')$$
$$=iq^{\alpha}\left[\frac{v_k{}^{\alpha}}{z_k{}^2}f_{q\omega}(k)+\sum_{k'}\gamma_{kk'}\left(\frac{k_{\alpha}+k_{\alpha}'}{m}\right)f_{q\omega}(k')\right.$$
$$\left.+v_k{}^{\alpha}\left(-\frac{\partial N_k}{\partial E_k}\right)\sum_{k'}I_{kk'}f_{q\omega}(k')\right], \quad (6.2)$$

$$-i\omega f_{q\omega}(k)-\omega\sum_{k'}\left.\frac{\partial W_{q\omega}(kk')}{\partial\omega}\right|_{00}f_{q\omega}(k')$$
$$=-i\omega[(1/z_k{}^2)f_{q\omega}(k)+\sum_{k'}\gamma_{kk'}f_{q\omega}(k')], \quad (6.3)$$

and

$$G_{00}{}^{\alpha}(k)=\frac{v_k{}^{\alpha}}{z_k}\left(-\frac{\partial N_k}{\partial E_k}\right)+\sum_{k'}\gamma_{kk'}\frac{k_{\alpha}'}{m}\left(-\frac{\partial F_{k'}}{\partial\epsilon_{k'}}\right). \quad (6.4)$$

Here we have introduced some new quantities which simplify the writing and will prove extremely useful later for the quasiparticle formulation of the transport equation. These include the transformation and quasiparticle interaction matrices, defined by

$$\gamma_{kk'}=-\sum_{ll'}\mathrm{Re}\frac{A_{l'}{}^{kl}|V(kll'k')|^2}{(\epsilon_k{}^{+}+\epsilon_l-\epsilon_{l'}-\epsilon_{k'})^2}$$
$$+\tfrac{1}{2}\sum_{ll'}\mathrm{Re}\frac{A_k{}^{ll'}|V(kk'll')|^2}{(\epsilon_k{}^{+}+\epsilon_{k'}-\epsilon_l-\epsilon_{l'})^2}, \quad (6.5)$$

$$I_{kk'}=V(kk'k'k)-\sum_{ll'}|V(kll'k')|^2$$
$$\times\mathrm{Re}\frac{F_l(1-F_{l'})-(1-F_l)F_{l'}}{\epsilon_k{}^{+}+\epsilon_l-\epsilon_{l'}-\epsilon_{k'}}-\tfrac{1}{2}\sum_{ll'}|V(kk'll')|^2$$
$$\times\mathrm{Re}\frac{F_lF_{l'}-(1-F_l)(1-F_{l'})}{\epsilon_k{}^{+}+\epsilon_{k'}-\epsilon_l-\epsilon_{l'}}, \quad (6.6)$$

where Re denotes the "real part of," $\epsilon_k{}^{\pm}=\epsilon_k\pm i\eta$, with η a positive infinitesimal, and

$$A_k{}^{ll'}=F_k(1-F_l)(1-F_{l'})+(1-F_k)F_lF_{l'}. \quad (6.7)$$

In addition, we have introduced the renormalization constant z_k, the quasiparticle energy E_k, the quasiparticle velocity $v_k{}^\alpha$, and the equilibrium quasiparticle distribution function

$$N_k=[e^{\beta(E_k-\mu)}+1]^{-1}. \tag{6.8}$$

These are conveniently defined by means of the self-energy function $K_k(\zeta)$, which is just the self-energy part of the temperature Green's function. Up to second order in V_p this function is

$$K_k(\zeta)=K_k{}^{(1)}+K_k{}^{(2)}(\zeta), \tag{6.9}$$

where

$$K_k{}^{(1)}=\sum_l V(kllk)F_l \tag{6.10}$$

is just the "thermal Hartree-Fock" correction to the single-particle energy, and

$$K_k{}^{(2)}(\zeta)=\sum_l V(kllk)K_l{}^{(1)}\partial F_l/\partial\epsilon_l + \tfrac{1}{2}\sum_{ll'l''}\frac{A_l{}^{l'l''}|V(kll'l'')|^2}{\zeta+\epsilon_l-\epsilon_{l'}-\epsilon_{l''}}. \tag{6.11}$$

Note that the first term of $K_k{}^{(2)}(\zeta)$ is real and independent of ζ and corresponds to a second-order Hartree-Fock correction to the single-particle energy. In terms of this function $K_k(\zeta)$, the quasiparticle quantities are defined by

$$E_k=\epsilon_k+\mathrm{Re}K_k(E_k{}^+)\cong\epsilon_k+K_k{}^{(1)}+K_k{}^{(2)}(\epsilon_k{}^+), \tag{6.12}$$

$$z_k{}^{-1}=1-\mathrm{Re}\frac{\partial K_k(\zeta)}{\partial\zeta}\bigg|_{E_k{}^+}\cong 1-\mathrm{Re}\frac{\partial K_k{}^{(2)}(\zeta)}{\partial\zeta}\bigg|_{\epsilon_k{}^+}, \tag{6.13}$$

$$v_k{}^\alpha=\frac{\partial E_k}{\partial k_\alpha}=z_k\left[\frac{k_\alpha}{m}+\mathrm{Re}\frac{\partial K_k(\zeta)}{\partial k_\alpha}\bigg|_{E_k{}^+}\right] \cong z_k\frac{k_\alpha}{m}+\frac{\partial}{\partial k_\alpha}[K_k{}^{(1)}+\mathrm{Re}K_k{}^{(2)}(\zeta)|_{\epsilon_k{}^+}]. \tag{6.14}$$

