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Phase-Space Analysis of Time-Correlation Functions*

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Using the recently developed phase-space techniques for the treatment of quantum-mechanical problems, we set up a procedure for calculating multitime-correlation functions in terms of the joint distribution functions. The correlation functions are then expressed simply as integrals over the associated phase space. Explicit expressions are given for these joint distribution functions, in terms of Green's functions of the c-number equations of motion for the phase-space equivalent of the density operator. Using these joint distribution functions, an exact regression theorem is rederived, and the connection with the multitime correspondence between classical and quantum stochastic processes is discussed.

I. INTRODUCTION

During the last several years, increasing use has been made of time-correlation functions in the description of the behavior of physical systems. Recently Zwanzig[1] summarized the main results in this area and discussed some applications of time-correlation functions to nonequlibrium problems (see also Ref. 2). In the calculation of quantum correlation functions, use has been made of phase-space techniques.[3-6] In this connection the Wigner distribution function has played a preferential role[3] and has been used to obtain first quantum corrections[4-6] to time-correlation functions calculated classically. This in turn permits one to obtain quantum corrections to transport coefficients. Similar procedures have been useful in discussing a wide variety of problems such as nuclear magnetic relaxation,[4] neutron scattering,[5] hydrodynamic transport coefficients, etc. However, in the discussion of certain problems in quantum optics, it is useful to use other distribution functions[7] based on different rules of association between functions of noncommuting operators and c-number functions. Recently a general technique, for the derivation of the different distribution functions from a unified point of view, was developed[8-10] and has been used to study dynamical problems.[10]

In the present investigation, we extend this analysis to construct various joint distribution functions. These functions are then used to express multitime-correlation functions as integrals over the associated phase space. An exact regression theorem,[11] relating two-time averages to one-time averages, which follows readily from our analysis, is also discussed.

II. ORDERING DELTA OPERATORS AND SOME RELATED RESULTS

Let us recall a few results which have been presented elsewhere.[8-10] In these papers, certain classes of ordering delta operators were introduced. These ordering delta operators are defined as

$$\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger)=\hat{\Omega}\delta^{(2)}(z-z_0) \tag{2.1}$$

$$=\pi^{-2}\int d^2\alpha\ \Omega(\alpha,\alpha^*)\times\exp[\alpha(z_0^*-\hat{a}^\dagger)-\alpha^*(z_0-\hat{a})]. \tag{2.2}$$

Here $\hat{\Omega}$ denotes an ordering operator (one for each rule of association), and the function $\Omega(\alpha,\beta)$ appearing in Eq. (2.2) is an entire analytic function of two complex variables α and β, which has no zeros and satisfies the requirements

(i) $\Omega(\alpha,\beta)=\Omega(-\alpha,-\beta)$, (ii) $\Omega(0,0)=1$.

Then it was shown that any operator function $G(\hat{a},\hat{a}^\dagger)$ of Boson creation and annihilation operators can be expanded in the following manner:

$$G(\hat{a},\hat{a}^\dagger)=\int F^{(\Omega)}(z,z^*)\Delta^{(\Omega)}(z-\hat{a},z^*-\hat{a}^\dagger)d^2z, \tag{2.3}$$

The function $F^{(\Omega)}(z,z^*)$ appearing in the right-hand side of Eq. (2.3) is the c-number function associated with the operator $G(\hat{a},\hat{a}^\dagger)$ via the Ω correspondence and is given by

$$F^{(\Omega)}(z,z^*)=\pi\,\mathrm{Tr}[G(\hat{a},\hat{a}^\dagger)\Delta^{(\bar{\Omega})}(z-\hat{a},z^*-\hat{a}^\dagger)], \tag{2.4}$$

wnere

$$\Delta^{(\bar{\Omega})}(z-\hat{a},z^*-\hat{a}^\dagger)=\pi^{-2}\int d^2\alpha[\Omega(\alpha,\alpha^*)]^{-1}\times\exp[\alpha(z_0^*-\hat{a}^\dagger)-\alpha^*(z_0-\hat{a})]. \tag{2.5}$$

Using the properties of the ordering delta operators, it was shown that the trace of the product of two operators $G_1(\hat{a},\hat{a}^\dagger)$ and $G_2(\hat{a},\hat{a}^\dagger)$ is given by

$$\mathrm{Tr}[G_1(\hat{a},\hat{a}^\dagger)G_2(\hat{a},\hat{a}^\dagger)]=\pi^{-1}\int d^2z\ F_1^{(\Omega)}(z,z^*)F_2^{(\bar{\Omega})}(z,z^*), \tag{2.6}$$

