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Finite-Temperature Corrections to the Transport Coefficients of a Normal Fermi Liquid*

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Finite-temperature corrections to the limiting low-temperature behavior of the thermal conductivity and spin-diffusion coefficient of a normal Fermi liquid are calculated using a quasiparticle Boltzmann equation which is solved by the variational method. For liquid He^3 the most important corrections come from small momentum transfer processes, the amplitude for which can be determined exactly by using Landau theory. The expressions obtained are evaluated in detail, assuming that all Landau parameters vanish for $l \geq 2$, and comparison of theory with experiment yields an estimate for the previously undetermined Landau parameter $F_1{}^a$. The calculations are compared with calculations based on paramagnon theory.

1. INTRODUCTION

Although the limiting low-temperature behavior of the specific heat, staticresponse functions, and transport coefficients were predicted over 10 years ago on the basis of Landau's theory of normal Fermi liquids,[1] it is only recently that the properties of a Fermi liquid at finite temperatures have been studied theoretically. The main stimulus to recent theoretical work was the experimental observation[2] that even at 50 m°K the specific heat of liquid He^3 showed appreciable deviations from the linear temperature dependence predicted by Landau theory. Doniach and Engelsberg[3] were able to account for the observed specific-heat data by using a model in which persistent spin fluctuations play an important role. They found that the specific heat calculated using their model had a contribution of order $T^3 \ln T$ as well as the term linear in T predicted by Landau theory. Other calculations of the $T^3 \ln T$ term in the specific heat have been performed by Brenig and Mikeska[4] and Amit, Kane, and Wagner[5] using Landau theory, by Brenig, Mikeska, and Riedel[6] and Brinkman and Engelsberg[7] using the random-phase approximation, and by Riedel[8] using the shielded potential approximation of Baym and Kadanoff.[9]

The low-temperature behavior of the thermal conductivity K and the viscosity η were first discussed in detail by Abrikosov and Khalatnikov,[10] and the spin-diffusion coefficient D was considered by Hone.[11] These authors used a quasiparticle Boltzmann equation with a collision term to take into account binary collisions and found that as $T\to 0$ the quantities $1/KT$, $1/DT^2$, and $1/\eta T^2$ tend to constants. Finite-temperature corrections to the limiting low-temperature forms of the transport coefficients have been calculated by Rice[12] using a model in which bare fermions are scattered by persistent spin fluctuations ("paramagnons"). In this model the fermion-paramagnon interaction is treated in much the same way as the interaction of electrons and phonons in the electron-phonon problem.[13] The part of the paramagnon propagator is played by the wave-number- and frequency-dependent susceptibility. Rice found that the leading finite-temperature corrections to $1/KT$ and $1/DT^2$ were of order T and that the leading corrections to $1/\eta T^2$ were of order T^3.

By inspecting the paramagnon theory calculations,[12] one can see that the terms of order T in $1/KT$ and $1/DT^2$ come from processes in which fermions emit or absorb small momentum paramagnons. Since a paramagnon is nothing more than an interacting particle-hole pair, such scattering processes may alternatively be described as small momentum transfer binary collisions between quasiparticles, and therefore one can evaluate transport coefficients using the standard quasiparticle Boltzmann equation[1] with a binary collision term. The observation which is crucial to the work to be described below is that the amplitude for collisions in which the momentum transfer $\vec{q}$ is small may be determined exactly in terms of Landau parameters without invoking the concept of a paramagnon. Thus one finds that corrections to $1/KT$ and $1/DT^2$ of order T are a general feature of any Fermi liquid irrespective of whether or not it is almost ferromagnetic or, more generally, almost unstable against a spontaneous deformation of the Fermi surface.[14]

Thus the linear correction terms to $1/KT$ and $1/DT^2$ coming from small momentum transfer processes, far from heralding a breakdown in Landau theory, may in fact be calculated in terms of Landau parameters; and one is led to the remarkable conclusion that one can pin down some of the finite-temperature corrections to $1/KT$ and $1/DT^2$ better than the low-temperature limits of these quantities. Our results enable us to predict a relationship between the finite-temperature corrections to $1/KT$ and those to $1/DT^2$ without making any assumptions about the size of the spin-antisymmetric Landau parameters. This relationship is well satisfied for liquid He^3 at low pressures.

The basic calculations of the finite-temperature corrections to $1/KT$ and $1/DT^2$ are described in Sec. 2. Certain mathematical details relating to the evaluation of these corrections in terms of Landau parameters are given in Appendices A and B. In Sec. 3 applications of the theory to almost ferromagnetic Fermi liquids and to liquid He^3 are considered, and an estimate of the Landau parameter F_1^a is obtained. A brief discussion of the finite-temperature corrections to $1/KT$ and $1/DT^2$ arising from processes in which the momentum transfer is not small is given in Sec. 4, and we argue that these corrections should be small in liquid He^3 and in almost ferromagnetic Fermi liquids. Section 5 is a conclusion, and in Appendix C we discuss the relationship between our work and the paramagnon theory calculations of Rice.[12]

2. CORRECTIONS RESULTING FROM SMALL MOMENTUM TRANSFER PROCESSES

In this section we describe the variational calculation of the transport coefficients, and for definiteness we consider the calculation of the thermal conductivity. If it is assumed that the collision integral in the quasiparticle Boltzmann equation contains only terms corresponding to binary collisions, the variational expression for the thermal conductivity is given by[15]

$$\frac{1}{KT}=\frac{1}{4T}\sum_{1,2,3,4}W_{12}{}^{34}n_1n_2(1-n_3)(1-n_4)(\phi_1+\phi_2-\phi_3-\phi_4)^2\left(\sum_1\phi_1\epsilon_1\vec{v}_1\cdot\vec{u}\,\frac{\partial n_1}{\partial\epsilon_1}\right)^{-2}$$
$$\times\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)\delta_{\sigma_1+\sigma_2,\,\sigma_3+\sigma_4}\delta_{\vec{p}_1+\vec{p}_2,\,\vec{p}_3+\vec{p}_4} \qquad (1)$$

Here ϵ_i and $\vec{v}_i$ are the free energy and velocity of a quasiparticle of momentum $\vec{p}_i$ and spin σ_i and $W_{12}{}^{34}$ is the transition probability for the process $1+2\to 3+4$ with $i\equiv(\vec{p}_i,\sigma_i)$. Also ϕ_i is the variational trial function and $n_i\equiv n(\epsilon_i)$ is the Fermi distribution function $[\exp(\beta\epsilon_i)+1]^{-1}$, where β is the inverse temperature.[16] $\vec{u}$ is an arbitrary unit vector, and the sum in Eq. (1) is to be taken over all distinct final states.