The names of these quantities E_k, z_k, $v_k{}^\alpha$ are justified by the roles they play in a quasiparticle interpretation of the transport equation. It is easily checked that if we introduce the quasiparticle distribution function, defined by

$$n_{q\omega}(k)=\sum_{k'}(\delta_{kk'}/z_{k'}+\gamma_{kk'})f_{q\omega}(k'), \tag{6.15}$$

the left-hand side of (6.1) can be written in the form (as Résibois[22] showed)

$$\text{left-hand side (6.1)}=\sum_{k'}(\delta_{kk'}/z_{k'}+\gamma_{kk'}) \times[-i\omega n_{q\omega}(k')+iq^\alpha v_{k'}{}^\alpha\bar{n}_{q\omega}(k') +F_{q\omega}{}^\alpha v_{k'}{}^\alpha(\partial N_{k'}/\partial E_{k'})], \tag{6.16}$$

where $\bar{n}_{q\omega}(k)$ denotes the "local" quasiparticle distribution function, defined by

$$\bar{n}_{q\omega}(k)=n_{q\omega}(k) +(-\partial N_k/\partial E_k)\sum_{k'}z_kz_{k'}I_{kk'}n_{q\omega}(k'), \tag{6.17}$$

with $I_{kk'}$ given by (6.6). $\bar{n}_{q\omega}(k)$, which is seen to be a functional of $n_{q\omega}(k)$, measures the departure of the actual distribution function $N_k+n_{q\omega}(k)$ from the *local* equilibrium distribution function.[15] In (6.16), only terms up to second order in V_p are to be kept. Let us recall from the earlier discussion, that to the order of the parameter λ introduced there, (6.16) is accurate for the left-hand side of (6.1) even in the presence of the impurities.

We must now calculate the right-hand side of (6.1), i.e., $W_{00}(kk')$, again up to second order in V_p, and up to second order in the impurity interaction V_i. We find,[22,27] denoting by superscripts the orders of V_p and V_i, respectively, that in the absence of impurities

$$W_{00}{}^{(1,0)}(kk')=0, \tag{6.18}$$

$$\sum_{k'}W_{00}{}^{(2,0)}(kk')f_{q\omega}(k')=\sum_{ll'l''}\pi|V(kll'l'')|^2 \times\delta(\epsilon_k+\epsilon_l-\epsilon_{l'}-\epsilon_{l''})S_{kl}{}^{l'l''}[F_k+f_{q\omega}(k)], \tag{6.19}$$

where the functional $S[F_k+f_{q\omega}(k)]$ denotes the usual statistical factors appropriate for a two-particle collision process, i.e., if we put $f_k=F_k+f_{q\omega}(k)$,

$$S_{kl}{}^{l'l''}[f_k]=\mathfrak{L}\{f_lf_{l'}(1-f_k)(1-f_l) -(1-f_{l'})(1-f_{l''})f_kf_l\}, \tag{6.20}$$

with $\mathfrak{L}$ denoting the linearization operator with respect to $f_{q\omega}(k)$.

Since this term is of second order in V_p, we have no way of knowing the appropriate expression of (6.19) in terms of the quasiparticle distribution function $n_{q\omega}(k)$. To determine this, the evaluation of the scattering term up to fourth order in V_p is required. This calculation was carried out by Watabe and Dagonnier.[23] They conclude that up to $(V_p)^4$ we may write, ignoring three-particle collisions,

$$\sum_{k'}W_{00}(kk')f_{q\omega}(k') =\sum_{k'}(\delta_{kk'}/z_{k'}+\gamma_{kk'})C_{k'}{}^{(p)}[\bar{n}_{q\omega}(k')], \tag{6.21}$$

where the interquasiparticle collision operator is

$$C_k{}^{(p)}[\bar{n}_{q\omega}(k)]=\sum_{ll'l''}\pi|T(kll'l'')|^2 \times\delta(E_k+E_l-E_{l'}-E_{l''})S_{kl}{}^{l'l''}[N_k+\bar{n}_{q\omega}(k)]. \tag{6.22}$$

In (6.22), $T(kll'l'')$ is the t matrix for two-quasiparticle scattering, i.e.,

$$|T(kll'l'')|^2=z_kz_lz_{l'}z_{l''}|V(kll'l'')|^2 +[\text{corrections to the two-quasiparticle scattering matrix of order higher than } V_p{}^2]. \tag{6.23}$$

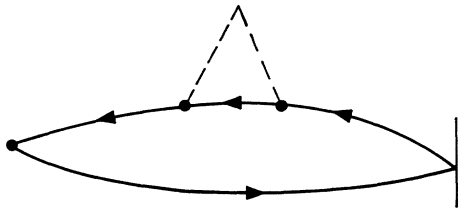

FIG. 6. A shorthand diagram representing all "scattering-out" terms in $W_{00}^{(0,2)}(kk')$.