Here $F_1^{(\Omega)}(z,z^*)$ and $F_2^{(\bar{\Omega})}(z,z^*)$ are the c-number functions associated with the operators and $G_1(\hat{a},\hat{a}^\dagger)$ and $G_2(\hat{a},\hat{a}^\dagger)$ via the Ω correspondence and $\bar{\Omega}$ correspondence, respectively.[12]

For a wide class of associations of interest $\Omega(\alpha,\alpha^*)$ is of the form

$$\Omega(\alpha,\alpha^*)=\exp(\mu\alpha^2+\nu\alpha^{*2}+\lambda\alpha\alpha^*), \tag{2.7}$$

where μ,ν,λ are parameters. The Ω-ordered equivalent $F_{12}^{(\Omega)}(z,z^*)$ of the product $G_1(\hat{a},\hat{a}^\dagger)G_2(\hat{a},\hat{a}^\dagger)$ is then given by

$$F_{12}^{(\Omega)}(z,z^*)=F_1^{(\Omega)}(z,z^*)\times\exp(\overleftrightarrow{\Lambda}_1+\overleftrightarrow{\Lambda}_2)F_2^{(\Omega)}(z,z^*), \tag{2.8}$$

with

$$\overleftrightarrow{\Lambda}_1=-2\nu\frac{\overleftarrow{\partial}}{\partial z}\frac{\overrightarrow{\partial}}{\partial z}-2\mu\frac{\overleftarrow{\partial}}{\partial z^*}\frac{\overrightarrow{\partial}}{\partial z^*}+\lambda\left(\frac{\overleftarrow{\partial}}{\partial z}\frac{\overrightarrow{\partial}}{\partial z^*}+\frac{\overleftarrow{\partial}}{\partial z^*}\frac{\overrightarrow{\partial}}{\partial z}\right), \tag{2.9}$$

$$\overleftrightarrow{\Lambda}_2=\frac{1}{2}\left(\frac{\overleftarrow{\partial}}{\partial z}\frac{\overrightarrow{\partial}}{\partial z^*}-\frac{\overleftarrow{\partial}}{\partial z^*}\frac{\overrightarrow{\partial}}{\partial z}\right). \tag{2.10}$$

Throughout this paper we consider rules of associations for which $\Omega(\alpha,\alpha^*)$ is of the form (2-7).

III. THE c-NUMBER FUNCTION ASSOCIATED WITH THE HEISENBERG OPERATOR $\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger,t)$

The Heisenberg operator $\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger,t)$ is defined by

$$\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger,t)=\hat{U}^\dagger(t,t_0)\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger,t_0)\hat{U}(t,t_0), \tag{3.1}$$

where $\hat{U}(t,t_0)$ is the unitary time-evolution operator. Let $F_\rho^{(\Omega)}(z,z^*,t)$ be the Ω-ordered equivalent of the density operator $\hat{\rho}(t)$ associated with a quantum-mechnical system, i.e.,

$$\hat{\rho}(t)=\hat{\Omega}[F_\rho^{(\Omega)}(z,z^*,t)]. \tag{3.2}$$

The Ω-ordered equivalents of the density operator at times t_0 and $t>t_0$ are related by an equation of the form

$$F_\rho^{(\Omega)}(z,z^*,t)=\int d^2z_0K^{(\Omega)}(z,z^*,t\,|\,z_0,z_0^*,t_0)F_\rho^{(\Omega)}(z_0,z_0^*,t_0), \tag{3.3}$$

where $K^{(\Omega)}(z,z^*,t\,|\,z_0,z_0^*,t_0)$ is the Green's function for the equation of motion for $F_\rho^{(\Omega)}(z,z^*,t)$. The

Green's function $K^{(\Omega)}(z,z^*,t\,|z_0,z_0^*,t_0)$ satisfies the initial condition

$$K^{(\Omega)}(z,z^*,t_0|z_0,z_0^*,t_0)=\delta^{(2)}(z-z_0). \tag{3.4}$$

From (3.2) and (3.3) it follows that

$$\hat{\rho}(t)=\int d^2z_0F_\rho^{(\Omega)}(z_0,z_0^*,t_0)[\hat{\Omega}K^{(\Omega)}(z,z^*,t|z_0,z_0^*,t_0)]. \tag{3.5}$$

According to (2.3), we can also express the density operator at time t_0 in terms of ordering delta operators, i.e.,

$$\hat{\rho}(t_0)=\int d^2z_0F_\rho^{(\Omega)}(z_0,z_0^*,t_0)\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger,t_0). \tag{3.6}$$

Therefore the time evolution of the density operator is given by

$$\begin{aligned}\hat{\rho}(t)&=\hat{U}(t,t_0)\hat{\rho}(t_0)\hat{U}^\dagger(t,t_0)\\&=\int d^2z_0F_\rho^{(\Omega)}(z_0,z_0^*,t_0)\hat{U}(t,t_0)\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger,t_0)\hat{U}^\dagger(t,t_0).\end{aligned} \tag{3.7}$$