To calculate $1/KT$ we need to know the quasiparticle scattering amplitude, and in general this depends on both the energies and the momenta of the quasiparticles. However, when the momentum transfer $\vec{q}\,(=\vec{p}_1-\vec{p}_3)$ is small, and $\vec{p}=\frac{1}{2}(\vec{p}_1+\vec{p}_3)$ and $\vec{p}\,'=\frac{1}{2}(\vec{p}_2+\vec{p}_4)$ are nearly equal to the Fermi momentum p_F, the scattering amplitude depends chiefly on the relative orientation of the vectors $\vec{p}$, $\vec{p}\,'$, and $\vec{q}$, and on the variable $s=\omega/v_Fq$, where ω is the energy transfer corresponding to the momentum transfer q and v_F is the Fermi velocity. For the purposes of the present calculation the dependence on other variables may be neglected.

The possible spin orientations of the initial and final quasiparticles are shown in Fig. 1, where for simplicity we show only processes in which the quasiparticle of momentum $\vec{p}+\vec{q}/2(=\vec{p}_1)$ has spin up. From the invariance of the quasiparticle interaction under rotation of the spins, it follows that the amplitude for the process shown in Fig. 1(c) is the difference between the amplitudes for the other two processes. We denote the amplitudes for these processes by the expressions

$$\begin{aligned}&(a)\ [A^s(\cos\theta,\cos\theta',\phi,s)+A^a(\cos\theta,\cos\theta',\phi,s)]/\nu(0)\,,\\&(b)\ [A^s(\cos\theta,\cos\theta',\phi,s)-A^a(\cos\theta,\cos\theta',\phi,s)]/\nu(0)\,,\\&(c)\ 2A^a(\cos\theta,\cos\theta',\phi,s)/\nu(0)\,,\end{aligned}\tag{2}$$

where $\nu(0)=m^*p_F/\pi^2$ is the density of quasiparticle states at the Fermi surface. ($m^*=p_F/v_F$ is the effective mass.) θ is the angle between $\vec{p}$ and $\vec{q}$, θ' is the angle between $\vec{p}'$ and $\vec{q}$, and ϕ is the angle between the plane containing $\vec{p}$ and $\vec{q}$ and the plane containing $\vec{p}'$ and $\vec{q}$. A^s and A^a may be determined in terms of Landau parameters, and we give certain of the mathematical details of this calculation in Appendix A. Finally we remark that in evaluating the scattering amplitude for real processes one must put the initial and final quasiparticles on the energy shell, which is equivalent to the condition $\omega=\epsilon_{\vec{p}+\frac{1}{2}\vec{q}}-\epsilon_{\vec{p}-\frac{1}{2}\vec{q}}=\epsilon_{\vec{p}'+\frac{1}{2}\vec{q}}-\epsilon_{\vec{p}'-\frac{1}{2}\vec{q}}$.

Let us now evaluate $1/K_LT$, the contribution to $1/KT$ which comes from processes in which the momentum transfer q is less than some cut-off momentum $q_L(\ll p_F)$. For ϕ_i we take the usual trial function $\epsilon_i\vec{v}_i\cdot\vec{u}$,[15] and to the order to which we are working the quasiparticle velocity may be replaced by v_F. The transition probabilities W are given by 2π times the squared moduli of the amplitudes 2(a)–(c). On substituting the transition probabilities into Eq. (1), one finds

$$\begin{aligned}\frac{1}{K_LT}=\frac{27\pi^5}{m^{*2}p_F^4T^5}\sum_{\vec{q}\,(q<q_L),\,\vec{p},\vec{p}'}\int_{-\infty}^{\infty}\frac{d\omega(|A^s(\cos\theta,\cos\theta',\phi,s)|^2+3|A^a(\cos\theta,\cos\theta',\phi,s)|^2)}{(e^{\beta\omega}-1)(1-e^{-\beta\omega})}\\ \times[n(\epsilon_{\vec{p}-\frac{1}{2}\vec{q}})-n(\epsilon_{\vec{p}+\frac{1}{2}\vec{q}})]\,\delta(\omega-\vec{p}\cdot\vec{q}/m^*)[n(\epsilon_{\vec{p}'-\frac{1}{2}\vec{q}})-n(\epsilon_{\vec{p}'+\frac{1}{2}\vec{q}})]\\ \times\delta(\omega-\vec{p}'\cdot\vec{q}/m^*)\{(\vec{p}\omega+\vec{q}\bar{\epsilon}_{\vec{p}})\cdot[(\vec{p}-\vec{p}')\omega+\vec{q}(\bar{\epsilon}_{\vec{p}}-\bar{\epsilon}_{\vec{p}'})]\}\,,\end{aligned}\tag{3}$$