The function $S[\ ]$ is given by (6.20), where now the statistical factors involve the quasiparticle equilibrium distribution function N_k, defined by (6.8), and the "local" quasiparticle distribution function $\bar{n}_{q\omega}(k)$ defined by (6.17).

Taking (6.21) as the appropriate generalization of (6.19), which agrees with it to second order in V_p, and multipliying (6.16) and (6.21) by the inverse of the matrix $(\delta_{kk'}/z_{k'}+\gamma_{kk'})$, we get for the Fermi liquid in the absence of impurities the Landau transport equation for the quasiparticle distribution function $n_{q\omega}(k)$,

$$-i\omega n_{q\omega}(k)+iq^{\alpha}v_k{}^{\alpha}\bar{n}_{q\omega}(k)+F_{q\omega}{}^{\alpha}v_k{}^{\alpha}(\partial N_k/\partial E_k) = C_k{}^{(p)}[\bar{n}_{q\omega}(k)]. \quad (6.24)$$

Note that the second term on the left-hand side as well as the collision term on the right-hand side involve the "local" quasiparticle distribution function $\bar{n}_{q\omega}(k)$, as given by (6.17).

Let us now discuss the temperature dependence of the collision term in (6.24). According to our earlier discussion, we are to evaluate the terms of the left-hand side of (6.24) to lowest order in λ; therefore, they should be evaluated at $T=0°\mathrm{K}$. The inhomogeneous term and, consequently, $n_{q\omega}(k)$ and $\bar{n}_{q\omega}(k)$ are then singular functions like $\delta(E_k-\mu)$. This gives us easily the result that the scattering term is proportional to T^2, as was anticipated earlier.

In order to complete the quasiparticle description, we must calculate the particle density and the current density in terms of the quasiparticle distribution function $n_{q\omega}(k)$. By comparison of (6.5), (6.10), and (6.12) we find that $\gamma_{kk'}$ has the properties

$$\sum_k \gamma_{kk'}=\mathrm{Re}\frac{\partial K_{k'}(\zeta)}{\partial\zeta}\bigg|_{\epsilon_{k'}{}^{+}}=-z_{k'}{}^{-1}+1, \quad (6.25)$$

$$\sum_k k_{\alpha}\gamma_{kk'}=k_{\alpha}'(-z_{k'}{}^{-1}+1). \quad (6.26)$$

With the help of these relations it is immediately seen that the induced particle and current densities are

$$\rho_{q\omega}=\sum_k f_{q\omega}(k)=\sum_k n_{q\omega}(k), \quad (6.27)$$

$$j_{q\omega}{}^{\alpha}=\sum_k \frac{k_{\alpha}}{m}f_{q\omega}(k)=\sum_k \frac{k_{\alpha}}{m}n_{q\omega}(k), \quad (6.28)$$

where we have used the definition (6.14) of $n_{q\omega}(k)$.

In the following we shall examine the way in which the impurities modify all these results and to what extent a quasiparticle description of the problem can be recovered.

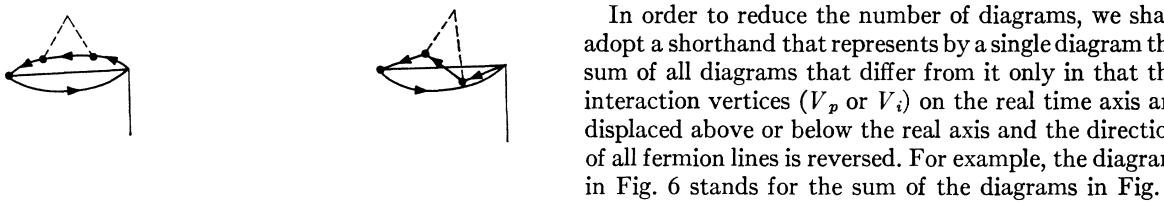

FIG. 7. Some of the diagonal fragments represented by Fig. 6.

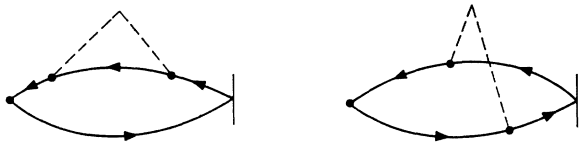

FIG. 8. All diagrams for $W_{00}^{(0,2)}(kk')$.

B. Scattering in the Presence of Impurities

We recall that for a random distribution of impurities the terms of first order in V_i vanish. Thus, in our approximation of weak scattering, we have to evaluate only the terms of second order in V_i. In the diagrammatic representation we have adopted, this means that we consider only diagrams for $W_{00}(kk')$ with one bunch of only two-impurity (dashed) lines. As before, we shall evaluate all such diagrams up to second order in the interparticle interaction V_p.