On comparison of (3.5) and (3.7) one finds that

$$\hat{\Omega}[K^{(\Omega)}(z,z^*,t|z_0,z_0^*,t_0)]=\hat{U}(t,t_0)\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger,t_0)\hat{U}^\dagger(t,t_0). \tag{3.8}$$

On inverting the relation (3.8), with the help of theorems II and III of Ref. 9, we have

$$\begin{aligned}&K^{(\Omega)}(z,z^*,t|z_0,z_0^*,t_0)\\&=\pi\mathrm{Tr}[\hat{U}(t,t_0)\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger,t_0)\hat{U}^\dagger(t,t_0)\Delta^{(\bar{\Omega})}(z-\hat{a},z^*-\hat{a}^\dagger,t_0)]\\&=\pi\mathrm{Tr}[\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger,t_0)\hat{U}^\dagger(t,t_0)\Delta^{(\bar{\Omega})}(z-\hat{a},z^*-\hat{a}^\dagger,t_0)\hat{U}(t,t_0)]\\&=\pi\mathrm{Tr}[\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger,t_0)\Delta^{(\bar{\Omega})}(z,-\hat{a},z^*-\hat{a}^\dagger,t)].\end{aligned} \tag{3.9}$$

On making use of theorem II of Ref. 9, we find that

$$\pi\mathrm{Tr}[\Delta^{(\Omega)}(z_0-\hat{a},z_0^*-\hat{a}^\dagger,t_0)\Delta^{(\bar{\Omega})}(z-\hat{a},z^*-\hat{a}^\dagger,t)]$$

is the $\bar{\Omega}$ equivalent of $\Delta^{(\bar{\Omega})}(z-\hat{a},z^*-\hat{a},t)$, i.e.,

$$\Delta^{(\bar{\Omega})}(z_2-\hat{a},z_2^*-\hat{a}^\dagger,t)=\hat{\bar{\Omega}}[K^{(\Omega)}(z_2,z_2^*,t|z,z^*,t_0)]. \tag{3.10}$$

We may summarize the result of this section in the form of the following theorem:

The $\bar{\Omega}$ equivalent of the Heisenberg operator $\Delta^{(\bar{\Omega})}(z_2-\hat{a},z_2^*-\hat{a}^\dagger,t)$ defined by (3.1), is given by $K^{(\Omega)}(z_2,z_2^*,t\,|z,z^*,t_0)$, which is the Green's function of the equation of motion for the Ω equivalent of the density operator $\hat{\rho}(t)$.

IV. JOINT DISTRIBUTION FUNCTION FOR TWO-TIME-CORRELATION FUNCTIONS

Let us now consider the correlation function $R(t_1,t_2)$ defined by

$$R(t_1,t_2)\equiv\langle G_2(\hat{a},\hat{a}^\dagger,t_2)G_1(\hat{a},\hat{a}^\dagger,t_1)\rangle,\quad(t_2\geq t_1), \tag{4.1}$$

where the sharp bracket denotes the quantum-mechanical average. We make use of Eq. (2.3) to expand $G_1(\hat{a},\hat{a}^\dagger,t_1)$ and $G_2(\hat{a},\hat{a}^\dagger,t_2)$ in terms of the ordering delta operators. This gives

$$G_1(\hat{a},\hat{a}^\dagger,t_1)=\int d^2z_1F_1^{(\bar{\Omega})}(z_1,z_1^*,t_1)\Delta^{(\bar{\Omega})}(z_1-\hat{a},z_1^*-\hat{a}^\dagger,t_1), \tag{4.2}$$

$$G_2(\hat{a},\hat{a}^\dagger,t_2)=\int d^2z_2F_2^{(\Omega)}(z_2,z_2^*,t_1)\Delta^{(\Omega)}(z_2-\hat{a},z_2^*-\hat{a}^\dagger,t_2). \tag{4.3}$$