where $2\bar{\epsilon}_{\vec{p}}=\epsilon_{\vec{p}+\frac{1}{2}\vec{q}}+\epsilon_{\vec{p}-\frac{1}{2}\vec{q}}$. In deriving Eq. (3) we have averaged over all directions of $\vec{p}$, $\vec{p}'$, and $\vec{q}$ relative to $\vec{u}$ for fixed values of θ, θ', and ϕ, and we have also used the identity $n_1(1-n_3)=(n_1-n_3)/\{1-\exp[-\beta(\epsilon_3-\epsilon_1)]\}$. The term in braces in Eq. (3) may be replaced by $\vec{p}\cdot(\vec{p}-\vec{p}')\omega^2+q^2\bar{\epsilon}_{\vec{p}}^{\,2}$, since the other terms vanish on performing the integrations. This term is expanded in powers of q, and the summation over $\vec{q}$ in Eq. (3) is transformed into an integral over s between the limits $|\omega|/v_Fq_L$ and unity. The zeroth-order term in this expansion, $p_F^2(1-s^2)(1-\cos\phi)\omega^2$, leads to a contribution to $1/K_LT$ which may be written in the form

$$\frac{27}{8\pi}\frac{m^{*3}}{p_F^7}T\int_{-\infty}^{\infty}\frac{dx|x|^5}{(e^x-1)(1-e^{-x})}\int_0^{2\pi}\frac{d\phi}{2\pi}(1-\cos\phi)\int_{s_0}^{1}\frac{ds}{s^2}(1-s^2)\,[\,|\alpha^s(s,\phi)|^2+3|\alpha^a(s,\phi)|^2]\,,\tag{4}$$

where $\alpha^i(s,\phi)=A^i(s,s,\phi,s)$ and $s_0=|x|T/v_Fq_L$. Strictly speaking the limits of the x integration in (4) are $\pm v_Fq_L/T$ but these may be replaced by $\pm\infty$ without altering the term linear in T. The divergence at the lower limit of the integral over s is separated out by rewriting the integral in the following way:

$$\begin{aligned}\int_{s_0}^{1}ds\,s^{-2}|\alpha^i(s,\phi)|^2(1-s^2)=(v_Fq_L/|x|T)\,|\alpha^i(0,\phi)|^2\\ -\left[\int_0^1 ds\left(\frac{|\alpha^i(0,\phi)|^2-|\alpha^i(s,\phi)|^2}{s^2}+|\alpha^i(s,\phi)|^2\right)+|\alpha^i(0,\phi)|^2\right]+O\left(\frac{|x|T}{v_Fq_L}\right).\end{aligned}\tag{5}$$

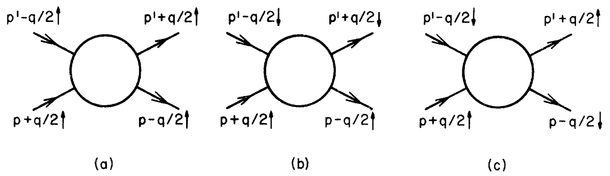

FIG. 1. The possible spin orientations of quasiparticles in a binary collision. Only processes in which the quasiparticle of momentum $\vec{p}+\vec{q}/2$ has spin up are shown.

The first term in Eq. (5) gives a temperature-independent contribution to $1/K_L T$, and the term in square brackets gives a contribution linear in T. Higher-order terms in the expansion in powers of q of the expression in braces in Eq. (3) do not give contributions linear in T, although they do give temperature-independent terms and contributions of order T^2 and higher. Thus, if we retain only terms of order T, we may write Eq. (3) in the form

$$(1/K_L T) - (1/K_L T)_{T=0} = -[810\zeta(5)m^{*3}/\pi p_F^{7}]\,(\Xi^s + 3\Xi^a)T, \tag{6}$$

where

$$\Xi^i = \int_0^{2\pi} \frac{d\phi}{2\pi}(1-\cos\phi)\left[\int_0^1 ds\left(\frac{|a^i(0,\phi)|^2 - |a^i(s,\phi)|^2}{s^2} + |a^i(s,\phi)|^2\right) + |a^i(0,\phi)|^2\right] \tag{7}$$

and $\zeta(n)$ is the Riemann zeta function of order n. We remark that the right-hand side of Eq. (6) is independent of the cutoff q_L. Calculations of a number of contributions to Ξ^i will be found in Appendix B but we defer discussion of these results until the next section.

If one now neglects correction terms of order T coming from processes in which the momentum transfer exceeds q_L, one can immediately rewrite Eq. (6) in the form

$$(1/KT) - (1/KT)_{T=0} = -[810\zeta(5)m^{*3}/\pi p_F^{7}]\,\hbar^2 k_B(\Xi^s + 3\Xi^a)T, \tag{8}$$

where we have inserted the appropriate powers of $\hbar$ and Boltzmann's constant k_B.

The calculations of the finite-temperature corrections to $1/DT^2$ closely parallel those for $1/KT$, and one finds

$$(1/DT^2) - (1/DT^2)_{T=0} = -36\pi\zeta(3)[m^{*4}k_B^{3}/p_F^{6}\hbar\,(1+F_0^{a})]\Xi^a T\quad . \tag{9}$$

In Sec. 4 we shall return to the problem of processes in which q is not small, and we shall argue that the other correction terms linear in T are small in liquid He^3 and in almost ferromagnetic Fermi liquids. However, for the present we accept the validity of Eqs. (8) and (9) and in the next section we consider applications of these results.

3. APPLICATION OF THE RESULTS

(i) A general result

By comparing Eqs. (8) and (9) one can see that if it is possible to estimate Ξ^s, two independent "experimental" values of Ξ^a may be obtained from measurements of the thermal conductivity and the spin-diffusion coefficient. If the theory gives a consistent account of the data the two values of Ξ^a will be the same. We shall not discuss this result further here, but will return to it when we consider the experimental data for liquid He^3 later in this section.