In order to reduce the number of diagrams, we shall adopt a shorthand that represents by a single diagram the sum of all diagrams that differ from it only in that the interaction vertices (V_p or V_i) on the real time axis are displaced above or below the real axis and the direction of all fermion lines is reversed. For example, the diagram in Fig. 6 stands for the sum of the diagrams in Fig. 7 plus the "conjugate" diagrams, i.e., those obtained from Fig. 7 by reversing the directions of all the fermion lines while moving all real time vertices above (below) the real axis below (above). It is easily proved from the rules of Sec. 3 that the contributions of two "conjugate" diagrams for $W_{00}(kk')$ are just complex conjugates of each other. In addition, a diagram with vertices on the imaginary time axis (as one with right insertions) will denote the sum of all similar diagrams

which differ only in the relative ordering of these vertices.

In this shorthand, the diagrams for $W_{00}(kk')$ in the absence of interparticle interaction are the two shown in Fig. 8. Their contribution is simply

$$W_{00}^{(0,2)}(kk')=n_i 2\pi[|u(k-k')|^2\delta(\epsilon_k-\epsilon_{k'}) -\delta_{kk'}\sum_l |u(k-l)|^2\delta(\epsilon_k-\epsilon_l)], \quad (6.29)$$

i.e., the usual "scattering-in" minus "scattering-out" terms of the collision kernel of the transport equation for dynamically independent electrons in the Born approximation.

To the first order in V_p, there are 16 diagrams for $W_{00}^{(1,2)}(kk')$ of the shorthand type, some of which are shown in Fig. 9. Two of them make vanishing contributions and the sum of the contributions of four others is zero. Of the remaining, four are of the type in Fig. 9(a) and give the Hartree-Fock modification of the one-particle energies appearing in $W_{00}^{(0,2)}(kk')$, as given by (6.29). Thus, the contribution of the diagram in Fig. 9(a) and a similar one with the V_p vertex on the lower fermion line is

$$W_{00}^{[9a]}(kk')=n_i 2\pi|u(k-k')|^2 \times\left[K_k^{(1)}\frac{\partial}{\partial\epsilon_k}+K_{k'}^{(1)}\frac{\partial}{\partial\epsilon_{k'}}\right]\delta(\epsilon_k-\epsilon_{k'}), \quad (6.30)$$

whereas the other two diagrams of this group give the "scattering-out" term corresponding to (6.30). Another set of four diagrams of the type shown in Fig. 9(b) give the modification of the particle-impurity interaction $u(k-k')$ due to the interparticle forces, i.e., the "screening." [In Fig. 9(b) the placement of the vertex to the right of the small vertical line indicates that the vertex lies on the imaginary time axis.] Thus, the contribution of the diagram in Fig. 9 (b) and a similar one with its vertices on the real time axis permuted is

$$W_{00}^{[9b]}(kk')=n_i 2\pi[u(k-k')U_{k'k}^{(1)}+U_{kk'}^{(1)}u(k'-k)] \times\delta(\epsilon_k-\epsilon_{k'}), \quad (6.31)$$

whereas the other two diagrams of this group give the "scattering-out" term corresponding to (6.31). In (6.31) we have introduced $U_{kk'}^{(1)}$, the modification of $u(k-k')$ to first order in V_p, which is defined by

$$U_{kk'}^{(1)}=u(k'-k)\sum_{ll'} V(k'll'k)(F_l-F_{l'}/\epsilon_l-\epsilon_{l'}). \quad (6.32)$$

Finally, the last set of two diagrams, of the type shown in Fig. 9(c), give a contribution

$$W_{00}^{[9c]}(kk')=n_i 2\pi\sum_l |u(k-l)|^2[V(kk'k'k) -V(lk'k'l)](\partial F_l/\partial\epsilon_l)\delta(\epsilon_k-\epsilon_l), \quad (6.33)$$

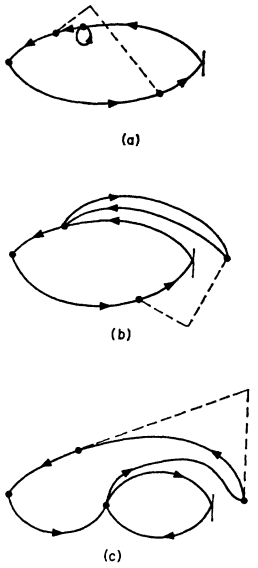

FIG. 9. (a) A diagonal fragment representing the Hartree-Fock renormalization of the energies in $W_{00}^{(0,2)}(kk')$. (b) A diagonal fragment representing the screening of the impurity potential in $W_{00}^{(0,2)}(kk')$. (c) A diagonal fragment representing the linearization of the fermion occupation in the Hartree-Fock energy with respect to the external field.

which arises from the Hartree-Fock renormalization of the one-particle energy in the equlibrium distribution function F_k and the subsequent linearization with respect to $f_{q\omega}(k)$ in (6.29). It should be noted that only the exchange part of the interparticle potential contributes to (6.33).