Using (4.2) and (4.3), the correlation function (4.1) can be expressed in the form

$$R(t_1,t_2)=\iint d^2z_1d^2z_2F_1^{(\bar{\Omega})}(z_1,z_1^*,t_1)F_2^{(\Omega)}(z_2,z_2^*,t_1)\,\mathcal{P}(z_2,z_2^*,t_2;z_1,z_1^*,t_1), \tag{4.4}$$

where

$$\mathcal{P}(z_2,z_2^*,t_2;z_1,z_1^*,t_1)=\langle\Delta^{(\Omega)}(z_2-\hat{a},z_2^*-\hat{a}^\dagger,t_2)\Delta^{(\bar{\Omega})}(z_1-\hat{a},z_1^*-\hat{a}^\dagger,t_1)\rangle \tag{4.5}$$

is the joint distribution function for two-time-correlation functions. Now using the result (2.8) and the result of Sec. 3, we find that the Ω-ordered equivalent $F_{\rho\Delta}^{(\Omega)}$ of the operator $\hat{\rho}\Delta^{(\Omega)}(z_2-\hat{a},z_2^*-\hat{a}^\dagger,t_2)$, such that

$$\hat{\rho}\Delta^{(\Omega)}(z_2-\hat{a},z_2^*-\hat{a}^\dagger,t_2)=\hat{\Omega}[F_{\rho\Delta}^{(\Omega)}],$$

is given by

$$F_{\rho\Delta}^{(\Omega)}=F_\rho^{(\Omega)}(z,z^*,t_1)\exp(\overleftrightarrow{\Lambda}_1+\overleftrightarrow{\Lambda}_2)K^{(\bar{\Omega})}(z_2,z_2^*,t_2\,|\,z,z^*,t_1). \tag{4.6}$$

Similarly the $\bar{\Omega}$-ordered equivalent $F_\Delta^{(\bar{\Omega})}$ of the operator $\Delta^{(\bar{\Omega})}(z_1-\hat{a},z_1^*-\hat{a}^\dagger,t_1)$, such that

$$\Delta^{(\bar{\Omega})}(z_1-\hat{a},z_1^*-\hat{a}^\dagger,t_1)=\hat{\bar{\Omega}}[F_\Delta^{(\bar{\Omega})}]$$

is given by

$$F_\Delta^{(\bar{\Omega})}=\delta^{(2)}(z-z_1). \tag{4.7}$$

Now using Eq. (2.6), (4.6), and (4.7), we find that (4.5) may be expressed in the following form:

$$\begin{aligned}\mathcal{P}&(z_2,z_2^*,t_2;z_1,z_1^*,t_1)\\&=\pi^{-1}\int d^2z\,\delta^{(2)}(z-z_1)[F_\rho^{(\Omega)}(z,z^*,t_1)\exp(\overleftrightarrow{\Lambda}_1+\overleftrightarrow{\Lambda}_2)K^{(\bar{\Omega})}(z_2,z_2^*,t_2\,|\,z,z^*,t_1)]\\&=\pi^{-1}F_\rho^{(\Omega)}(z_1,z_1^*,t_1)\exp(\overleftrightarrow{\Lambda}_1+\overleftrightarrow{\Lambda}_2)K^{(\bar{\Omega})}(z_2,z_2^*,t_2\,|\,z_1,z_1^*,t_1),\end{aligned} \tag{4.8}$$

where now the operators $\overleftrightarrow{\Lambda}_1$ and $\overleftrightarrow{\Lambda}_2$ act on functions of z_1 and z_1^*. We now introduce the distribution function $\Phi_\rho^{(\Omega)}(z,z^*,t)$ for Ω ordering associated with the density operator $\hat{\rho}(t)$, via the relation

$$\Phi_\rho^{(\Omega)}(z,z^*,t)=\pi^{-1}F_\rho^{(\Omega)}(z,z^*,t), \tag{4.9}$$

so that $\Phi_\rho^{(\Omega)}(z,z^*,t)$ is properly normalized ($\mathrm{Tr}\rho=1$). It is then seen from (4.8) and (4.9) that the joint distribution function is given by

$$\mathcal{P}(z_2,z_2^*t_2;z_1,z_1^*,t_1)=\Phi_\rho^{(\Omega)}(z_1,z_1^*,t_1)\exp(\overleftrightarrow{\Lambda}_1+\overleftrightarrow{\Lambda}_2)K^{(\bar{\Omega})}(z_2,z_2^*,t_2\,|\,z_1,z_1^*,t_1). \tag{4.10}$$

We now use the joint distribution function to find a relation between two-time averages and one-time averages. For this purpose, we take $G_1(\hat{a},\hat{a}^\dagger,t_1)$ to be the identity operator. For the identity operator $F_1^{(\bar{\Omega})}(z_1,z_1^*,t_1)=1$ and (4.4) gives the following expression for the one-time average:

$$\langle G_2(\hat{a},\hat{a}^\dagger,t_2)\rangle=\iint d^2z_1d^2z_2F_2^{(\Omega)}(z_2,z_2^*,t_1)\mathcal{P}(z_2,z_2^*,t_2;z_1,z_1^*,t_1). \tag{4.11}$$

On comparing (4.4) and (4.11) we find that, in both the cases, the essential time dependence, i.e., the dependence on t_2, is contained in the joint distribution function $\mathcal{P}(z_1,z_2^*,t_2;z_1,z_1^*,t_1)$ and this can be taken as the statement of the exact regression theorem. The same theorem has been proved by Louisell and Margburger[13] in a different way. Lax has extensively used the regression theorem, in a slightly different form, to discuss Markoff processes in quantum-mechanical context.