(ii) The almost ferromagnetic limit

The results of our calculations are rather complicated, in general, but they may be simplified considerably in the case of an almost ferromagnetic Fermi liquid ($F_0^a \to -1$). In this limit the parameter $A_0^a = F_0^a/(1+F_0^a)$ is large, and it will therefore be a good approximation to consider only the leading term in an expansion of Ξ^a in descending powers of A_0^a. Ξ^s is independent of the antisymmetric Landau parameters and may therefore be neglected. In Appendix B we give expressions for Ξ^a calculated on the assumption that Landau parameters for $l \geq 2$ may be neglected. By examining these, one can see that the dominant contribution to Ξ^a in the almost ferromagnetic limit is $-\frac{1}{4}\pi^2(A_0^a)^3$ and comes from the first term in the expression for Ξ_{00}^a [Eq. (B.3)]. The dominant contribution to Ξ_{01}^a [Eq. (B.6)] is of order $(A_0^a)^2 A_1^a$ and may therefore be neglected. If one includes higher Landau parameters in the calculation of Ξ^a, the dominant contribution is still $-\frac{1}{4}\pi^2(A_0^a)^3$. Substituting this value into Eqs. (8) and (9) one finds

$$\left.\frac{1}{KT} - \frac{1}{KT}\right|_{T=0} = \frac{1215}{2}\pi\zeta(5)\frac{m^{*3}}{p_F^{7}}\hbar^2 k_B (A_0^a)^3 T \tag{10}$$

$$= \frac{45}{2}\pi\frac{\zeta(5)}{p_F}\frac{\hbar^2}{k_B^2}\frac{(F_0^a)^3}{T_M^3}T, \tag{11}$$

and

$$\frac{1}{DT^2} - \frac{1}{DT^2}\bigg|_{T=0} = 9\pi^3\zeta(3)\frac{m^{*4}}{p_F^6}\frac{k_B^3}{\hbar}\frac{(A_0^a)^3}{1+F_0^a}\,T \quad (12)$$

$$= \frac{\pi^3}{9}\zeta(3)\frac{p_F^2}{\hbar k_B}\frac{(F_0^a)^3}{T_M^4}\,T, \quad (13)$$

where T_M, the effective magnetic temperature of the Fermi liquid,[17] is given by the expression

$$T_M = p_F^2(1+F_0^a)/3m^* k_B\,. \quad (14)$$

T_M may be obtained directly from measurements of the magnetic susceptibility without making use of a value for m^*. The expressions (11) and (13) will be useful in Appendix C where we compare our calculations with those based on paramagnon theory.

We notice that according to Eqs. (10) and (12) the finite-temperature corrections to $1/KT$ and $1/DT^2$ vary as $(A_0^a)^3\,T$, and we now try to explain this dependence in simple terms. As one can see by examining the discussion at the end of Appendix B, the normalized quasiparticle scattering amplitude for small momentum transfer processes is well approximated by the expression

$$A_0^a/(1+\tfrac{1}{2}i\pi A_0^a s) \text{ when } |s| = |\omega|/v_F q \ll 1.$$

Thus the transition probability falls off quickly as $|s|$ increases beyond the value $1/|A_0^a|$. In the limit $T\to 0$ only scattering processes in which $s\to 0$ are important. However, at finite temperatures the energy transfer in a collision is typically of the order of $T(s\sim T/v_F q)$, and therefore for processes in which $q \lesssim |A_0^a|\,T/v_F$ the transition probability will on the average be considerably less than its value in the limit $T\to 0$. Thus in Eqs. (10) and (12) a factor $|A_0^a|\,T$ is accounted for by the number of "ineffective" values of q weighted with a factor $1/q^2$, and two remaining factors of A_0^a come from the transition probability. The weight factor is most easily explained by looking on the binary-collision process as the scattering of a quasiparticle-quasihole pair from one state to another. The density of initial and final pair states of a given energy each vary as $1/q$, which accounts for the weight factor $1/q^2$.

(iii) Application to liquid He^3

Although the results for the almost ferromagnetic limit are not applicable to liquid He^3 there are large finite-temperature corrections to $1/KT$ and $1/DT^2$ whose magnitude is consistent with theory. The thermal conductivity data of Abel, Johnson, Wheatley, and Zimmermann[18] have recently been re-analyzed by Abel and Wheatley[19] in connection with some thermometry studies. Instead of fitting $1/KT^*$ versus T^* (T^* is the effective magnetic temperature) to a straight line, Abel and Wheatley fitted values of $1/K(T^*+\Delta)$ versus $T^*+\Delta$ to a straight line for different values of Δ. A Δ of $+0.3$ m°K gives a smaller mean-square deviation from the line than either 0.2 or 0.4 m°K. This value of Δ is consistent with values determined by other methods.[19] In Wheatley's opinion $T^*+\Delta$ with a Δ of 0.3 m°K is very close to T above 3 m°K, and we assume in our analysis that the difference may be neglected. For $\Delta = 0.3$ m°K a fit to the data gives

$$1/K(T^*+\Delta) = [(2.87\pm 0.1) - (18\pm 3)(T^*+\Delta)(\mathrm{K}^\circ)^{-1}]\times 10^{-2}\ \text{sec cm/erg}.$$

A possible error of $\pm$ 0.1 m°K in Δ was included in estimating the errors in the parameters of the line. A plot of the data is given in Fig. 2. The spin-diffusion data of Anderson, Reese, Sarwinski, and Wheatley[20] were analyzed using $\Delta = 0.3$ m°K, and the fit to the data is given by

$$1/D(T^*+\Delta)^2 = [(0.695\pm 0.012) - (2.0\pm 0.3)(T^*+\Delta)(\mathrm{K}^\circ)^{-1}]\times 10^6\ \text{sec/cm}^2\ (\mathrm{K}^\circ)^2.$$

These data are displayed in Fig. 3.