The calculation of $W_{00}^{(2,2)}(kk')$, i.e., of order $V_p^2V_i^2$, is unfortunately very long. There are 252 diagrams (in our shorthand) involved, although some of them are quite similar. These terms describe the effects of energy renormalization and of the modification of the impurity-particle interaction due to the interparticle forces on the impurity scattering, and conversely, the analogous effects of the impurity-particle interaction on the interparticle scattering. Furthermore, there are terms that cannot be simply interpreted as renormalization of energies or modification of interactions. Here we shall present a few diagrams as examples, which at the same time help us define the modification of the impurity-particle interaction to second order in V_p, and give the final results of this lengthy calculation. For more details the interested reader is referred to the thesis of one of us.[27]

In Fig. 10 we present a diagram for $W_{00}^{(2,2)}(kk')$, which is typical of a group of 12 diagrams which are

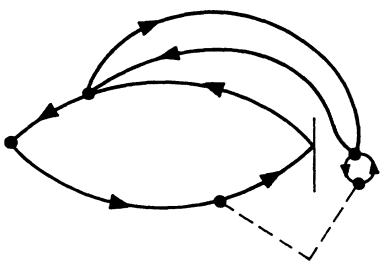

FIG. 10. A diagram representing a simple contribution to the effective impurity potential of order V_p^2 in $W_{00}(kk')$.

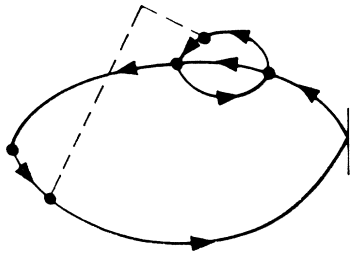

FIG. 11. A somewhat more complicated contribution to the effective impurity potential of order V_p^2 in $W_{00}(kk')$.

obtained from this one by placing the impurity-particle vertex occurring in the right insertion in other allowed positions. The contribution of this group of diagrams is, according to the rules of Sec. 3,

$$W_{00}^{[10]}(kk')=n_i 2\pi[u(k-k')U_{k'k}^{(2a)}+U_{kk'}^{(2a)}u(k-k')]\times\delta(\epsilon_k-\epsilon_{k'})-[\text{scattering-out term}]. \quad (6.34)$$

This is analogous to (6.31), but with $U_{kk'}^{(2a)}$ given by

$$U_{kk'}^{(2a)}=\sum_{ll'}u(l'-l)V(kll'k')\left[\frac{K_l^{(1)}}{\epsilon_{l'}-\epsilon_l}\left(\frac{F_l-F_{l'}}{\epsilon_l-\epsilon_{l'}}-\frac{\partial F_l}{\partial\epsilon_l}\right)+\frac{K_{l'}^{(1)}}{\epsilon_l-\epsilon_{l'}}\left(\frac{F_l-F_{l'}}{\epsilon_l-\epsilon_{l'}}-\frac{\partial F_l}{\partial\epsilon_{l'}}\right)+\sum_{l''l'''}[V(l'l''l'''l)\left(\frac{F_l-F_{l'}}{\epsilon_l-\epsilon_{l'}}\right)\left(\frac{F_l-F_{l'''}}{\epsilon_l-\epsilon_{l'''}}\right)\right]. \quad (6.35)$$

The diagram in Fig. 11 is typical of a group of diagrams obtained from it by placing the impurity-particle vertices in all possible relative arrangements. Their contribution is found to be

$$W_{00}^{[11]}(kk')=n_i 2\pi[u(k-k')U_{k'k}^{(2b)}(\epsilon_k^-)+U_{kk'}^{(2b)}(\epsilon_k^+)u(k'-k)]\delta(\epsilon_k-\epsilon_{k'}), \quad (6.36)$$

where now

$$U_{kk'}^{(2b)}(\zeta)=u(k-k')\sum_{ll'l''l'''}V(kll'l')\left[\frac{V(l'''l'lk')}{\epsilon_{l''}-\epsilon_{l'''}}\times\left(\frac{A_l^{l'l'''}}{\zeta-\epsilon_{l'}-\epsilon_{l'''}+\epsilon_l}-\frac{A_l^{l'l''}}{\zeta-\epsilon_{l'}-\epsilon_{l''}+\epsilon_l}\right)+\frac{1}{2}\frac{V(l''l'l'''k')}{\epsilon_l-\epsilon_{l'''}}\times\left(\frac{A_{l'''}^{l'l''}}{\zeta-\epsilon_{l'}-\epsilon_{l''}+\epsilon_{l'''}}-\frac{A_l^{l'l''}}{\zeta-\epsilon_{l'}-\epsilon_{l''}+\epsilon_l}\right)\right]. \quad (6.37)$$

The quantities $U_{kk'}^{(1)}$, $U_{kk'}^{(2a)}$, $U_{kk'}^{(2b)}(\zeta)$, given by (6.32), (6.35), and (6.37), have a simple connection with the temperature Green's-function diagrams. For convenience in the following we define, up to second order in V_p, the quantity

$$U_{kk'}(\zeta)=u(k-k')+U_{kk'}^{(1)}+U_{kk'}^{(2a)}+U_{kk'}^{(2b)}(\zeta), \quad (6.38)$$

which plays the role of an impurity-quasiparticle interaction, as we shall see presently.