V. CONNECTION WITH THE MULTITIME CORRESPONDENCE

Let us write down the expression (4.4) for the case when $\hat{\Omega}$ represents normal and antinormal orderings, respectively. For normal ordering it follows from (2.9), (2.10), and (4.10) that the correlation function $R(t_1,t_2)$ is given by

$$R(t_1,t_2)=\iint d^2z_1d^2z_2F_1^{(A)}(z_1,z_1^*,t_1)F_2^{(N)}(z_2,z_2^*,t_1)\times\left[\Phi_\rho^{(N)}(z_1,z_1^*,t_1)\exp\left(\frac{\overleftarrow{\partial}}{\partial z_1}\frac{\overrightarrow{\partial}}{\partial z_1^*}\right)K^{(A)}(z_2,z_2^*,t_2|z_1,z_1^*,t_1)\right]. \tag{5.1}$$

For antinormal ordering, one has from (2.9), (2.10), and (4.10)

$$R(t_1,t_2)=\iint d^2z_1d^2z_2F_1^{(N)}(z_1,z_1^*,t_1)F_2^{(A)}(z_2,z_2^*,t_1)\times\left[\Phi_\rho^{(A)}(z_1,z_1^*,t_1)\exp\left(-\frac{\overleftarrow{\partial}}{\partial z_1^*}\frac{\overrightarrow{\partial}}{\partial z_1}\right)K^{(N)}(z_2,z_2^*,t_2|z_1,z_1^*,t_1)\right]. \tag{5.2}$$

In particular, using (5.1), the correlation function $\langle a^\dagger(t_2)\hat{a}(t_1)\rangle$ is given by

$$\langle\hat{a}^\dagger(t_2)\hat{a}(t_1)\rangle=\iint d^2z_1d^2z_2(z_2^*z_1)\times\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\partial^n}{\partial z_1^n}\Phi_\rho^{(N)}(z_1,z_1^*,t_1)\right)\left(\frac{\partial^n}{\partial z_1^{*n}}K^{(A)}(z_2,z_2^*,t_2|z_1,z_1^*,t_1)\right). \tag{5.3}$$

On integration by parts Eq. (5.3) leads to

$$\langle\hat{a}^\dagger(t_2)\hat{a}(t_1)\rangle=\iint d^2z_1d^2z_2(z_2^*z_1)K^{(A)}(z_2,z_2^*,t_2|z_1,z_1^*,t_1)\exp\left(-\frac{\overrightarrow{\partial^2}}{\partial z_1\partial z_1^*}\right)\Phi\rho^{(N)}(z_1,z_1^*,t_1)$$

$$=\iint d^2z_1d^2z_2(z_2^*z_1)K^{(A)}(z_2,z_2^*,t_2|z_1,z_1^*,t_1)\Phi\rho^{(A)}(z_1,z_1^*,t_1), \tag{5.4}$$

where the relation[14]

$$\exp\left(-\frac{\overrightarrow{\partial^2}}{\partial z_1\partial z_1^*}\right)\Phi_\rho^{(N)}(z_1,z_1^*,t_1)=\Phi_\rho^{(A)}(z_1,z_1^*,t_1) \tag{5.5}$$

has been used. Similarly the relation (5.2) gives

$$\langle\hat{a}^\dagger(t_2)\hat{a}(t_1)\rangle=\iint d^2z_2d^2z_1\,z_2^*[(z_1+\partial/\partial z_1^*)\Phi_\rho^{(N)}(z_1,z_1^*,t_1)]K^{(N)}(z_2,z_2^*,t_2|z_1,z_1^*,t_1). \tag{5.6}$$

The relations (5.4) and (5.6) follow easily from the multitime correspondence between classical and quantum stochastic processes. For it has been shown elsewhere[15,16] that the generating function S, for the time ordered, normally order correlation functions

$$\langle[\hat{a}^\dagger(t_1)]^{i_1}\cdots[\hat{a}^\dagger(t_n)]^{i_n}[\hat{a}(t_n)]^{j_n}\cdots[\hat{a}(t_1)]^{j_1}\rangle,\quad(t_j\geq t_{j-1}),$$

is given by

$$S=\int\cdots\int d^2z_1\cdots d^2z_n\prod_{\lambda=1}^{n}e^{i\xi_\lambda z_\lambda^*+i\xi_\lambda^*z_\lambda}K^{(A)}(z_\lambda,z_\lambda^*,t_\lambda|z_{\lambda-1},z_{\lambda-1}^*,t_{\lambda-1}). \tag{5.7}$$