To check the prediction made in (i) above we need a value of Ξ^s. We calculated this assuming that all spin-symmetric Landau parameters vanish for $l \geq 2$. The general formulas given in Appendix B are not particularly useful for numerical calculations for liquid He^3 because of the large amount of cancellation between the terms from the poles corresponding to collective modes and the other contributions. However, we do not need to know Ξ^s with great accuracy since $\Xi^s \ll \Xi^a$. For $F_1^s = 6.0$, the value obtained from the data of Wheatley,[21] and $F_0^s\to\infty$, Ξ^s may be evaluated analytically by the techniques described in Appendix B and we find

$$\Xi^s = \frac{24}{5} - \pi^2/4 \simeq 2.333.$$

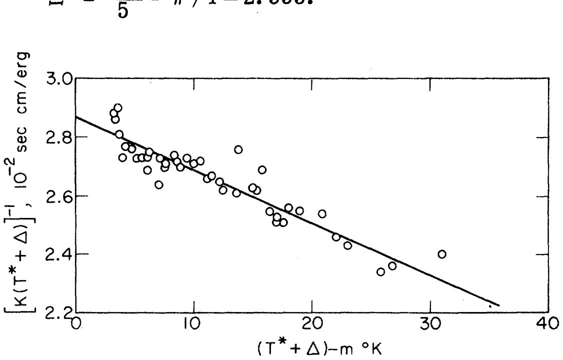

FIG. 2. The thermal conductivity of liquid He^3 at low pressure (the saturated vapor pressure). The data points are those of Abel, Johnson, Wheatley, and Zimmermann (Ref. 18) and the line is given by $[K(T^*+\Delta)]^{-1} = [2.87 - 18(T^*+\Delta)\ (\mathrm{K}^\circ)^{-1}]\times 10^{-2}$ sec cm/erg ($\Delta = 0.3$ m°K) which is obtained by fitting a straight line to the data.

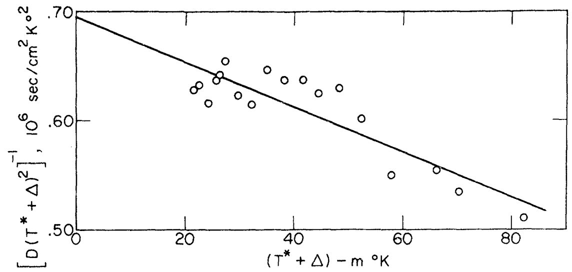

FIG. 3. The spin-diffusion coefficient of liquid He^3 at low pressure (0.17 atm). The data points are those of Anderson, Reese, Sarwinski, and Wheatley (Ref. 20), and the line is given by $[D(T^*+\Delta)^2]^{-1} = [0.695 - 2.0(T^* + \Delta)(K°)^{-1}] \times 10^6$ sec/cm^2(K°)^2 ($\Delta = 0.3$ m°K) which is obtained by fitting a straight line to the data.

The corrections to the strong-coupling approximation are of order $(F_0^s)^{-2}$ for $F_1^s = 6$ and are therefore completely negligible for liquid He^3. Using Eqs. (8) and (9) and the values $m^*/m = 3.0$ and $F_0^a = -0.67$ obtained from the work of Wheatley,[2,21] we find that the "experimental" values of Ξ^a obtained from the thermal conductivity and spin-diffusion data are 11 ± 2 and 13 ± 2, respectively.[22] The agreement between these numbers is remarkably good. We can now go one stage further and estimate F_1^a if we make the popular assumption that Landau parameters with $l \geq 2$ may be neglected. The values we find are

$$F_1^a = -0.66 \pm 0.09 \text{ (thermal conductivity)}$$

and $F_1^a = -0.58 \pm 0.09$ (spin diffusion).

The quoted errors are only those due to uncertainties in the parameters of the lines – the true errors are certainly larger and include contributions from approximations in the theory.

The values we obtain for F_1^a are consistent with the inequality $F_1^s \geq F_1^a$ derived by Leggett[23] and are not very different from the estimate $F_1^a = -0.70$ based on the forward-scattering sum rule,[24] $A_0^s + A_1^s + A_0^a + A_1^a = 0$ assuming higher Landau parameters vanish.

4. OTHER FINITE-TEMPERATURE CORRECTIONS

In this section we briefly discuss contributions to $1/KT$ and $1/DT^2$ from processes in which the momentum transfer is not small. For a system of fermions interacting via a weak local potential one can calculate the contribution to $1/DT^2$ of order T exactly, and the result depends only on the Fourier transform of the potential for a wave number $2p_F$. If one calculates the finite-temperature corrections coming from processes in which the momentum transfer (in either the direct or the exchange channel) is small, one obtains a different answer. The procedure employed in this paper is therefore inconsistent for the weak-coupling problem and presumably the inconsistencies will persist under more general conditions, although we have not isolated the particular sorts of process which give rise to the other contributions linear in T. However, we note that in liquid He^3 and in almost ferromagnetic Fermi liquids the linear correction terms coming from small momentum transfer processes are greatly enhanced as a result of the strong dependence of the scattering amplitude on the variable s (essentially the energy transfer). By studying the Bethe-Salpeter equations for particle-particle and particle-hole scattering we have sought in other parts of momentum space for a strongly energy-dependent quasiparticle scattering amplitude which could lead to a large enhancement of the finite-temperature corrections to the transport coefficients. We did not detect any energy dependence which would alter the linear correction term[25] to $1/KT$ and $1/DT^2$ and conclude that in liquid He^3 and in almost ferromagnetic Fermi liquids only the contributions from small q processes will be enhanced. It should therefore be a good approximation to neglect all but small q processes.

Our conclusions are consistent with the results of Amit, Kane, and Wagner[5] which indicate that if one neglects processes in which the momentum transfer is not small the inconsistencies introduced into the calculations of non-analytic terms in the self-energy are only of the order of a few percent for liquid He^3.