Collecting all the terms for $W_{00}(kk')$ up to second order in V_p and V_i, we note that they are of two types: (1) terms with a δ function of two energy arguments or derivatives of it; (2) terms with a δ function of four energy arguments or derivatives thereof. Terms of the first type will be designated as impurity-scattering terms, while terms of the second type will be referred to as interparticle-scattering terms. This designation will be justified below.

All impurity-scattering terms can be expressed quite simply, after some algebra, in terms of the quasiparticle quantities introduced earlier and the new quantity $U_{kk'}(\zeta)$, given by (6.38). We find for the impurity-scattering terms, denoted by $W_{00}^{(i)}(kk')$, up to second order in both V_p and V_i,

$$\sum_{k'}W_{00}^{(i)}(kk')f_{q\omega}(k')=\sum_{k'}(\delta_{kk'}/z_{k'}+\gamma_{kk'})C_{k'}^{(i)}[\bar{n}_{q\omega}(k')], \quad (6.39)$$

where the impurity-quasiparticle collision operator is

$$C_k^{(i)}[\bar{n}_{q\omega}(k)]=\sum_{k'}n_i 2\pi z_k z_{k'}|U_{kk'}(E_k^+)|^2\times\delta(E_k-E_{k'})[\bar{n}_{q\omega}(k')-\bar{n}_{q\omega}(k)], \quad (6.40)$$

with the "local" quasiparticle distribution function $\bar{n}_{q\omega}(k)$ given by (6.17).

Note that the form of the impurity-quasiparticle collision operator as given by (6.40) is the same as that suggested[8-10] on phenomenological grounds. The scattering amplitude $U_{kk'}$ is explicitly given in this approximation by (6.38).

Among the interparticle-scattering terms, there are some that are easily described in terms of impurity modifications of the particle energy, $\Delta\epsilon_k^i$, the renormalization constant Δz_k^i, and the transformation matrix $\gamma_{kk'}^i$. Up to second order in V_i, these are defined by

$$\Delta\epsilon_k^i=n_i\sum_{k'}\mathrm{Re}\frac{|u(k-k')|^2}{\epsilon_k^+-\epsilon_{k'}}, \quad (6.41)$$

$$\Delta z_k^i=-n_i\sum_{k'}\mathrm{Re}\frac{|u(k-k')|^2}{(\epsilon_k^+-\epsilon_{k'})^2}, \quad (6.42)$$

$$\gamma_{kk'}^i=-n_i\,\mathrm{Re}\frac{|u(k-k')|^2}{(\epsilon_k^+-\epsilon_{k'})^2}. \quad (6.43)$$

Reasoning as before, in connection with the replacement of (6.18) by (6.20) on the basis of the work of Watabe and Dagonnier,[23] we can reorganize these interparticle scattering terms, $W_{00}^{(p)}(kk')$, up to second order in V_p

and V_i, as

$$\sum_{k'} W_{00}{}^{(p)}(kk')f_{q\omega}(k') = \sum_{k'}\left(\frac{\delta_{kk'}}{\tilde{z}_{k'}}+\tilde{\gamma}_{kk'}\right)\tilde{C}_{k'}{}^{(p)}[\hat{n}_{q\omega}(k')]. \quad (6.44)$$

Here $\tilde{C}_k{}^{(p)}[\hat{n}_{q\omega}(k)]$ is identical to (6.22) with all quasiparticle quantities z_k, E_k, $\tilde{n}_{q\omega}(k)$ replaced by $\tilde{z}_k$, $\tilde{E}_k$, $\hat{n}_{q\omega}(k)$. The tildes or carets on these quantities denote the corresponding quantities for a new quasiparticle, more appropriate in the presence of the impurities. That is, if we define

$$\tilde{z}_k = z_k+\Delta z_k{}^i, \qquad \tilde{E}_k = E_k+\Delta\epsilon_k{}^i,$$
$$\tilde{\gamma}_{kk'} = \gamma_{kk'}+\gamma_{kk'}{}^i, \quad \tilde{N}_k = [e^{\beta(\tilde{E}_k-\mu)}+1]^{-1}, \quad (6.45)$$

an appropriate quasiparticle distribution function in the presence of both interparticle and impurity interactions is

$$\tilde{n}_{q\omega}(k) = \sum_{k'} (\delta_{kk'}/\tilde{z}_{k'}+\tilde{\gamma}_{kk'})f_{q\omega}(k'), \quad (6.46)$$

and its "local" value is

$$\hat{n}_{q\omega}(k) = \tilde{n}_{q\omega}(k)+\left(-\frac{\partial\tilde{N}_k}{\partial\tilde{E}_k}\right)\sum_{k'}\tilde{z}_k\tilde{z}_{k'}I_{kk'}\tilde{n}_{q\omega}(k'). \quad (6.47)$$

The remaining interparticle scattering terms can be written in the form

$$\sum_{k'} (\delta_{kk'}/\tilde{z}_{k'}+\tilde{\gamma}_{kk'})\Lambda_{k'}, \quad (6.48)$$

where $\Lambda_{k'}$ consists of the impurity modifications of the interparticle scattering which cannot be interpreted as simple renormalization. $\Lambda_{k'}$ is, of course, proportional to $n_i|u|^2$. An analysis of the temperature dependence of this term shows that for low temperatures it is also proportional to T^2. We conclude that (6.48) is of order $\lambda^2 f_{q\omega}$. According to our earlier discussion, in our approximation such terms may be ignored.