Here $K^{(A)}(z_\lambda,z_\lambda^*,t_\lambda|z_{\lambda-1},z_{\lambda-1}^*,t_{\lambda-1})$ for $\lambda\neq1$ is the Green's function of the equation of motion satisfied by the antinormally ordered equivalent of the density operator, and for $\lambda=1$, $K^{(A)}$ is given by

$$K^{(A)}(z_1,z_1^*,t_1|z_0,z_0^*,t_0)=\Phi_\rho^{(A)}(z_1,z_1^*,t_1). \tag{5.8}$$

The generating function S can also be expressed in terms of the Green's function $K^{(N)}(z_\lambda,z_\lambda^*,t_\lambda|z_{\lambda-1},z_{\lambda-1}^*,t_{\lambda-1})$, $(\lambda\neq1)$, i.e.,

$$S=\iint d^2z_1\cdots d^2z_n\prod_{\lambda=1}^{n}K^{(N)}(z_\lambda,z_\lambda{}^*,t_\lambda\,|\,z_{\lambda-1}-i\xi_{\lambda-1},z_{\lambda-1}{}^*-i\xi_{\lambda-1}{}^*,t_{\lambda-1})$$

$$\times\exp(i\xi_\lambda{}^*z_\lambda+i\xi_\lambda z_\lambda{}^*+\xi_\lambda\xi_\lambda{}^*),\tag{5.9}$$

where $$K^{(N)}(z_1,z_1{}^*,t_1\,|\,z_0,z_0{}^*,t_0)=\Phi_\rho^{(N)}(z_1,z_1{}^*,t_1).\tag{5.10}$$

Expressions of the type (5.1), (5.2), (5.4), and (5.6) are useful in computing the spectrum of amplitude and phase fluctuations of, say, laser beams[17] One usually calculates the Green's function appearing in (5.4) and (5.6) by eigenfunction-expansion methods[17,18]. In some cases one can also obtain closed expressions for the Green's functions. As an example we consider a system for which the Hamiltonian is given by

$$H=\omega(t)\hat{a}^\dagger\hat{a}+g(t)\hat{a}+g^*(t)\hat{a}^\dagger,\quad(\hbar=1).\tag{5.11}$$

In this case, the distribution function Φ_ρ for normal as well as antinormal orderings satisfies the equation of motion[10]

$$i\frac{\partial\Phi_\rho}{\partial t}=[-\omega(t)z-g^*(t)]\frac{\partial\Phi_\rho}{\partial z}+[\omega(t)z^*+g(t)]\frac{\partial\Phi_\rho}{\partial z^*}.\tag{5.12}$$

The Green's function is given by (see the Appendix)

$$K^{(N)}(z_2,z_2{}^*,t_2\,|\,z_1,z_1{}^*,t_1)=K^{(A)}(z_2,z_2{}^*,t_2\,|\,z_1,z_1{}^*,t_1)$$
$$=\delta^{(2)}\Big[z_2-e^{-i\beta(t_2,t_1)}z_1+ie^{-i\beta(t_2,t_1)}\int_{t_1}^{t_2}dt'g^*(t')e^{i\beta(t',t_1)}\Big],\tag{5.13}$$

where $$\beta(t_2,t_1)=\int_{t_1}^{t_2}\omega(t')dt'.\tag{5.14}$$

Then Eq. (5.4) and (5.6) lead to the following expressions:

$$\langle\hat{a}^\dagger(t_2)\hat{a}(t_1)\rangle=\int d^2z_1\Phi_\rho^{(A)}(z_1,z_1{}^*,t_1)z_1$$
$$\times\Big(e^{-i\beta(t_2,t_1)}z_1-ie^{-i\beta(t_2,t_1)}\int_{t_1}^{t_2}dt'g^*(t')e^{i\beta(t',t_1)}\Big)^*\tag{5.15}$$
$$=\int d^2z_1\Big(z_1+\frac{\partial}{\partial z_1{}^*}\Big)\Phi_\rho^{(N)}(z_1,z_1{}^*,t_1)$$
$$\times\Big(e^{-i\beta(t_2,t_1)}z_1-ie^{-i\beta(t_2,t_1)}\int_{t_1}^{t_2}dt'g^*(t')e^{i\beta(t',t_1)}\Big)^*.\tag{5.16}$$

One could have also obtained the result (5.15) by using the coherent-state techniques, for it is well known[19] that for the Hamiltonian (5.11) a coherent state remains a coherent state as the system evolves in time.