Finally we note that collisions involving three or more quasiparticles make no contribution to the linear corrections to $1/KT$ or $1/DT^2$.

5. CONCLUSION

The theory we have devoloped gives a consistent account of the available data for D and K for liquid He^3. By fitting our theory to the experimental results for K and D we find $F_1^a = -0.66$ and $F_1^a = -0.58$, respectively. The estimates of F_1^a are likely to contain appreciable errors due to our use of a variational method to calculate the transport coefficients,[26] the neglect of processes involving a momentum transfer $q > q_L$ and the neglect of Landau parameters for $l \geq 2$.

If one tries to calculate finite-temperature corrections to the viscosity using the methods described here, one finds that the dominant terms do not come from small-angle scattering processes and therefore cannot be calculated in terms of Landau parameters alone.

The calculations we have described are very close in spirit to recent work of Emery,[14] but the results differ by appreciable numerical coefficients as a result of approximations made in the latter work and a different procedure for solving the Boltzmann equation.

Finally we stress the value of more extensive measurements of the thermal conductivity and spin-diffusion coefficient of liquid He^3 at low temperatures.

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APPENDIX A. SOLUTION OF THE LANDAU EQUATIONS

In this Appendix we describe the evaluation of $A^s(\cos\theta,\ \cos\theta',\ \phi, s)$ and $A^a(\cos\theta,\ \cos\theta',\ \phi, s)$ in terms of Landau parameters. Landau's version of the Bethe-Salpeter equation[27] for a particle-hole pair may be written in the form

$$A^i(\mu,\mu',\phi,s) = A^i(\mu,\mu',\phi,\infty) + (4\pi)^{-1}\int d\Omega'' A^i(\mu,\mu'',\phi-\phi'',s)[\mu''/(s-\mu'')]A^i(\mu'',\mu',\phi'',\infty), \quad \text{(A.1)}$$

where $d\Omega''$ is the element of solid angle $d\mu''\,d\phi''$. To solve Eq. (A.1) the A^i are expanded in terms of Legendre polynomials:

$$A^i(\mu,\mu',\phi,s) = \sum_{l_1 l_2,\, m} A_{l_1 l_2}{}^{im}(s) P_{l_1}{}^m(\mu) P_{l_2}{}^m(\mu') e^{im\phi} \ . \quad \text{(A.2)}$$

The values of $A_{l_1 l_2}{}^{im}(\infty)$ are related to the Landau parameters by the expression

$$A_{l_1 l_2}{}^{im}(\infty) = \delta_{l_1 l_2}[(l-|m|)!/(l+|m|)!]F_{l_1}{}^i \ . \quad \text{(A.3)}$$

Combining Eqs. (A.1)–(A.3) we find

$$A_{l_1 l_2}{}^{im}(s) = \delta_{l_1 l_2}[(l-|m|)!/(l+|m|)!]F_{l_1}{}^i - \sum_{l_3} F_{l_1}{}^i \Omega_{l_1 l_3}{}^m A_{l_3 l_2}{}^{im}(s) \ , \quad \text{(A.4)}$$

where $\Omega_{ll'}{}^m(s) = [(l-|m|)!/(l+|m|)!]\int \tfrac{1}{2} d\mu' P_l{}^m(\mu')[\mu'/(\mu'-s)]P_{l'}{}^m(\mu')$.

We solve Eq. (A.4) assuming that only the Landau parameters $F_0{}^i$ and $F_1{}^l$ are nonzero. The solutions are

$$A_{00}{}^{i0}(s) = F_0{}^i[1+F_1\Omega_{11}{}^0(s)]/\Delta^i, \quad A_{10}{}^{i0}(s) = A_{01}{}^{i0} = -F_0{}^i\Omega_{01}{}^0(s)F_1{}^i/\Delta^i,$$
$$A_{11}{}^{i0}(s) = F_1{}^i[1+F_0\Omega_{00}{}^0(s)]/\Delta^i, \quad A_{11}{}^{i1}(s) = \tfrac{1}{2}F_1{}^i[1+F_1\Omega_{11}{}^1(s)]^{-1}, \quad \text{(A.5)}$$

where $\Delta^i = [1+F_1\Omega_{11}{}^0(s)][1+F_0\Omega_{00}{}^0(s)] - F_0F_1[\Omega_{01}{}^0(s)]^2$.

The functions Ω^m are given by the expressions

$$\Omega_{00}{}^0 = \chi, \quad \Omega_{10}{}^0 = \Omega_{01}{}^0 = s\chi, \quad \Omega_{11}{}^0 = \tfrac{1}{3} + s^2\chi, \quad \Omega_{11}{}^1 = \tfrac{1}{2}[(1-s^2)\chi - \tfrac{1}{3}], \quad \text{(A.6)}$$

where $\chi(s) = 1 - \tfrac{1}{2}s\ln[(s+1)/(s-1)]$.

Substituting Eq. (A.5) into Eq. (A.2) one finds

$$\mathcal{A}^i(s,\phi) \equiv A^i(s,s,\phi,s) = \mathcal{A}_0{}^i(s) + 2\mathcal{A}_1{}^i(s)\cos\phi, \quad \text{(A.7)}$$

where $\mathcal{A}_0{}^i(s) = (F_0{}^i + A_1{}^i s^2)/[1+(F_0{}^i + A_1{}^i s^2)\chi]$ (A.8)

and $\mathcal{A}_1{}^i(s) = F_1{}^i(1-s^2)/2\{1+\tfrac{1}{2}F_1{}^i[(1-s^2)\chi - \tfrac{1}{3}]\}$.

$A_l{}^i$ is defined by the equation $A_l{}^i = F_l{}^i[1+F_l{}^i/(2l+1)]^{-1}$.