We may now collect all the terms that enter the transport equation (6.1) for small λ [the parameter representative of the quantities ω/μ, qv_F/μ, Γ_i/μ, and $\Gamma_p/\mu \propto (kT)^2$]. Keeping only terms of order $\lambda f_{q\omega}$ for the homogeneous terms and of order λ^0 for the inhomogeneous term, we find that the left-hand side of (6.1) is given by (6.16) and the right-hand side consists of the sum of (6.39) and (6.44). When solved this equation enables us to calculate the mean induced quantities up to order λ^{-1}. In this form, the equation for $f_{q\omega}(k)$ is unrevealing; we can, however, transform it to a simpler and physically more transparent equation for the quasiparticle distribution function $n_{q\omega}(k)$, the solution of which will enable us to evaluate the mean induced quantities to the same order in λ.

To achieve this, we note that insertion of terms of order $\lambda^2 f_{q\omega}$ and λ into the terms of (6.16) and (6.39) can have no effect on the results of the calculation of $f_{q\omega}(k)$ to order λ^{-1}. Thus, we may rewrite (6.16) and (6.39), respectively, as

$$\text{left-hand side (6.1)} = \sum_{k'} (\delta_{kk'}/\tilde{z}_{k'}+\tilde{\gamma}_{kk'}) \times[-i\omega\tilde{n}_{q\omega}(k')+iq^\alpha\tilde{v}_{k'}{}^\alpha\hat{n}_{q\omega}(k') +F_{q\omega}{}^\alpha\tilde{v}_{k'}{}^\alpha(\partial\tilde{N}_{k'}/\partial\tilde{E}_{k'})] \quad (6.49)$$

and

$$\sum_{k'} W_{00}{}^{(i)}(kk')f_{q\omega}(k') = \sum_{k'} (\delta_{kk'}/\tilde{z}_{k'}+\tilde{\gamma}_{kk'})\tilde{C}_{k'}{}^{(i)}[\hat{n}_{q\omega}(k')], \quad (6.50)$$

where $\tilde{C}_k{}^{(i)}[\hat{n}_{q\omega}(k)]$ is identical to (6.40) with z_k, E_k, $\tilde{n}_{q\omega}(k)$ replaced by $\tilde{z}_k$, $\tilde{E}_k$, $\hat{n}_{q\omega}(k)$, and $\tilde{v}_k{}^\alpha = \partial\tilde{E}_k/\partial k_\alpha$. The transport equation consists now of equating (6.49) to the sum of (6.50) and (6.44). If we multiply this equation by the inverse of the matrix $(\delta_{kk'}/\tilde{z}_{k'}+\tilde{\gamma}_{kk'})$, we have simply

$$-i\omega\tilde{n}_{q\omega}(k)+iq^\alpha\tilde{v}_k{}^\alpha\hat{n}_{q\omega}(k)+F_{q\omega}{}^\alpha\tilde{v}_k{}^\alpha(\partial\tilde{N}_k/\partial\tilde{E}_k) = \tilde{C}_k{}^{(i)}[\hat{n}_{q\omega}(k)]+\tilde{C}_k{}^{(p)}[\hat{n}_{q\omega}(k)]. \quad (6.51)$$

This looks very much like the Landau transport equation[11] for a quasiparticle distribution function $\tilde{n}_{q\omega}(k)$, defined by (6.46), with terms for impurity and interparticle scattering added. Furthermore, it is easily checked from (6.41)–(6.43) that, to the order of interest here, the mean induced particle and current densities are simply

$$\rho_{q\omega} = \sum_k \tilde{n}_{q\omega}(k), \quad j_{q\omega}{}^\alpha = \sum_k (k_\alpha/m)\tilde{n}_{q\omega}(k). \quad (6.52)$$

Alternatively, now that we have derived Eq. (6.51) for $\tilde{n}_{q\omega}$, we may drop terms of order $\lambda^2 f_{q\omega}$ in the homogeneous term of (6.51) and of order λ in the inhomogeneous term; again, this will not affect the results to order λ^{-1}. This procedure simply replaces $\tilde{n}_{q\omega}(k)$ by $n_{q\omega}(k)$, and similarly for all other quantities with tildes, and (6.51) becomes

$$-i\omega n_{q\omega}(k)+iq^\alpha v_k{}^\alpha\tilde{n}_{q\omega}(k)+F_{q\omega}{}^\alpha v_k{}^\alpha(\partial N_k/\partial E_k) = C_k{}^{(i)}[\tilde{n}_{q\omega}(k)]+C_k{}^{(p)}[\tilde{n}_{q\omega}(k)], \quad (6.53)$$

where now $n_{q\omega}(k)$ is the quasiparticle distribution function, defined by (6.15), appropriate for the pure system. The left-hand side of (6.53) evaluated at $T=0$°K, when equated to zero, is just the Landau transport equation for a pure Fermi liquid at $T=0$°K. The right-hand side gives the impurity scattering, as defined by (6.40), and the (low-temperature) interparticle scattering, as defined by (6.22). The quasiparticle distribution function $n_{q\omega}(k)$ can be used to calculate the induced quantities according to (6.27)–(6.28), to the lowest order in λ.