VI. HIGHER-ORDER JOINT DISTRIBUTION FUNCTIONS

Using again the method of Sec. 4, we can show that the multitime correlation function, defined by

$$R(t_1,t_2,\ldots,t_n)\equiv\langle\hat{G}_n(t_n)\ldots\hat{G}_1(t_1)\rangle,\quad(t_n\geqslant t_{n-1}\cdots\geqslant t_1),$$

is given by

$$R(t_1,t_2,\ldots,t_n)=\int\cdots\int d^2z_1\cdots d^2z_nF_n^{(\Omega)}(z_n,z_n{}^*,t_1)\cdots F_2^{(\Omega)}(z_2,z_2{}^*,t_1)$$
$$\times F_1^{(\bar\Omega)}(z_1,z_1{}^*,t_1)\,\mathcal{P}(z_n,z_n{}^*,t_n;\ldots;z_1,z_1{}^*,t_1),\tag{6.1}$$

where $\mathcal{P}(z_n,z_n{}^*,t_n;\ldots;z_1,z_1{}^*,t_1)$

$$=\langle\Delta^{(\Omega)}(z_n-\hat{a},z_n{}^*-\hat{a}^\dagger,t_n)\cdots\Delta^{(\Omega)}(z_2-\hat{a},z_2{}^*-\hat{a}^\dagger,t_2)\Delta^{(\bar\Omega)}(z_1-\hat{a},z_1{}^*-\hat{a}^\dagger,t_1)\rangle.\tag{6.2}$$

To simplify (6.2) further, we make use of the following *theorem*:
The Ω-ordered equivalent of $G_1(\hat{a},\hat{a}^\dagger)\cdots G_n(\hat{a},\hat{a}^\dagger)$ is given by[16]

$$\exp\left(\sum_{i<j}\Lambda_1^{ij}+\sum_{i<j}\Lambda_2^{ij}\right)F_1^{(\Omega)}(\alpha_1,\alpha_1^*)\cdots F_n^{(\Omega)}(\alpha_n,\alpha_n^*)\Big|_{\substack{\alpha_l=z\\ \alpha_l^*=z^*}},\quad (l=1,2\ldots,n) \tag{6.3}$$

with

$$\Lambda_1^{ij}=-2\mu\frac{\partial}{\partial\alpha_i^*}\frac{\partial}{\partial\alpha_j^*}-2\nu\frac{\partial}{\partial\alpha_i}\frac{\partial}{\partial\alpha_j}+\lambda\left(\frac{\partial}{\partial\alpha_i}\frac{\partial}{\partial\alpha_j^*}+\frac{\partial}{\partial\alpha_i^*}\frac{\partial}{\partial\alpha_j}\right), \tag{6.4}$$

$$\Lambda_2^{ij}=\frac{1}{2}\left(\frac{\partial}{\partial\alpha_i}\frac{\partial}{\partial\alpha_j^*}-\frac{\partial}{\partial\alpha_i^*}\frac{\partial}{\partial\alpha_j}\right). \tag{6.5}$$

Then using (2.6), (3.10), (4.7), and (6. 3) we find that the joint distribution function is given by

$$\mathcal{P}(z_n,z_n^*,t_n;z_{n-1},z_{n-1}^*,t_{n-1};\ldots;z_1,z_1^*,t_1)=\exp\left(\sum_{i<j}\Lambda_1^{ij}+\sum_{i<j}\Lambda_2^{ij}\right)$$

$$\times\Phi_\rho^{(\Omega)}(\alpha_1,\alpha_1^*,t_1)K^{(\bar{\Omega})}(z_n,z_n^*,t_n|\alpha_2,\alpha_2^*,t_1)\cdots K^{(\bar{\Omega})}(z_2,z_2^*,t_2|\alpha_n,\alpha_n^*,t_1)\Big|_{\substack{\alpha_l=z\\ \alpha_l^*=z^*}},\quad (l=1,\ldots,n). \tag{6.6}$$

These higher-order joint distribution functions are useful especially in defining quantum mechanically a Markoff process. We can define quantum mechanically the Markoff process as a process for which these higher-order joint distribution functions factorize, in the following way[11]:

$$\mathcal{P}(z_n,z_n^*,t_n;\ldots;z_1,z_1^*,t_1)=\mathcal{P}(z_n,z_n^*,t_n|z_{n-1},z_{n-1}^*,t_{n-1})\,\mathcal{P}(z_{n-1},z_{n-1}^*,t_{n-1};\ldots;z_1,z_1^*,t_1). \tag{6.7}$$

Here the first factor is independent of $t_{n-2},t_{n-3},\ldots,t_1$.