APPENDIX B. EVALUATION OF Ξ^i

In this Appendix we describe the calculation of some of the terms which occur in the expressions for Ξ^i

[Eq. (7)]. Substituting Eq. (A.7) into Eq. (7) one finds

$$\Xi^i = \Xi_{00}{}^i + 2\Xi_{11}{}^i - 2\,\mathrm{Re}\,\Xi_{01}{}^i\,, \tag{B.1}$$

where

$$\Xi_{\lambda\mu}{}^i = \int_0^1 ds \left(\frac{\alpha_\lambda{}^i(0)\,\alpha_\mu{}^i(0) - \alpha_\lambda{}^i(s)\alpha_\mu{}^{*i}(s)}{s^2} + \alpha_\lambda{}^i(s)\alpha_\mu{}^{i*}(s) \right) + \alpha_\lambda{}^i(0)\alpha_\mu{}^i(0)\,. \tag{B.2}$$

$\Xi_{00}{}^i$ and $\Xi_{11}{}^i$ may be evaluated analytically by using the fact that

$$\mathrm{Im}\,\alpha_\lambda{}^i(s) = -\tfrac{1}{2}\pi s\,\alpha_\lambda{}^i(s)\alpha_\lambda{}^{i*}(s)\theta(1-|s|), \quad \mathrm{Im}\chi(s) = \tfrac{1}{2}\pi s\theta(1-|s|)\;,$$

where $\theta(s)$ is the Heaviside step function. The first of these results is essentially a form of the optical theorem. Equation (B.2) may therefore be written in the following way:

$$\Xi_{\lambda\lambda}{}^i = \int_0^1 \frac{2ds}{\pi s} \left(\frac{[\alpha_\lambda{}^i(0)]^2\,\mathrm{Im}\chi(s) + \mathrm{Im}\alpha_\lambda{}^i(s)}{s^2} - \mathrm{Im}\alpha_\lambda{}^i(s) \right) + [\alpha_\lambda{}^i(0)]^2\;.$$

This integral is easily converted into a contour integral around the contour C shown in Fig. 4, and the result is

$$\Xi_{\lambda\lambda}{}^i = -\int_C \frac{ds}{2\pi i s} \left(\frac{[\alpha_\lambda{}^i(0)]^2\chi(s) + \alpha_\lambda{}^i(s)}{s^2} - \alpha_\lambda{}^i(s) \right) + [\alpha_\lambda{}^i(0)]^2\,.$$

The function in the braces has a cut along the real axis from $s=-1$ to $s=+1$ but the discontinuity across the cut vanishes as $s\to 0$. It it therefore permissible for the contour to cross the cut as $s\to 0$. $\alpha_\lambda{}^i(s)$ is analytic away from the real axis but it may have poles on the real axis corresponding to collective modes (e.g., zero sound or the $m=1$ collective mode in liquid He^3). The contour can be deformed to pass around the pole at the origin, the poles on the real axis and the circle at infinity as shown in Fig. 5. The integrals may be evaluated easily, and one finds

$$\begin{aligned}\Xi_{00}{}^i &= (A_0{}^i)^2[-\tfrac{1}{4}\pi^2 A_0{}^i + 1 + A_1{}^i] - 2A_0{}^i A_1{}^i + F_0{}^i A_0{}^i - \tfrac{1}{3}F_1{}^i A_1{}^i \\ &\quad + \tfrac{2}{3}F_0{}^i F_1{}^i + \tfrac{1}{9}F_0{}^i (F_1{}^i)^2 + \tfrac{1}{5}(F_1{}^i)^2 + 2R_0{}^i(s_0{}^i) \\ \Xi_{11}{}^i &= \tfrac{1}{2}(A_1{}^i)^2[1-(\pi^2/16)A_1{}^i] + \tfrac{1}{3}F_1{}^i A_1{}^i - \tfrac{1}{30}(F_1{}^i)^2 + 2R_1(s_1{}^i)\,.\end{aligned} \tag{B.3}$$

$R_0{}^i(s_0{}^i)$ and $R_1(s_1{}^i)$ are contributions from the poles corresponding to collective modes and are given by the equations

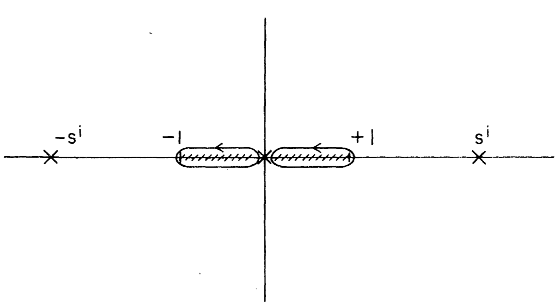

FIG. 4. The contour C. The crosses correspond to poles, those at $\pm s^i$ being due to possible collective modes.

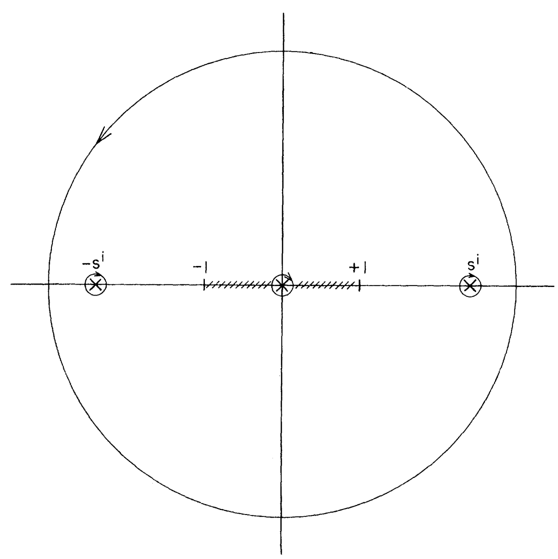

FIG. 5. The contour obtained by deforming the contour C (Fig. 4).