The transport equation (6.53), the left-hand side of which should be taken at $T=0$°K for consistency, along with (6.27)–(6.28), constitutes the main result of the application of the general theory established in the

earlier sections. Let us recall that we have proved these results only up to order V_p^2.

It may be wondered why our calculation led to Eq. (6.53) for $n_{q\omega}(k)$ via Eq. (6.51) for $\tilde{n}_{q\omega}(k)$. Observe that the method we have used gives a transport equation for the bare-particle distribution function $f_{q\omega}(k)$. If we had gone directly from $f_{q\omega}(k)$ to $n_{q\omega}(k)$, i.e., had we transformed from bare to quasiparticle distributions with the matrix $(\delta_{kk'}/z_{k'}+\gamma_{kk'})$, we would have found terms involving $\gamma_{kk'}{}^i$, the physical significance of which would have been obscure. Using the transformation matrix $(\delta_{kk'}/\tilde{z}_{k'}+\tilde{\gamma}_{kk'})$, on the other hand, we found that these terms can be eliminated without any consequence, to the order of the parameter λ in which we treat the problem. That is, these terms just effect the correct transformation from bare-particle to quasiparticle distribution function; since this produces terms in the current density, etc., of order λ^0, it can be dropped altogether.

C. Results for the Case of Coulomb Interactions

As we saw in general in Sec. 5, for the important case of Coulomb interactions among the electrons, the coefficients of the transport equation for $f_{q\omega}(k)$ consist of the proper diagrams for $W_{q\omega}(kk')$, denoted by $\tilde{W}_{q\omega}(kk')$, and the proper diagrams for $G_{q\omega}{}^{\alpha}(k)$, denoted by $\tilde{G}_{q\omega}{}^{\alpha}(k)$, multiplied by the total field $\bar{F}_{q\omega}{}^{\alpha}$ in the medium. Thus, for small $\mathbf{q}$ and ω the transport equation is identical to (6.1), except for the replacements just indicated.

Examining $G_{q\omega}{}^{\alpha}(k)$ to order λ^0 and up to second order in V_p, we have found that all improper graphs give vanishing contributions, i.e., $G_{q\omega}{}^{\alpha}$ is the same as $\tilde{G}_{q\omega}{}^{\alpha}$. Using this result and the fact that for Coulomb interactions

$$V(kk'k'k)=v(0)-v(k-k')=-v(k-k'), \quad (6.54)$$

we have that $\tilde{G}_{00}{}^{\alpha}(k)$ is formally identical to $G_{00}{}^{\alpha}(k)$ calculated for a short-range interaction. Similarly, we find, up to the orders considered in this section, that with the identification (6.54) $\tilde{W}_{00}(kk')$, $\partial\tilde{W}_{q\omega}/\partial\omega|_{00}$, and $\partial\tilde{W}_{q\omega}/\partial q^{\alpha}|_{00}$ are identical to the corresponding quantities for the short-range case. We can thus proceed as before and derive a transport equation for the quasiparticle distribution function $n_{q\omega}(k)$, given by (6.15) in conjunction with (6.54), identical to (6.53), except for $F_{q\omega}{}^{\alpha}$ which is replaced by $\bar{F}_{q\omega}{}^{\alpha}$, the total force field in the medium. In fact, to the order of interest here, this modification arises from the improper diagrams for $W_{q\omega}(kk')$. There are only three such, the contribution of which for small $\mathbf{q}$ in the transport equation can be shown to amount to the replacement of $F_{q\omega}{}^{\alpha}$ by

$$\bar{F}_{q\omega}{}^{\alpha}=F_{q\omega}{}^{\alpha}+i^{-1}q^{\alpha}v(q)\sum_{k} n_{q\omega}(k), \quad (6.55)$$

where use of (6.27) was made. That is, in this approximation the driving field for the quasiparticles is the sum of the external field and the average field associated with the induced quasiparticle density.

Thus, the transport equation for $n_{q\omega}(k)$, which involves $\bar{F}_{q\omega}{}^{\alpha}$, is to be solved self-consistently in conjunction with (6.55), if the values of $\rho_{q\omega}$ and $j_{q\omega}{}^{\alpha}$ are desired for a given external field. If, however, the conductivity or dielectric constant is desired, one has only to solve the transport equation treating $\bar{F}_{q\omega}{}^{\alpha}$ as given.