Finally we mention that throughout this paper we considered functions of Boson annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$. However the same problem could have been treated by considering functions of position and momentum operators $\hat{q}$ and $\hat{p}$. Then the joint distribution functions are functions of p and q variables, where p and q are the c numbers associated with the operators $\hat{p}$ and $\hat{q}$ respectively. These joint distribution function of p and q may be used to find quantum corrections to classically defined correlation functions. A different approach has been employed by other authors (see, for example, Ref. 4 - 6).

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APPENDIX

In this Appendix, we wish to determine the Green's function associated with the equation (5.12). The Green's function $K(z,z^*,t\,|z_0,z_0^*,t_0)$ is the solution of the differential equation

$$i\frac{\partial K}{\partial t}=[-\omega(t)z-g^*(t)]\frac{\partial K}{\partial z}+[\omega(t)z^*+g(t)]\frac{\partial K}{\partial z^*}, \tag{A.1}$$

subject to the initial condition

$$K(z,z^*,t_0\,|z_0,z_0^*,t_0)=\delta^{(2)}(z-z_0). \tag{A.2}$$

We express $\delta^{(2)}$ in the form

$$\delta^{(2)}(z-z_0)=\lim_{\epsilon\to\infty}\epsilon e^{-\epsilon(z-z_0)(z-z_0)^*}. \tag{A.3}$$

Following Louisell and Marburger,[13] we try a solution of the form

$$K(z,z^*,t\,|z_0,z_0^*,t_0)=\exp\left(-\frac{[z-\gamma(t)][z-\gamma(t)]^*}{\xi(t)}+\ln\Gamma(t)\right). \tag{A.4}$$

From the equation (A.4) at $t=t_0$ and the formula (A.3), it follows that

$$\gamma(t_0)=z_0,\quad\gamma^*(t_0)=z_0^*,\quad\xi(t_0)=1/\epsilon,\quad\Gamma(t_0)=\epsilon. \tag{A.5}$$

We now substitute (A.4) in (A.1) and equate the coefficients of equal powers of z and z^* on both sides. We then obtain the equations

$$\frac{\partial\xi(t)}{\partial t}=0, \tag{A.6}$$

$$\frac{\partial\gamma(t)}{\partial t}+i\omega(t)\gamma(t)=-ig^*(t), \tag{A.7}$$

$$\frac{i}{\Gamma(t)}\frac{\partial\Gamma(t)}{\partial t}-i\epsilon\frac{\partial}{\partial t}|\gamma(t)|^2$$

$$=\epsilon g(t)\gamma(t)-\epsilon g^*(t)\gamma^*(t). \quad (A.8)$$

Using (A.7) and (A.8) we find that

$$\frac{\partial\Gamma(t)}{\partial t}=0. \quad (A.9)$$

The solution of (A.6) - (A.9) under the initial condition (A.5) is given by

$$\xi(t)=\xi(t_0)=1/\epsilon, \quad (A.10)$$

$$\Gamma(t)=\Gamma(t_0)=\epsilon, \quad (A.11)$$

$$\gamma(t)=e^{-i\beta(t,t_0)}z_0 - i\,e^{-i\beta(t,t_0)}\int_{t_0}^{t}dt'g^*(t')e^{i\beta(t',t_0)}, \quad (A.12)$$

$$\beta(t,t_0)=\int_{t_0}^{t}\omega(t')dt'. \quad (A.13)$$

On substituting (A.10)–(A.12) in (A.4) and taking the limit $\epsilon\to\infty$, we obtain

$$K(z,z^*,t\,|z_0,z_0^*,t_0) = \lim_{\epsilon\to\infty}\epsilon e^{-\epsilon[z-\gamma(t)][z-\gamma(t)]^*} = \delta^{(2)}[z-\gamma(t)], \quad (A.14)$$

where $\gamma(t)$ is given by (A.12).

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[12]By $\bar{\Omega}$ correspondence we mean a correspondence reciprocal to the Ω correspondence. Thus the c-number function associated with the operator $G(\hat{a},\hat{a}^\dagger)$ via the $\bar{\Omega}$ correspondence will be given by

$$F^{(\bar{\Omega})}(z,z^*) = \pi\,\mathrm{tr}[G(\hat{a},\hat{a}^\dagger)\Delta^{(\Omega)}(z-\hat{a},z^*-\hat{a}^\dagger)].$$

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[14]The relation (5.5) is obtained by inverting the relation[9]:

$$\Phi_\rho^{(N)}(z',z'^*,t) = \pi^{-1}\int\Phi_\rho^{(A)}(z,z^*,t)e^{-|z-z'|^2}d^2z.$$

This inversion can be readily performed by using the theory of Weierstrass transforms. [See, for example, I.I. Hirschman and D.V. Widder, The Convolution Transform (Princeton University Press, Princeton, N.J., 1955), Chap. VIII.

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