$$R_0{}^i(s) = -[(s^2-1)/s^3](F_0{}^i + A_1{}^i s^2)\left(\frac{d}{ds}[(F_0{}^i + A_1{}^i s^2)\chi(s)]\right)^{-1} \tag{B.4}$$

and $R_1(s) = [(s^2-1)^2/s^3]\left(\frac{d}{ds}[(1-s^2)\chi(s)]\right)^{-1}$,

where $s_0{}^i$ and $s_1{}^i$, the velocities of the collective modes measured in units of the Fermi velocity, are the roots of the equations

$$1 + (F_0{}^i + A_1{}^i s_0{}^{i2})\chi(s_0{}^i) = 0, \quad \text{and} \quad 1 + \tfrac{1}{2}F_1{}^i[(1 - s_0{}^{i2})\chi(s_0{}^i) - \tfrac{1}{3}] = 0 . \tag{B.5}$$

$\Xi_{01}{}^i$ has not been evaluated analytically but analytical approximations suitable for calculating $\Xi_{01}{}^s$ for liquid He^3 at low pressures are described in Sec. 3. Here we give an analytical approximation for $\Xi_{01}{}^i$ when $A_0{}^i$ is large. This result is used in our discussion of an almost ferromagnetic Fermi liquid in Sec. 3(ii). When A_0^i is large $\mathcal{a}_0{}^i(s)$ [Eq. (A.8)] has a strong dependence on s for small values of s, and may be approximated by $A_0{}^i/(1+\frac{1}{2}i\pi A_0{}^i s)$, which is obtained from Eq. (A.8) by putting $A_1{}^i = 0$ and $\mathrm{Re}\chi = 1$. Where $\mathcal{a}_0{}^i(s)$ is large and rapidly varying, $\mathcal{a}_1{}^i(s)$ varies only slowly and may be replaced by $A_1{}^i/2$, its value at $s = 0$. Making these approximations $\mathrm{Re}\Xi_{01}{}^i$ may be evaluated by using Eq. (B.2). The dominant term as $A_0{}^i \to -\infty$ is given by

$$\mathrm{Re}\Xi_{01}{}^i \sim -(\pi^2/8)A_1{}^i(A_0{}^i)^2 \ . \tag{B.6}$$

APPENDIX C. COMMENTS ON PARAMAGNON THEORY

The most important differences between our calculations and those of paramagnon theory[12] are due to the fact that the latter calculations start from a Boltzmann equation for bare fermions and therefore contain the bare fermion mass m whereas our calculations start from a quasiparticle Boltzmann equation, which contains the effective mass m^*. Since the quasiparticle Boltzmann equation has a well-defined microscopic basis[28] it would appear that m^* is the quantity that should appear in the transport equation. The quantitative effects of this difference are most easily seen by considering the results for the almost ferromagnetic limit given in Sec. 3 (ii). The paramagnon theory analogs of Eqs. (11) and (13) differ from these equations only in the replacement of $F_0{}^a$ by $-\bar{I}$, where $\bar{I}$ is related to $F_0{}^a$ by the equation

$$m^*/m = (1 + F_0{}^a)/(1 - \bar{I}) .$$

The predictions of Eqs. (11) and (13) differ from those of their paramagnon analogs by a factor of about 2.5 if one uses values of the Landau parameters appropriate for liquid He^3 at low pressure.

Results similar to those of paramagnon theory may easily be derived from Eq. (1) if one makes a number of further approximations. Firstly, one must neglect the term $2(\phi_1 - \phi_3)(\phi_2 - \phi_4)$ in the expansion of $(\phi_1 + \phi_2 - \phi_3 - \phi_4)^2$ and, secondly, one must neglect all Landau parameters except $F_0{}^a$. As to the latter approximation we have already seen that it is not justified in liquid He^3, and the first approximation is equivalent to the assumption in paramagnon theory that paramagnons are in equilibrium. By explicit calculation one can see that the first approximation leads to errors in Ξ^a of order $A_1{}^a(A_0{}^a)^2$ and of order $A_0{}^a$ but that it gives correctly the result in the almost ferromagnetic limit.

In conclusion we note that our results for D agree in the appropriate limit with the calculations of Ma, Béal-Monod, and Fredkin[29] based on a study of spin waves in the paramagnon model.

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Phase-Space Analysis of Time-Correlation Functions*

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Using the recently developed phase-space techniques for the treatment of quantum-mechanical problems, we set up a procedure for calculating multitime-correlation functions in terms of the joint distribution functions. The correlation functions are then expressed simply as integrals over the associated phase space. Explicit expressions are given for these joint distribution functions, in terms of Green's functions of the c-number equations of motion for the phase-space equivalent of the density operator. Using these joint distribution functions, an exact regression theorem is rederived, and the connection with the multitime correspondence between classical and quantum stochastic processes is discussed.

I. INTRODUCTION

During the last several years, increasing use has been made of time-correlation functions in the description of the behavior of physical systems. Recently Zwanzig[1] summarized the main results in this area and discussed some applications of time-correlation functions to nonequlibrium problems (see also Ref. 2). In the calculation of quantum correlation functions, use has been made of phase-space techniques.[3-6] In this connection the Wigner distribution function has played a preferential role[3] and has been used to obtain first quantum corrections[4-6] to time-correlation functions calculated classically. This in turn permits one to obtain quantum corrections to transport coefficients. Similar procedures have been useful in discussing a wide variety of problems such as nuclear magnetic relaxation,[4] neutron scattering,[5] hydrodynamic transport coefficients, etc. However, in the discussion of certain problems in quantum optics, it is useful to use other distribution functions[7] based on different rules of association between functions of noncommuting operators and c-number functions. Recently a general technique, for the derivation of the different distribution functions from a unified point of view, was developed[8-10] and has been used to study dynamical problems.[10]

In the present investigation, we extend this analysis to construct various joint distribution functions. These functions are then used to express multitime-correlation functions as integrals over the associated phase space. An exact re-